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Appendix 1:
Chapterwise Solved JEE Main 2019 Questions (16 Sets) **A.1–A.8**

Appendix 2:
Chapterwise Solved JEE Main January 2020 Questions (6 Sets) **A.9–A.12**

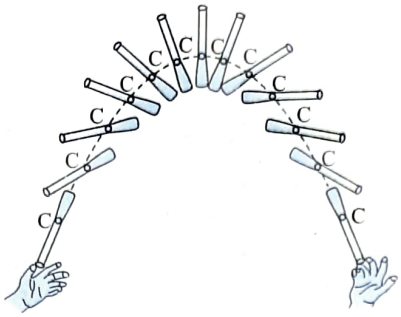
Appendix 3:
JEE Advanced 2019 Solved Questions **A.13–A.16**

1 System of Particles and Centre of Mass

CENTRE OF MASS

In this section, we will discuss how the motion of a complicated system of objects, can be simplified in terms of a very special point called the **centre of mass of the system**. This notion gives us confidence in the particle model because we will see that the centre of mass accelerates as if all the system's mass were concentrated at that point and all external forces act there.

If you throw a ball into the air without spin, its motion is simple—it follows a parabolic path, as we had discussed in projectile motion in kinematics, and the ball can be treated as a particle. If, instead, let us throw a baseball bat into the air (figure), holding it from one of its ends so that it rotates also Snapshots taken after regular intervals of time are shown in the figure. As the bat moves all of its particles move in a complex manner except one special point C, which follows a parabolic trajectory as if it was a particle of mass equal to that of the bat and force of gravity was acting on it. This special point is called **centre of mass** of the bat.



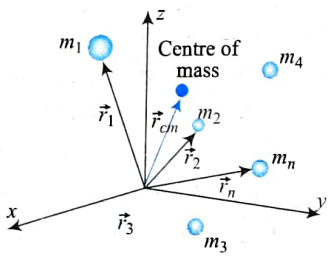
Concept of centre of mass provides us an easier way to look into motion of the system as a whole as superposition of translation of the centre of mass and motion of all the particles relative to the centre of mass. This concept is used to represent net translation of the system. Therefore total linear momentum of the whole system must be equal to the linear momentum of its centre of mass.

CENTRE OF MASS OF A SYSTEM OF N DISCRETE PARTICLES

Consider a system of N point masses m_1, m_2, m_3, ... m_n whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_n$, respectively. The position vector of the centre of mass C of the system is given by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$



where $M\left(=\sum_{i=1}^{n} m_i\right)$ is the total mass of the system.

Consider a system of point masses m_1, m_2, m_3, ..., located at the coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), ..., respectively. The centre of mass of this system of particles is a point whose coordinates are (x_{cm}, y_{cm}, z_{cm}), which are given by

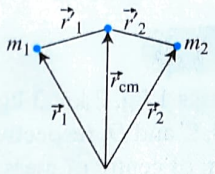
$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + \cdots}{m_1 + m_2 \cdots}$$

and $$z_{cm} = \frac{m_1z_1 + m_2z_2 + \cdots}{m_1 + m_2 + \cdots}$$

Important Points:

- The centre of mass of two particles lies on the line joining them.
  To verify this, let $\vec{r}_1'$ and $\vec{r}_2'$ be position vector of m_1 and m_2 relative to centre of mass.



  Then, $m_1\left(\vec{r}_{cm} + \vec{r}_1'\right) + m_2\left(\vec{r}_{cm} + \vec{r}'\right) = \left(m_1 + m_2\right)\vec{r}_{cm}$

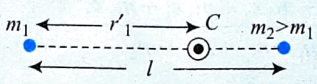
  $\Rightarrow \quad m_1\vec{r}_1' + m_2\vec{r}_2' = 0$ or $m_1\vec{r}_1' = -m_2\vec{r}_2'$

  Thus, $\vec{r}_1'$ and $\vec{r}_2'$ are directed opposite to each other. This shows that $\vec{r}_1'$ and $\vec{r}_2'$ are at 180° with each other but their tails lie at a common point. Thus, C lies in the line joining the particles.
- The centre of mass divides the distance between two particles in the inverse ratio of masses.

  $$m_1\vec{r}_1' + m_2\vec{r}_2' = 0 \Rightarrow \frac{|\vec{r}_1'|}{|\vec{r}_2'|} = \frac{m_2}{m_1} \qquad \text{[taking modulus]}$$

  Thus, the centre of mass is dividing the distance in inverse ratio of their masses, i.e., $r' \propto \frac{1}{m}$. Hence, the centre of mass is closer to more massive particle.

  $$m_1r_1' = m_2\left(l - r_1'\right)$$

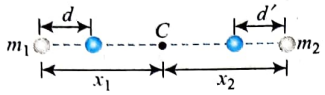


  $$r_1' = \left(\frac{m_2}{m_1 + m_2}\right)l$$

ILLUSTRATION 1.1

Consider a two-particle system with the particles having masses m_1 and m_2. If the first particle is pushed towards the centre of mass through a distance d, by what distance should the second particle be moved so as to keep the centre of mass at the same position?

Sol. We know the centre of mass of two particle systems lie on the line joining them and divides the distance between two particles in the inverse ratio of masses.



Suppose the distance of m_1 from the centre of mass C is x_1 and that of m_2 from C is x_2.

Here, $\frac{m_1}{m_2} = \frac{x_2}{x_1}$

Then, $m_1 x_1 = m_2 x_2$ ...(i)

If the first particle is pushed towards the centre of mass through a distance d, to keep the centre of mass at the same position suppose mass m_2 has to move through a distance d' towards C. It means we can write

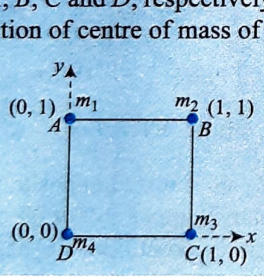
$$m_1(x_1 - d) = m_2(x_2 - d') \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$m_1 d = m_2 d' \text{ or } d' = (m_1/m_2)\, d$$

ILLUSTRATION 1.2

Four particles of masses 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D, respectively, of a square of side 1 m. Find the position of centre of mass of the particles.



Sol. Let us first write the coordinates of the positions of the particles

Position of $m_1 = 1$ kg, $(x_1, y_1) = (0, 1\text{ m})$

Position of $m_2 = 2$ kg, $(x_2, y_2) = (1\text{ m}, 1\text{ m})$

Position of $m_3 = 3$ kg, $(x_3, y_3) = (1\text{ m}, 0)$

Position of $m_4 = 4$ kg, $(x_4, y_4) = (0, 0)$

Now coordinates of their CM are calculated as follows.

x-coordinate of CM

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1 + 2 + 3 + 4} = \frac{5}{10} = \frac{1}{2}\text{ m} = 0.5\text{ m}$$

Similarly, y-coordinate of CM

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

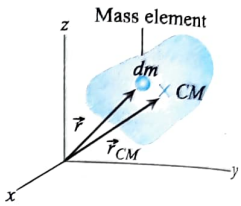
$$= \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1 + 2 + 3 + 4} = \frac{3}{10}\text{ m} = 0.3\text{ m}$$

Hence the coordinate of centre of mass of the particle system,

$$(x_{CM}, y_{CM}) = (0.5\text{ m}, 0.3\text{ m})$$

CENTRE OF MASS OF A CONTINUOUS MASS DISTRIBUTION

For a continuous mass distribution, the centre of mass can be located by replacing summation sign with an integral sign. Proper limits for the integral are chosen according to the situation.



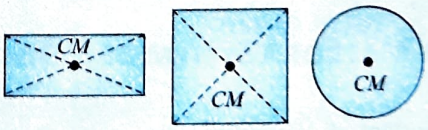
$$x_{CM} = \frac{\int x\,dm}{\int dm},\quad y_{CM} = \frac{\int y\,dm}{\int dm},\quad z_{CM} = \frac{\int z\,dm}{\int dm}$$

$$\int dm = M \text{ (mass of the body)}$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

Important Points:

- **Symmetry as a Shortcut:** We can bypass one or more of these integrals if an object has a point, a line, or a plane of symmetry. The centre of mass of such an object then lies at that point, on that line, or in that plane. For example, the centre of mass of a uniform sphere (which has a point of symmetry) is at the centre of the sphere (which is the point of symmetry). The centre of mass of a uniform cone (whose axis is a line of symmetry) lies on the axis of the cone. Centre of mass of a uniform rectangular, square or circular plate lies at its centre.



- The centre of mass of an object need not lie within the object. There is no dough at the CM of a doughnut, and no iron at the CM of a horseshoe.
- An extended object is a continuous distribution of mass, each small mass element is acted upon by the gravitational force. The net effect of all these forces is equivalent to the effect of a single force $M\vec{g}$ acting through a special point, called the **centre of gravity**. If $\vec{g}$ is constant over the mass distribution, the centre of gravity coincides with the centre of mass. If an extended object is pivoted at its centre of gravity, it balances in any orientation.

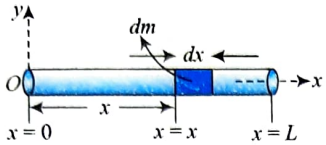
CENTRE OF MASS OF A UNIFORM ROD

Consider a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$.

Mass per unit length of the rod $= \dfrac{M}{L}$

Hence, the mass of the element dx situated at $x = x$ is

$$dm = \frac{M}{L}dx$$



The coordinates of the element dm are $(x,0)$. Therefore, x-coordinate of CM of the rod will be

$$x_{cm} = \frac{\int_0^L x\,dm}{\int dm} = \frac{\int_0^L x\frac{M}{L}dx}{M} = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

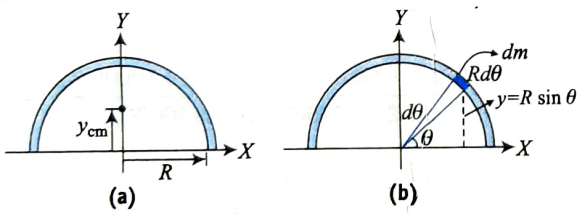
The y-coordinate of CM is zero.

That is, the coordinates of CM of the rod are $(L/2, 0)$, or it lies at the centre of the rod.

Note: We can locate the centre of mass of the rod by using symmetry property. We note that the rod is symmetric about the middle point. Thus, centre of mass must lies at the centre of the rod.

CENTRE OF MASS OF SEMICIRCULAR RING

If we draw y-axis such that it passes through the centre of curvature of the semicircular ring, it divide the ring into two equal halves. By symmetry property we can say that the x-coordinate of the centre of the mass of the ring is zero as the half ring is symmetrical on both side of the origin. Now it is required to find the y-coordinate of the centre of mass.



Therefore, y-coordinate of CM of the ring will be

$$y_{cm} = \frac{1}{M}\int dm\ y \qquad ...(i)$$

Let us consider an elemental arc of the ring at an angle θ from the x-direction of angular width $d\theta$. If radius of ring is R then its y coordinate will be $R\sin\theta$. The mass of the elemental arc dm is given as

$$dm = \frac{M}{\pi R} \times Rd\theta$$

So from Eq. (i), we have

$$y_{cm} = \frac{1}{M}\left(\int_0^\pi \frac{M}{\pi R}Rd\theta\right)R\sin\theta = \frac{R}{\pi}\int_0^\pi \sin\theta d\theta$$

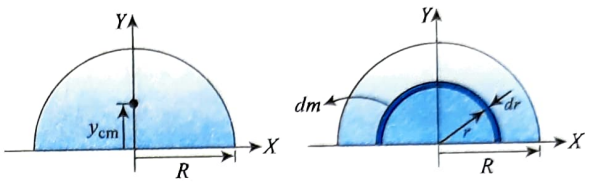
$$= \frac{R}{\pi}[-\cos\theta]_0^\pi = -\frac{R}{\pi}[\cos\pi - \cos 0]$$

Hence, $y_{cm} = \dfrac{2R}{\pi}$

CENTRE OF MASS OF A SEMICIRCULAR DISC

In the figure shown, a semicircular disc of mass M and radius R. In this case also we required to find the y-coordinate of the centre of mass of this disc. Here to find y_{cm}, we consider a small elemental ring of mass dm of radius r on the disc which will be integrated from 0 to R. Here dm is given as

$$dm = \frac{2M}{\pi R^2}\pi r\,dr$$



Now the y-coordinate of dm is taken as $\dfrac{2r}{\pi}$, as, we have derived that mass of a half ring is concentrated at $\dfrac{2R}{\pi}$ distance from its centre.

Here y_{cm} is given as,

$$y_{cm} = \frac{1}{M}\int_0^R dm\frac{2r}{\pi} = \frac{1}{M}\int_0^R\left(\frac{2M}{\pi R^2}\pi r\,dr\right)\frac{2r}{\pi} = \frac{4}{\pi R^2}\int_0^R r^2 dr$$

$$= \frac{4}{\pi R^2}\left[\frac{r^3}{3}\right]_0^R = \frac{4}{3\pi R^2}[R^3 - 0]$$

Hence, $y_{cm} = \dfrac{4R}{3\pi}$

The following points regarding centre of mass can be noted.

- For a lamina type (two-dimensional) body with uniform negligible thickness, the formulae for finding the position of centre of mass are as follows:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$= \frac{\rho A_1 t\vec{r}_1 + \rho A_2 t\vec{r}_2 + \cdots}{\rho A_1 t + \rho A_2 t + \cdots} \qquad (m = \rho At)$$

$$= \frac{A_1\vec{r}_1 + A_2\vec{r}_2 + \cdots}{A_1 + A_2 + \cdots}$$

  Here, A stands for area and ρ for density.

- If some mass is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formulae:

(a) $\vec{r}_{CM} = \dfrac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$ or $\vec{r}_{CM} = \dfrac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$

(b) $x_{CM} = \dfrac{m_1x_1 - m_2x_2}{m_1 - m_2}$ or $x_{CM} = \dfrac{A_1x_1 - A_2x_2}{A_1 - A_2}$

(c) $y_{CM} = \dfrac{m_1y_1 - m_2y_2}{m_1 - m_2}$ or $y_{CM} = \dfrac{A_1y_1 - A_2y_2}{A_1 - A_2}$

(d) $z_{CM} = \dfrac{m_1 z_1 - m_2 z_2}{m_1 - m_2}$ or $z_{CM} = \dfrac{A_1 z_1 - A_2 z_2}{A_1 - A_2}$

Here, $m_1, A_1, \vec{r}_1, x_1, y_1$ and z_1 are the values for the whole mass while $m_2, A_2, \vec{r}_2, x_2, y_2$ and z_2 are the values for the mass which has been removed.

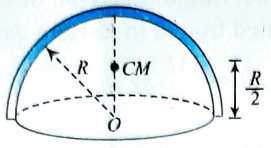
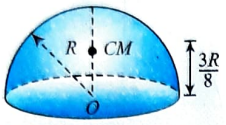
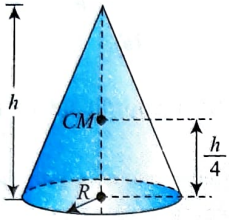
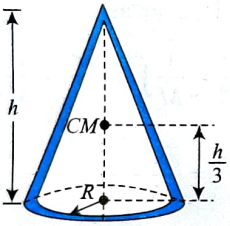
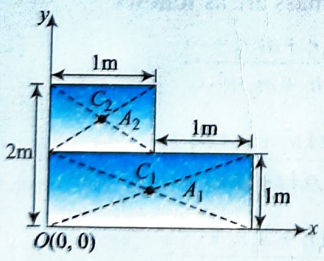
Shape	Position of centre of mass
Hemispherical shell	
Solid hemisphere	
Solid cone	
Hollow cone	

ILLUSTRATION 1.3

Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown in figure. The mass of lamina is 3 kg.



Sol. The plate has uniform density and same thickness everywhere. It means mass of the plate should be proportional to its surface area. So its CM will coincide with the centroid.

Divide the given plate into two parts of area A_1 and A_2 as shown in the figure. We have

$A_1 = 2 \times 1\ m^2$ with its centroid C_1 (1 m, 1/2 m) and

$A_2 = 1 \times 1\ m^2$ with its centroid C_2 (1/2 m, 3/2 m)

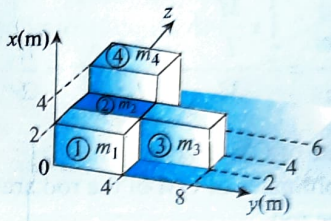
The centroid of the whole plate can be defined as

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} = \frac{2\times 1 + 1\times \frac{1}{2}}{2+1} = \frac{5}{6}\text{m}$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{2\times \frac{1}{2} + 1\times \frac{3}{2}}{2+1} = \frac{5}{6}\text{m}$$

ILLUSTRATION 1.4

The four freight containers shown in figure, sit on the deck of a container ship. The containers 1, 2 and 3 each have mass 5000 kg and size 4 m × 2 m × 2 m. The container 4 has mass 10000 kg and size 4 m × 4 m × 2 m. Assume that each of the containers has an individual centre of mass at its geometric centre. What are the x-coordinate, y-coordinate and the z-coordinate of the containers' combined centre of mass? Use the coordinate system shown in the figure to describe the location of this centre of mass.



Sol. The coordinates of CM of the containers

Containers	Mass	Coordinates
1	5000 kg	(1 m, 2 m, 1 m)
2	5000 kg	(1 m, 2 m, 3 m)
3	5000 kg	(1 m, 6 m, 3 m)
4	10000 kg	(2 m, 2 m, 5 m)

For the x-coordinate of the combined centre of mass, we find

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{5000\times 1 + 5000\times 1 + 5000\times 1 + 10000\times 2}{(5000 + 5000 + 5000 + 10000)} = \frac{7}{5}\text{m}$$

In the same way, we can calculate the y-coordinate:

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{5000\times 2 + 5000\times 2 + 5000\times 6 + 10000\times 2}{5000 + 5000 + 5000 + 10000} = \frac{14}{5}\text{m}$$

Similarly z-coordinate of CM

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + m_4 z_4}{m_1 + m_2 + m_3 + m_4}$$

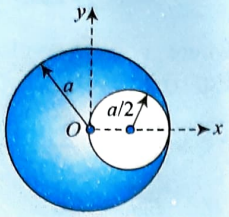
$$= \frac{5000\times 1 + 5000\times 3 + 5000\times 3 + 10000\times 5}{5000 + 5000 + 5000 + 10000} = \frac{17}{5}\text{m}$$

Hence the coordinate of centre of mass of the container is

$$\left(\frac{7}{5}\text{m}, \frac{14}{5}\text{m}, \frac{17}{5}\text{m}\right)$$

ILLUSTRATION 1.5

A uniform metal plate of radius a, from which a disk of diameter a has been stamped out (removed) as shown in figure. Locate the centre of mass of the remaining plate. Find the position of centre of mass of the plate.



Sol. We can roughly locate the centre of mass of the plate by using symmetry property. We note that the plate is symmetric about the x-axis. Thus, centre of mass must be on the x-axis. The plate (with the disk removed) is not symmetric about the y- axis. However, because there is somewhat more mass on the right of the y axis, the centre of mass must be somewhat to the right of that axis. We can assume that the mass of a uniform object (as we have here) is concentrated in a particle at the object's centre of mass. Thus we can treat the object as a particle and avoid any two-dimensional integration.

As the particle has uniform density and same thickness everywhere, so its mass is proportional to its area.

Here, A_1 = area of complete plate = πa^2

A_2 = area of small disk = $\pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$

(x_1, y_1) = coordinates of centre of mass of the complete plate = $(0, 0)$

(x_2, y_2) = coordinates of centre of mass of the small disk = $\left(\frac{a}{2}, 0\right)$

Using $x_{CM} = \frac{(A_1x_1 - A_2x_2)}{(A_1 - A_2)}$ we get,

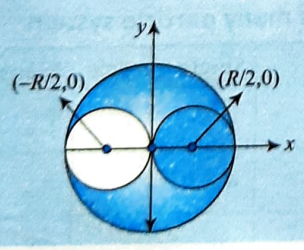
$$x_{CM} = \frac{\pi a^2 \times 0 - \frac{\pi a^2}{4} \times \frac{a}{2}}{\pi a^2 - \frac{\pi a^2}{4}} = -\frac{a}{6}$$

$y_{CM} = 0$ (as y_1 and y_2 both are zero)

Therefore, coordinates of CM of the lamina shown in figure are $(-a/6, 0)$.

ILLUSTRATION 1.6

Figure shows a uniform disc of radius R, from which a hole of radius $R/2$ has been cut out from left of the centre and is placed on the right of the centre of the disc. Find the CM of the resulting disc.



Sol. In this case the plate is symmetric about x-axis. Thus, centre of mass must be on the x-axis. In this situation the plate has uniform thickness hence its mass is proportional to its area.

The area of uncut plate $A_1 = \pi R^2$

and area of cut-out $A_2 = \pi\left(\frac{R}{2}\right)^2 = \frac{\pi R^2}{4}$

The x-coordinate of centre of mass of the system,

$$x_{cm} = \frac{A_1x_1 - A_2x_2 + A_3x_3}{A_1 - A_2 + A_3}$$

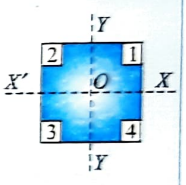
x_1 is the position of centre of mass of uncut plate which is at origin. x_2 is the position of centre of mass of the cutout which is at $\left(-\frac{R}{2}, 0\right)$ and x_3 is the position of the centre of mass of the cut-out plate which is placed on the right of the centre of the disc which is at $\left(\frac{R}{2}, 0\right)$.

$$\text{Hence } x_{cm} = \frac{\pi R^2 \times 0 - \frac{\pi}{4}R^2 \times \left(-\frac{R}{2}\right) + \frac{\pi}{4}R^2 \times \left(\frac{R}{2}\right)}{\pi R^2 - \frac{\pi R^2}{4} + \frac{\pi R^2}{4}} = \frac{R}{4}$$

Hence the coordinate of centre of mass of system will be $\left(\frac{R}{4}, 0\right)$.

CONCEPT APPLICATION EXERCISE 1.1

1. A uniform square plate from which four identical squares at the corners will be removed is shown in figure. Answer the following questions on the basis of above data.



(i) Where is the centre of mass of the uncut plate?

(ii) Where does CM lie after square 1 is removed?

(iii) Where does CM lie after squares 1 and 2 are removed?

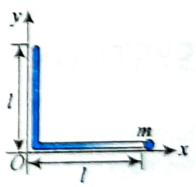
(iv) Where does CM lie after squares 1 and 3 are removed?

(v) Where does CM lie after squares 1, 2 and 3 are removed?

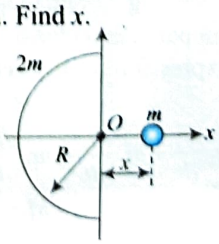
(vi) Where does CM lie after all the four squares are removed?

(Give Answers in terms of quadrants and axes.)

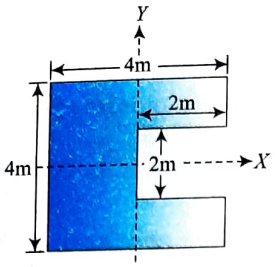
2. Find the CM of the system of a L-shaped rod of mass m and a particle of mass m attached at the end of the rod as shown.



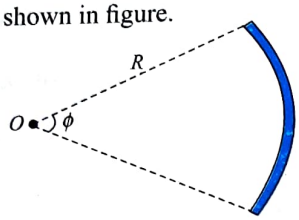
3. A ring and a particle of masses $2m$ and m are kept as shown in the figure, such that the CM of the combination lies at the origin. Find x.



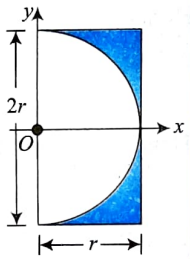
4. Find x and y coordinates of the centre of mass of the plate shown in figure from which a square of side 2 m is cut out.



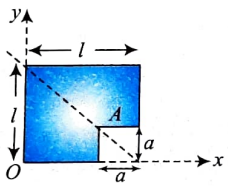
5. Find centre of mass distance from point O of the uniform circular arc shown in figure.



6. Determine the coordinate of centre of mass of a thin homogeneous plate having the form of a rectangle with sides r and $2r$ from which a semicircle with a radius r is cut out.



7. The figure shows a square metal plate of side l from which a square plate of side a has been cut as shown in the figure. Find the ratio (a/l) so that the centre of mass of the remaining L–shaped plate coincides with the point A.



ANSWERS

1. (i) at O (ii) III quadrant (iii) y-axis (OY) (iv) at O (v) at 4 (vi) at O

2. $\frac{5l}{8}, \frac{l}{8}$ **3.** $\frac{4R}{\pi}$ **4.** $\frac{-1}{3}$ m, 0 **5.** $2R\sin\frac{\phi}{2}$

6. $\left(\frac{2r}{3(4-\pi)}, 0\right)$ **7.** $\left(\frac{\sqrt{5}-1}{2}\right)l$

SYSTEMS OF MANY PARTICLES

Let us consider a system of two or more particles for which we have identified the centre of mass. Here we begin to understand the physical significance and utility of the centre of mass concept by taking the time derivative of the position vector for the centre of mass. The time derivative of a position vector is by definition the velocity vector. Assuming M remains constant for a system of particles—that is, no particles enter or leave the system—we obtain the following expression for the **velocity of the centre of mass** of the system:

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \cdots + m_n\frac{d\vec{r}_n}{dt}}{M}$$

$$\vec{v}_{CM} = \frac{1}{M}\sum_i m_i\frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_i m_i\vec{v}_i \quad \ldots(i)$$

where $\vec{v}_i$ is the velocity of the i^{th} particle. Rearrangement of Eq. (i) gives

$$M\vec{v}_{CM} = \sum_i m_i\vec{v}_i = \sum_i \vec{p}_i$$

$$\vec{p}_{tot} = M.\vec{v}_{CM} \quad \ldots(ii)$$

Therefore, the total linear momentum of the system equals the total mass multiplied by the velocity of the centre of mass. In other words, the total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity $\vec{v}_{CM}$.

Differentiating Eq. (ii) with respect to time, we obtain the **acceleration of the centre of mass** of the system:

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \cdots + m_n\frac{d\vec{v}_n}{dt}}{M}$$

$$= \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n}{M} = \frac{\sum\vec{F}_{net}}{M}$$

Rearranging this expression and using Newton's second law gives

$$\therefore \quad \vec{F}_{total} = M\vec{a}_{CM} \quad \ldots(iii)$$

The forces on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). By Newton's third law, however, the internal force exerted by particle 1 on particle 2, for example, is equal in magnitude and opposite in direction to the internal force exerted by particle 2 on particle 1. Therefore, when we sum over all internal force vectors in Eq. (iii), they cancel in pairs and we find that the net force on the system is caused only by external forces. We can then write Eq. (iii) in the form

$$\sum\vec{F}_{ext} = M\vec{a}_{CM} \quad \ldots(iv)$$

That is, the net external force on a system of particles equals the total mass of the system multiplied by the acceleration of the centre of mass. Comparing Eq. (iv) with Newton's second law for a single particle, we see that the particle model which have used in several chapters can be described in terms of the centre of mass.

The centre of mass of a system of particles having combined mass M moves like an equivalent particle of mass M would move under the influence of the net external force on the system.

Single particle vs many particle system

Item	Single particle	Many-particle system
Velocity	$\vec{v}$	$\vec{v}_{cm}$
Accelerationa	$\vec{a}$	$\vec{a}_{cm}$
Momentum	$m\vec{v}$	$m\vec{v}_{cm}$
Newton's second law of motion	$\vec{F} = m\vec{a}$	$\vec{F}_{ext} = m\vec{a}_{cm}$

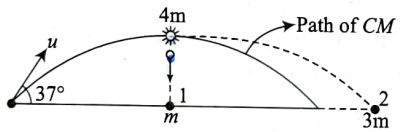
Note:

- Net force on a system of particles is equal to net external force.
- Sum of internal forces on particles of a system is zero.
- Centre of mass cannot be accelerated by internal forces.
- External forces equal mass multiplied by acceleration of centre of mass.
- Action point of external force is immaterial in producing acceleration in centre of mass.
- If $\vec{F}_{ext} = 0$, the CM moves with constant velocity. Hence, the CM (but not necessarily other points of the system which may be accelerating) can be treated as an inertial frame (point).
- The net effect of internal forces is zero on the CM whereas it is not zero (in general) for other points of the system. Inconsequence, the CM moves with a constant velocity whereas other points may move with same accelerations when $\vec{F}_{ext} = 0$.

ILLUSTRATION 1.7

A projectile is fired at a speed of 100 m/s at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1:3, after explosion the lighter part comes to rest. Find the distance from the launching point to the point where the heavier piece lands.

Sol. Neglecting air resistance, the only external force on the projectile is the gravitational force. Therefore, if the projectile did not explode, it would continue to move along the parabolic path indicated by the solid line drawn in figure. Because the forces caused by the explosion are internal, they do not affect the motion of the centre of mass of the system (the fragments). Therefore, after the explosion, the centre of mass of the fragments follows the same parabolic path the projectile would have followed if no explosion had occurred.



The range of the original projectile is

$$x_{CM} = \frac{2u^2 \sin\theta\cos\theta}{g} = \frac{2\times10^4 \times \frac{3}{5}\times\frac{4}{5}}{10}\,\text{m} = 960\,\text{m}$$

The centre of mass will hit the ground at this position. As the smaller block comes to rest after breaking, it falls down vertically and hits the ground at half of the range, i.e., at x = 480 m. If the heavier block hits the ground at x_2, then

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$\Rightarrow\ 960 = \frac{(m)(480) + (3m)(x_2)}{(m+3m)}$$

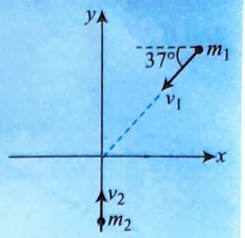
$$\therefore\ x_2 = 1120\,\text{m}$$

ILLUSTRATION 1.8

Two balls with masses m_1 = 3 kg and m_2 = 5 kg have initial velocities $v_1 = v_2$ = 5 m/s in the directions shown in figure. They collide at the origin.

(a) Find the velocity of the CM 3 s before the collision.

(b) Find the position of the CM 2 s after the collision.



Sol.

(a) If we take ball '1' and ball '2' as system, there is no external force acting on the system. Hence the velocity of the CM of the system will be constant. From given figure, we can calculate the velocity of the CM, which will also be the velocity of CM 3 s before the collision.

x-component of velocity of CM,

$$v_{CM(x)} = \frac{m_1v_{1x} + m_2v_{2x}}{m_1 + m_2}$$

$$= \frac{(3)(-5\cos37°) + (5)(0)}{8\,\text{kg}} = -1.5\,\text{m/s}$$

y-component of velocity of CM,

$$v_{CM(y)} = \frac{m_1v_{1y} + m_2v_{2y}}{m_1 + m_2}$$

$$= \frac{(3)(-5\sin37°) + (5\times5)}{8\ \text{kg}} = +2\,\text{m/s}$$

$$\therefore\ \vec{v}_{CM} = -1.5\hat{i} + 2\hat{j}\ \text{m/s}$$

(b) As collision occurs at the origin ($\vec{r}_i = 0$), it means the position of CM at the time of collision will be at origin. The position of the centre of mass 2 s after the collision is,

$$\vec{r}_{CM} = \vec{r}_i + \vec{v}_{CM}t = \vec{v}_{CM}t$$

$$= (-1.5\hat{i} + 2\hat{j})\times2 = \left(-3\hat{i} + 4\hat{j}\right)\text{m}$$

ILLUSTRATION 1.9

Two particles of masses 2 kg and 4 kg are approaching towards each other with accelerations 1 m/s² and 2 m/s², respectively, on a smooth horizontal surface. Then find the acceleration of centre of mass of the system.

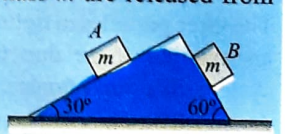
Sol. The acceleration of CM of the system is given by

$$a_{CM} = \frac{m_1a_1 + m_2a_2}{m_1 + m_2} \Rightarrow a_{CM} = \frac{2\times1 + 4(-2)}{2+4} = -1\ \text{m/s}^2$$

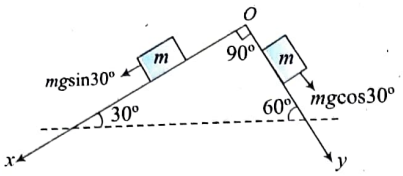
Negative sign indicates that acceleration of CM will be in the direction of acceleration of 4 kg mass.

ILLUSTRATION 1.10

Two blocks A and B each of equal mass m are released from the top of a smooth fixed wedge as shown in the figure. Find the magnitude of the acceleration of the centre of mass of the two blocks.



Sol. Let us consider incline surfaces of wedge as x-axis and y-axis respectively. We can write accelerations of blocks A and B.



$$\vec{a}_A = g\sin 30°\hat{i} \text{ and } \vec{a}_B = g\sin 60°\hat{j}$$

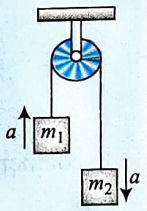
The acceleration of centre of mass of two blocks can be written as

$$\vec{a}_{cm} = \frac{m_A\vec{a}_A + m_B\vec{a}_B}{m_A + m_B} = \frac{mg\sin 30°\hat{i} + mg\sin 60°\hat{j}}{m+m}$$

$$= \frac{g\sin 30°\hat{i} + g\sin 60°\hat{j}}{2} = \frac{g}{4}\hat{i} + \frac{\sqrt{3}g}{4}\hat{j} \Rightarrow |\vec{a}_{cm}| = \frac{g}{2}$$

ILLUSTRATION 1.11

A pulley fixed to the ceiling carried a thread with bodies of masses m_1 and m_2 attached to its ends. The masses of the pulley and the thread are negligible and friction is absent. Find the acceleration of the centre of mass of this system.



Sol. Let us assume that $m_2 > m_1$. We can see that the masses have equal and opposite acceleration given by,

$$a = \frac{m_2 - m_1}{m_2 + m_1}g$$

and tension in string is $T = \frac{2m_1m_2g}{m_2 + m_1}$

Taking downward direction as positive,

$$a_1 = -a;\ a_2 = +a$$

$$a_{CM} = \frac{m_1a_1 + m_2a_2}{m_1 + m_2} = \frac{(m_2 - m_1)a}{m_2 + m_1}$$

Substituting for the value of a, we have

$$a_{CM} = \left(\frac{m_2 - m_1}{m_2 + m_1}\right)^2 g$$

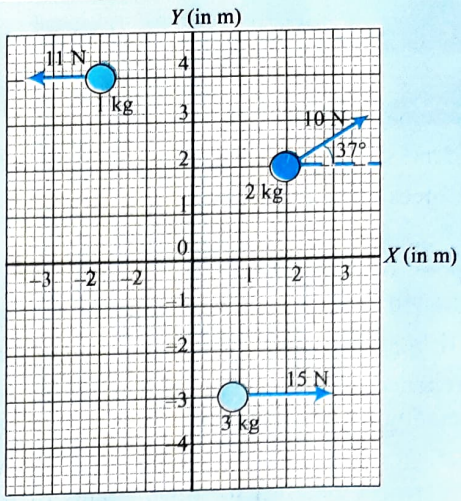
Alternative method:

$$a_{CM} = \frac{F_{ext}}{m_1 + m_2} = \frac{(m_1g + m_2g) - 2T}{m_1 + m_2} = g - \frac{2T}{m_1 + m_2}$$

$$= g - \frac{4m_1m_2g}{(m_2 + m_1)^2} = \left(\frac{m_2 - m_1}{m_2 + m_1}\right)^2 g \qquad \text{(downwards)}$$

ILLUSTRATION 1.12

The three particles in figure are initially at rest. Each experiences an external force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are $F_1 = 11.0$ N, $F_2 = 10$ N, and $F_3 = 15$ N.



(i) Find the position of centre of mass.
(ii) What is the acceleration of the centre of mass of the system, and in what direction does it move?

Sol. The centre of mass of the system of these 3 particles lies at a point, whose coordinates (x, y, z) are given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{1(-2) + 2(1) + 3(1)}{1+2+3} = \frac{1}{2}\text{m}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{1(4) + 2(2) + 3(-3)}{1+2+3} = -\frac{1}{6}\text{m}$$

We can treat the centre of mass as if it were a real particle, with a mass equal to the system's total mass $M = 6$ kg. We can also treat the three external forces as if they act at the centre of mass. We can now apply Newton's second law $(\vec{F}_{net} = m\vec{a})$ to the centre of mass, writing

$$\vec{F}_{net} = M\vec{a}_{cm} \text{ or } \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{cm}$$

$$\text{so } \vec{a}_{cm} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M} \qquad \text{...(i)}$$

Equation (i) tells us that the acceleration $\vec{a}_{cm}$ of the centre of mass is in the same direction as the net external force $\vec{F}_{net}$ on the system. We can rewrite Eq. (i) in component form,

$$a_{cm,x} = \frac{F_{1x} + F_{2x} + F_{3x}}{M}$$

$$= \frac{-11.0\text{ N} + (10\text{ N})\cos 37° + 15\text{ N}}{6\text{ kg}} = 2.0\text{ m/s}^2$$

Along the y axis, we have,

$$a_{cm,y} = \frac{F_{1y} + F_{2y} + F_{3y}}{M} = \frac{0 + (10\text{ N})\sin 37° + 0}{6\text{ kg}} = 1.0\text{ m/s}^2$$

Hence the acceleration of the centre of mass of the system

$$\vec{a}_{cm} = (2.0\hat{i} + 1.0\hat{j})\text{ m/s}^2$$

From these components, we find that $\vec{a}_{cm}$ has the magnitude

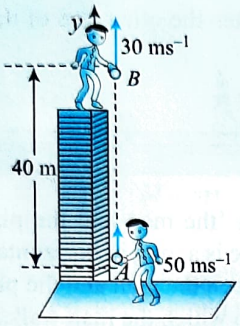
$$a_{cm} = \sqrt{(a_{cm,x})^2 + (a_{cm,y})^2} = \sqrt{(2)^2 + (1)^2} = \sqrt{5}\text{ m/s}^2$$

and the angle (from the positive direction of the x axis);

$$\theta = \tan^{-1}\frac{a_{cm,y}}{a_{cm,x}} = \tan^{-1}\left(\frac{1}{2}\right)$$

ILLUSTRATION 1.13

Two balls A and B of equal masses are projected upward simultaneously, one from the ground with speed 50 m s^{-1} and other from a tower of height 40 m above the first ball with initial speed 30 m s^{-1}. Find the maximum height attained by their centre of mass.



Sol. Let us take ball 'A' and 'B' as system the initial height of centre of mass from ground,

$$y_{ic} = \frac{m\times 0 + m\times 40}{2m} = 20\,\text{m}$$

Initial velocity of centre of mass

$$u_{cm} = \frac{m\times 50 + m\times 30}{2m} = 40\,\text{m s}^{-1}$$

Acceleration of centre of mass, $a_{cm} = -g$

Using kinematics equation: $v_{cm}^2 = u_{cm}^2 + 2a_{cm}H$

Here H is the maximum height reach by centre of mass of two balls from initial level.

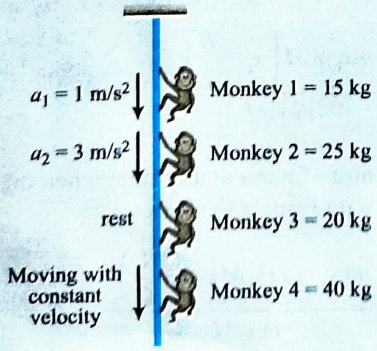
$$\therefore\ 0^2 = 40^2 - 2\times 10\times H \Rightarrow H = \frac{1600}{20} = 80\,\text{m}$$

Hence, maximum height reach by centre of mass from ground level will be

$$h_{max} = y_{ic} + H = 20 + 80 = 100\ \text{m}$$

ILLUSTRATION 1.14

A light rope supporting four monkeys, is hanging from roof as shown in figure. Monkey 1 of mass 15 kg is going upwards with an acceleration 1 m/s^2. Monkey 2 of mass 25 kg is going downwards with acceleration 3 m/s^2. Monkey 3 of mass 20 kg is at rest. Monkey 4 of mass 40 kg is coming downwards with constant speed. Find out the force exerted by ceiling on the rope.



Sol. Taking four monkeys together as system. The external forces acting on the system is tension and gravitational force.

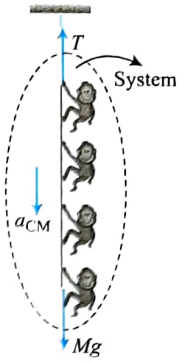
We know that, $F_{ext} = Ma_{cm}$ ...(i)

Now, a_{cm} can be calculated as:

$$a_{cm} = \frac{m_1a_1 + m_2a_2 + m_3a_3 + m_4a_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{15(1) + 25(-3) + 20(0) + 40(0)}{15 + 25 + 20 + 40}$$

$$\Rightarrow a_{cm} = -\frac{60}{100} = \frac{6}{10}\ \text{m/s}^2 \text{ downward}$$



So, by Eq. (i):

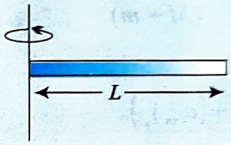
$$Mg - T = Ma$$

$$\Rightarrow 100\times 10 - T = 100\times 0.6$$

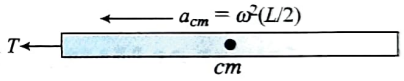
$$\Rightarrow T = 1000 - 60 = 940\text{N}$$

ILLUSTRATION 1.15

A uniform rod of mass M and length L is tied to a vertical shaft. It rotates in horizontal plane about the vertical axis at angular velocity ω. How much horizontal force does the shaft exert on the rod?



Sol. Let T be the force applied horizontally on the rod. The acceleration of centre of mass is $\omega^2\left(\frac{L}{2}\right)$, as it is moving in a circle of radius $L/2$.

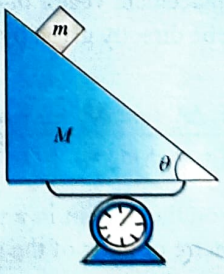


Using $\vec{F}_{ext} = M\vec{a}_{cm} \Rightarrow T = M\omega^2\left(\frac{L}{2}\right)$

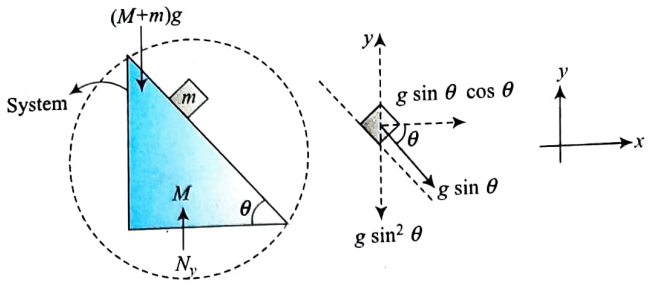
Thus, the horizontal force exerted is $\frac{1}{2}M\omega^2 L$.

ILLUSTRATION 1.16

A triangular wedge of mass M is kept on a spring balance. A small block of mass m can move along the frictionless incline of the wedge. What is the reading of the balance while the block slides? Ignore the recoil of the wedge.



Sol. We consider wedge and block as our system. Action–reaction force between block and wedge is internal force. N_x and N_y represent reactions on the wedge by the balance.



From Newton's second law, for system 'block and wedge'.

$$N_y - (Mg + mg) = (M + m)(a_{CM})_y \quad ...(i)$$

$$(a_{CM})_y = \frac{Ma_{1y} + ma_{2y}}{(M + m)} \quad ...(ii)$$

Note that the block has acceleration of magnitude $g \sin \theta$ along an incline y-component of this acceleration is $g \sin^2 \theta$.

The wedge is at rest. Hence the acceleration of wedge will be zero.

Hence $a_{1y} = 0$ and $a_{2y} = -g \sin^2 \theta$

It means $(a_{cm})_y = \frac{M \times 0 + m(-g \sin^2 \theta)}{(M + m)} = \frac{-mg \sin^2 \theta}{(M + m)}$ ...(iii)

From Eqs. (i) and (iii),

$$N_y = (M + m)\left(g + (a_{cm})_y\right)$$

$$= (M + m)\left[g - \frac{mg \sin^2 \theta}{(M + m)}\right]$$

$$= (M + m)g - mg \sin^2 \theta$$

DISPLACEMENT OF CENTRE OF MASS OF A SYSTEM OF PARTICLES

Consider a system of N point masses $m_1, m_2, m_3, ..., m_n$ whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$, respectively. Then, the position vector of the centre of mass of the system is given by

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n} \quad ...(i)$$

If $\Delta\vec{r}_1, \Delta\vec{r}_2, \Delta\vec{r}_3$ ..., are the displacement vectors of the particles of system, then the displacement vector of centre of mass of this system of particles can be directly given by the vector difference using Eq. (i) as

$$\Delta\vec{r}_{cm} = \frac{m_1\Delta\vec{r}_1 + m_2\Delta\vec{r}_2 + m_3\Delta\vec{r}_3 + ... + m_N\Delta\vec{r}_N}{m_1 + m_2 + m_3 + ... + m_N} \quad ...(ii)$$

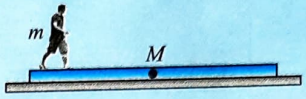
If net external force on the system in a particular direction is zero and initially the centre of mass of the system is at rest, then obviously the centre of mass will not move along that particular direction even though some particles (or bodies) of the system may move along that direction. The following illustrations will illustrate the above theory.

ILLUSTRATION 1.17

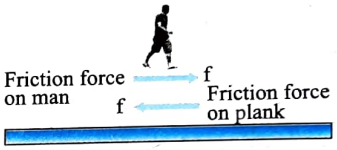
A plank of mass M and length L is at rest on a frictionless floor. The top surface of the plank has friction. At one end of it a man of mass m is standing as shown in figure. If the man walks towards the other end, find the distance, which the plank moves:

(a) till the man reaches the centre of the plank,

(b) till the man reaches the other end of the plank.

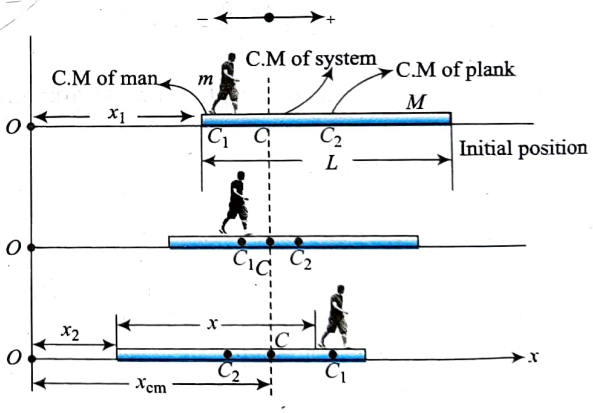


Sol. If we consider 'the man and the plank' as a system on which no external force is acting in horizontal direction. The only interaction force between the man and the plank is the friction as shown in figure, due to which the man walks along the plank and the friction on plank would be in opposite direction, due to which plank moves towards left, such that the centre of mass of 'plank plus man' remains at rest.



Method 1:

During motion of the man, this centre of mass must remain at dotted line only. As the man moves towards right, the plank will move towards left such that centre of mass remains at rest.



Just before the motion started, the initial distance of the centre of mass from point 'O'

$$x_{cm(initial)} = \frac{mx_1 + M\left(x_1 + \frac{L}{2}\right)}{m + M}$$

Position of centre of mass at any time when man has moved a displacement x, with respect to plank

$$x_{cm(time\ 't')} = \frac{m(x_2 + x) + M\left(x_2 + \frac{L}{2}\right)}{m + M}$$

As position of centre of mass of man is not changing then

$$x_{cm(Initial)} = x_{cm(time\ 't')}$$

$$\frac{mx_1 + M\left(x_1 + \frac{L}{2}\right)}{(m+M)} = \frac{m(x_2 + x) + M\left(x_2 + \frac{L}{2}\right)}{(m+M)}$$

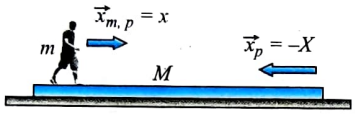
Which gives $(x_1 - x_2) = \frac{mx}{(M+m)}$ (towards left)

(a) When $x = \frac{L}{2} \Rightarrow \Delta x_{plank} = \frac{m(L/2)}{(M+m)}$

(b) When $x = L \Rightarrow \Delta x_{plank} = \frac{mL}{(M+m)}$

Method 2:

As there is no external force on the system (man + plank) and the system is initially at rest, there should not be displacement of centre of mass of the system during the motion of the man.



The displacement of centre of mass is Δx_{cm} and is given by

$$\Delta x_{cm} = \frac{m\Delta\vec{x}_m + M\Delta\vec{x}_p}{m+M} \quad ...(i)$$

$\Delta\vec{x}_m$ = displacement of the man with respect to the ground

$\Delta\vec{x}_p$ = displacement of the plank with respect to the ground

$= -X$ (towards left)

and $\Delta\vec{x}_{m,p}$ = displacement of man with respect to plank

$= x$ (towards right)

$$\Delta\vec{x}_m = \Delta\vec{x}_{m,p} + \Delta\vec{x}_p = (x - X)$$

As $\Delta\vec{x}_{cm} = 0$, from Eq. (i),

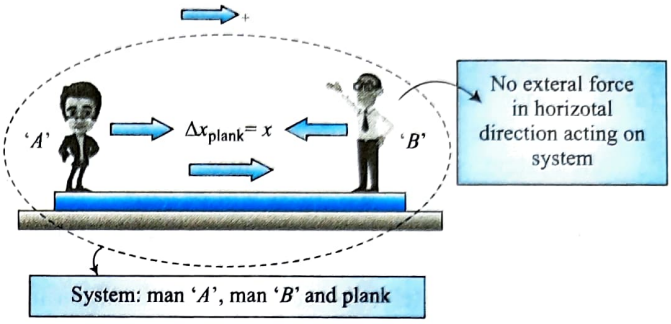
$$0 = \frac{m(x-X) - MX}{m+M} \Rightarrow X = \frac{mx}{m+M}$$

(a) When $x = \frac{L}{2}$, displacement of plank $X = \frac{m\frac{L}{2}}{m+M}$

(b) When $x = L$, displacement of plank $X = \frac{mL}{m+M}$

Sol.

(a) If we consider man 'A' + man 'B' and plank as system, there is no external force acting on system in horizontal direction. The system initially at rest, i.e., centre of mass is at rest. Hence the displacement of centre of mass should be zero.



$$\Delta\vec{x}_{cm} = \frac{m_A\Delta\vec{x}_A + m_B\Delta\vec{x}_B + m_{plank}\Delta\vec{x}_{plank}}{m_A + m_B + m_C} \quad ...(i)$$

Here $\Delta\vec{x}_{cm} = 0$

Displacement of 'A': $\Delta\vec{x}_A = \Delta\vec{x}_{A,\,plank} + \Delta\vec{x}_{Plank}$

$\Rightarrow \Delta\vec{x}_A = (10 + x)$

Similarly, $\Delta\vec{x}_B = (-10 + x)$

From Eq. (i), $0 = \frac{40(10+x) + 50(-10+x) + 10(x)}{(40+50+10)}$

$\Rightarrow -100 + 100x = 0$ or $x = 1$ m

Hence the plank will move 1m towards right.

(b) In this case also the displacement of the centre of mass will be zero or $\Delta x_{cm} = 0$. Let the displacement of the plank in this case is x'.

Here $\Delta\vec{x}_A = (5 + x')$ and $\Delta\vec{x}_B = (-5 + x')$

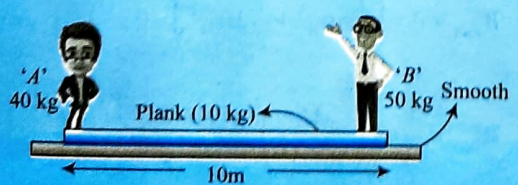
$$\Delta x_{cm} = 0 = \frac{40(5+x') + 50(-5+x') + 10(x')}{(40+50+10)}$$

$\Rightarrow -50 + 100x' = 0$ or $x' = \frac{1}{2}$m

Hence the displacement of the plank will be $\frac{1}{2}$m towards right.

ILLUSTRATION 1.18

Two men 'A' and 'B' are standing on the ends of a plank. The plank is resting on a smooth horizontal surface. The system is initially at rest and masses are shown in figure. Find the displacement of plank if



(a) both 'A' and 'B' interchange their positions.

(b) both 'A' and 'B' move towards each other and finally meet at the middle of the plank.

ILLUSTRATION 1.19

A man of mass m runs without skidding from rest from one end of a boat of mass M and length l with an acceleration a relative to the boat. If the friction between water and boat is neglected, find the:

(a) acceleration of CM of the system $(M + m)$

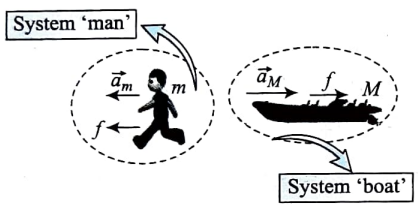
(b) acceleration of the man and boat

(c) position of man when he reaches to other end of the boat
(d) frictional force
(e) work done by friction on boat
(f) total work done by friction
(g) velocities of man and boat when the man reaches other end of the boat.
(h) work done by man

Sol.

(a) Consider 'boat and man' a system. No external force acts on the system in horizontal direction and initially the system is at rest i.e., the centre of mass is at rest. The acceleration of centre of mass will be zero.
There will not be any displacement of centre of mass.

(b) Considering the FBD of man and boat. If we consider man and boat separately, the static friction between man and boat will be responsible for the motion of man and boat. But as a whole the friction will be an internal force and will not move centre of mass of system



Let the acceleration and displacement of man be $\vec{a}_m$ and $\vec{x}_m$ respectively and for boat be $\vec{a}_M$ and $\vec{x}_M$.

$$\vec{a}_m = \vec{a}_{m,M} + \vec{a}_M = -a\hat{i} + a_M\hat{i} = (a_M - a)\hat{i}$$

But $\vec{a}_{cm} = \dfrac{m\vec{a}_m + M\vec{a}_M}{(m+M)}$

or $0 = \dfrac{m(a_M - a)\hat{i} + Ma_M\hat{i}}{(m+M)}$

$\Rightarrow \vec{a}_M = \dfrac{ma}{(m+M)}(\hat{i})$ (towards right)

Hence $a_m = \dfrac{ma}{(m+M)} - a$

$\Rightarrow \vec{a}_m = \dfrac{Ma}{(m+M)}(-\hat{i})$ (towards left)

(c) We know the displacement of CM

$$\vec{x}_{cm} = \frac{m\vec{x}_m + M\vec{x}_M}{(m+M)}$$

Here, $\vec{x}_m = \vec{x}_{m,M} + \vec{x}_M = (-l\hat{i} + x_M\hat{i})$

$\Rightarrow \vec{x}_m = (x_M - l)\hat{i}$

Hence $x_{cm} = 0 = \dfrac{m(x_M - l)\hat{i} + Mx_M\hat{i}}{(m+M)}$

$\Rightarrow \vec{x}_M = \dfrac{ml}{(m+M)}(\hat{i})$ (towards right)

and $\vec{x}_m = \dfrac{Ml}{(m+M)}(-\hat{i})$ (towards left)

Position of man when the man reaches to other end of the boat.

$$\vec{x} = (x_0 + l)\hat{i} + \vec{x}_m$$

$$= (x_0 + l)\hat{i} - \frac{Ml}{(m+M)}\hat{i} = \left(x_0 + \frac{ml}{(m+M)}\right)\hat{i}$$

(d) From FBD of man, friction force acting on the man,

$$f = ma_m = m\left(\frac{Ma}{(m+M)}\right) = \frac{Mma}{(m+M)}$$

(e) Work done by friction force on boat

$$(W_f)_{\text{boat}} = \vec{f}\cdot\vec{x}_m = \left[\frac{Mma}{(m+M)}\right]\left[\frac{ml}{(m+M)}\right]$$

$$\Rightarrow (W_f)_{\text{boat}} = \frac{Mm^2al}{(m+M)^2}$$

(f) As friction between man and boat in static nature and tot work done by static friction force on the system should b zero.

(g) For finding the velocity of man when he reaches other en of the boat, we can use kinematics.

Using $v_m^2 = u_m^2 + 2a_mx_m = 0 + 2\left(\dfrac{Ma}{(m+M)}\right)\left(\dfrac{Ml}{(m+M)}\right)$

$$\Rightarrow v_m^2 = \left(\frac{M}{m+M}\right)^2.2al$$

Hence velocity of man $v_m = \left(\dfrac{M}{m+M}\right)\sqrt{2al}$

Similarly velocity of boat $v_M = \left(\dfrac{m}{m+M}\right)\sqrt{2al}$

(h) If we consider 'man + boat' on system, friction will do n work on system as it in static nature. But we can see th kinetic energy of the system will increase. The questio arises here who does work to increase the kinetic energ of system. The answer is 'man'. The man does this wor through his internal energy to increase the kinetic energ of the system.
Using work energy theorem on the system, we get

$$W_{\text{man}} + W_{\text{friction}} = \Delta K = K_f - K_i$$

We know $W_{\text{friction}} = 0$

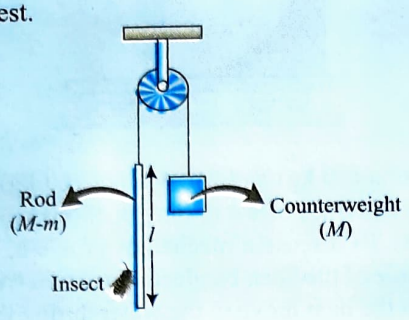
$$W_{\text{man}} + 0 = \left(\frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2\right) - 0$$

$$W_{\text{man}} = \frac{1}{2}m\left[\left(\frac{M}{m+M}\right)^2 2al\right] + \frac{1}{2}M\left[\left(\frac{m}{m+M}\right)^2 2al\right]$$

$$\Rightarrow W_{\text{man}} = \frac{mMal}{(m+M)}$$

ILLUSTRATION 1.20

A rod of mass 'M-m' and length 'l' carries an insect of mass 'm' at its bottom end and top end is connected with a string which passes over a smooth pulley and other end of the string is connected to a counter weight of mass 'M'. Initially the insect is at rest.



If we consider 'insect + rod + counter weight' as system.

(a) Is there any net external force acting on the system initially?

(b) If the insect starts crawling up on the rod with constant velocity, whether the centre of mass of the system accelerates or not?

(c) If the insect moves up with acceleration, does the centre of mass of the system accelerate?

(d) Find the displacement of centre of mass of the system when the insect reaches to the other end of the rod.

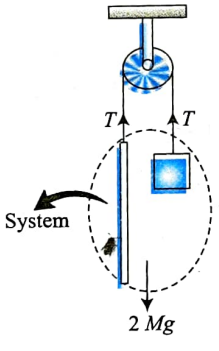
(e) Find the displacement of the rod when the insect reaches the other end of the rod.

Sol.

(a) Here we considering 'insect + rod + counter weight' as system. The external forces acting on the system are tension force in string and gravitational force. When insect is at rest:

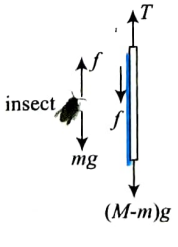
$$2T = 2Mg \text{ or } T = Mg$$

Hence no net external force is acting on the system.

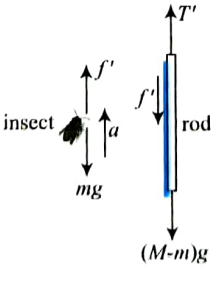


(b) If we consider free body diagram of the insect, it is the friction force which balance the insect when it is at rest.

When insect moves with constant velocity, the net force on insect should be zero. In this case the magnitude of the friction force will remain unchanged. It means the tension in the string will not change. Hence no external force acts on the system, the acceleration of the centre of mass of the system will be zero.



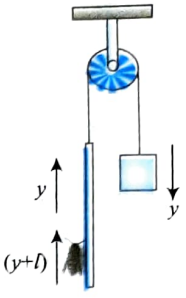
(c) When the insect moves up with acceleration, the friction force between the insect and rod increases. This increased friction, increases the tension in the string ($T > Mg$). Now in this case, there will be net force on the system in upward direction. Hence the centre of mass of the system will accelerate inward direction.



(d) Method 1: Let displacement of the rod when the insect reaches to the other end of the rod be y upward. Hence the displacement of the counter weight 'M' will be y downward as the rod and counter weight both are connected together by a single string.

During this time the displacement of the insect will be $(y + l)$.

We know



$$\Delta y_{cm} = \frac{m_1\Delta y_1 + m_2\Delta y_2 + m_3\Delta y_3}{(m_1 + m_2 + m_3)}$$

$$= \frac{m(y+l) + (M-m)y + M(-y)}{m + (M-m) + M}$$

$$\Rightarrow \quad \Delta y_{cm} = \frac{ml}{2M}$$

Method 2: All the objects of the system are initially at rest. The rope tension is the same on the left and the right-hand side, at every instant and consequently the momenta of the counter-balancing mass $(\vec{p}_1)$ and the rod with the insect $(\vec{p}_2)$ are equal at any moment of time. i.e.,

$$\vec{p}_1 = \vec{p}_2$$

or $$M\vec{v}_1 = m\vec{v} + (M-m)\vec{v}_1 \qquad ...(i)$$

Here $\vec{v}_1$ and $\vec{v}$ are the velocities of the counterbalancing mass and the insect respectively.

Velocity of insect $\vec{v}_{insect} = \vec{v}_{insect,\,rod} + \vec{v}_{rod} = \vec{v}' - \vec{v}_1$ ...(ii)

where $\vec{v}'$ is the insect's velocity relative to the rod.

From Eqs. (i) and (ii), we obtain $\vec{v}_1 = \left(\frac{m}{2M}\right)\vec{v}'$

On the other hand, the momentum of the centre of mass is

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = 2\vec{p}_1$$

or $$2M\,\vec{v}_C = 2M.\vec{v}_1 \text{ or } \vec{v}_C = \vec{v}_1 = \left(\frac{m}{2M}\right)\vec{v}'$$

and finally the desired displacement is

$$\Delta\vec{y}_{cm} = \int \vec{v}_{cm}\cdot dt = \frac{m}{2M}\int \vec{v}'\cdot dt = \frac{m}{2M}\Delta\vec{y}$$

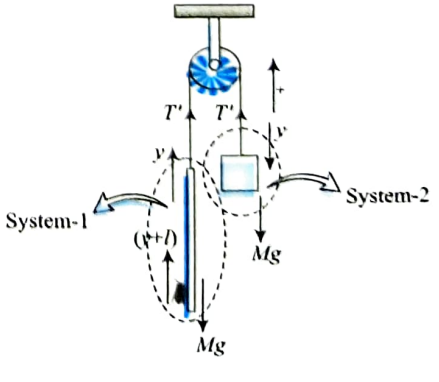
or displacement of centre of mass of system is

$$\Delta\,\vec{y}_{cm} = \frac{m}{2M}\Delta\,\vec{y} \text{ or } \Delta y_{cm} = \frac{ml}{2M}$$

(e) Let us consider two systems:

System-1: 'Rod + insect'

System2: counter weight



Both the systems have equal masses and equal net external force. Also, both the systems are initially at rest. It means the displacement of system-1 and system-2 should be same.

As system-2 consist of only one object, it means that the displacement of the object should be equal to displacement of centre of mass of system-1. Say the displacement of centre of mass of system-2 is 'y' downward.

For the displacement of centre of mass of system-1.

$$(\Delta y_{cm})_1 = \frac{m(y+l)+(M-m)y}{m+(M-m)} = -y$$

$$\Rightarrow \quad ml + My = -My$$

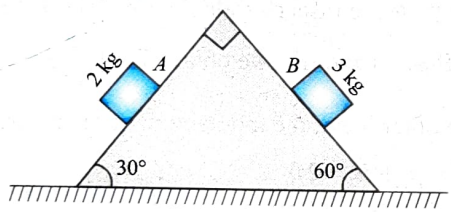
$$\Rightarrow \quad 2My = -ml$$

$$\Rightarrow \quad y = -\frac{ml}{2M} \quad \text{or} \quad \frac{ml}{2M} \text{ (upward)}$$

Hence the rod will displace by distance $\frac{ml}{2M}$ upward when the insect reaches the other end of the rod.

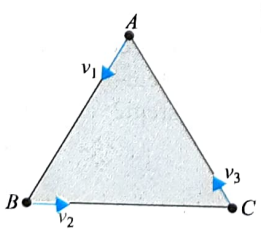
CONCEPT APPLICATION EXERCISE 1.2

1. Figure shows a fixed wedge on which two blocks of masses 2 kg and 3 kg are placed on its smooth inclined surfaces. When the two blocks are released from rest, find the acceleration of centre of mass of the two blocks.

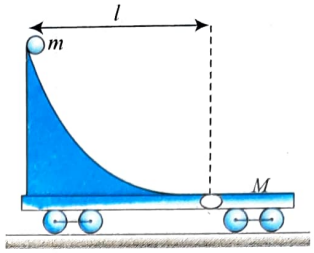


2. There are two masses m_1 and m_2 placed at a distance l apart, let the centre of mass of this system is at a point named C. If m_1 is displaced by l_1 towards C and m_2 is displaced by l_2 away from C, find the distance from C where the new centre of mass will be located.

3. Let there are three equal masses situated at the vertices of an equilateral triangle, as shown in figure. Now particle A starts with a velocity v_1 towards line AB, particle B starts with the velocity v_2 towards line BC and particle C starts with velocity v_3 towards line CA. Find the displacement of the centre of mass of the three particles A, B and C after time t. What would it be if $v_1 = v_2 = v_3$?

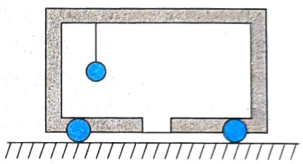


4. Figure shows a flat car of mass M on a frictionless road. A small massless wedge is fitted on it as shown. A small ball of mass m is released from the top of the wedge, it slides over it and falls in the hole at distance l from the initial position of the ball. Find the distance the flat car moves till the ball gets into the hole.

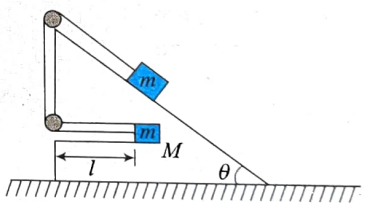


5. Mr. Verma (50 kg) and Mr. Mathur (60 kg) are sitting at the two extremes of a 4 m long boat (40 kg) standing still in water. To discuss a mechanics problem, they come to the middle of the boat. Neglecting friction with water, how far does the boat move in the water during the process?

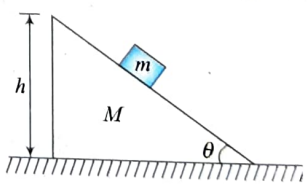
6. A cart of mass M is at rest on a frictionless horizontal surface and a pendulum bob of mass m hangs from the roof of the cart. The string breaks, the bob falls on the floor, makes several collisions on the floor and finally lands up in a small slot made on the floor. The horizontal distance between the string and the slot is L. Find the displacement of the cart during this process.



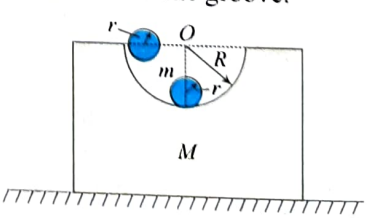
7. Find the displacement of the wedge when m comes out of the wedge. There is no friction anywhere.



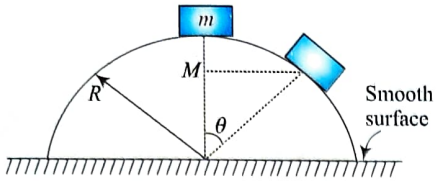
8. A block of mass m is initially lying on a wedge of mass M with an angle of inclination θ, as shown in figure. Calculate the displacement of the wedge when the block is released and reaches to the bottom of the wedge.



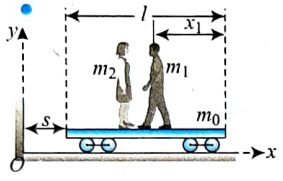
9. Calculate the displacement of the wedge when the ball reaches at the bottom of the groove.



10. A block is released on the convex surface of a hemispherical wedge as shown in figure. Determine the displacement of the wedge when the block reaches the angular position θ.

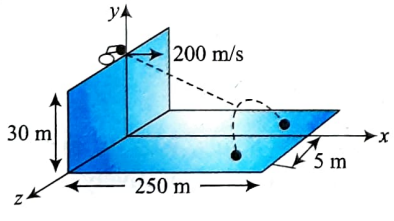


11. Figure shows the system is at rest initially with $x = 0$. A man and a woman both are initially at the extreme of carrier of the platform. The man and the woman start to move towards each other. Obtain an expression for the displacement s of the platform when the two meet in terms of the displacement x_1 of the man relative to the platform.

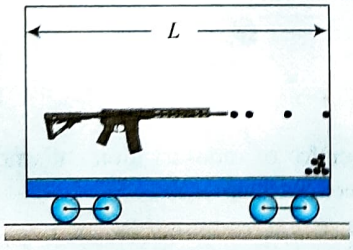


12. A 30 kg projectile moving horizontally with a velocity $\vec{v}_0 = (120 \text{ m/s})\,\hat{i}$ explodes into two fragments A and B of masses 12 kg and 18 kg, respectively. Taking point of explosion as origin and knowing that 3 s later the position of fragment A is (300 m, 24 m, –48 m), determine the position of fragment B at the same instant.

13. Two 20 kg cannon balls are chained together and fired horizontally with a velocity of 200 m/s from the top of a 30 m wall. The chain breaks during the flight of the cannon balls and one of them strikes the ground at $t = 2$ s, at a distance of 250 m from the foot of the wall, and 5 m to the right of the line of fire. Determine the position of the other cannon ball at that instant. Neglect the resistance of air.



14. A cannon and a supply of cannon balls are inside a sealed rail road car. The cannon fires to the right, the car recoils to the left. The canon balls remain in the car after hitting the far wall. Show that no matter how the cannon balls are fired, the rail road car cannot travel more than L, assuming it starts from rest.



ANSWERS

1. $\frac{\sqrt{31}}{10}g$ **2.** $\frac{m_1l_1 + m_2l_2}{m_1 + m_2}$ **4.** $\frac{ml}{M+m}$ **5.** $\frac{2}{15}$ m

6. $\frac{mL}{m+M}$ **7.** $\frac{ml(1-\cos\theta)}{2m+M}$ **8.** $\frac{mh\cot\theta}{M+m}$

9. $\frac{-m(R-r)}{M+m}$ **10.** $\frac{mR\sin\theta}{m+M}$ **12.** 400 m, –91 m, 32 m

CONSERVATION OF LINEAR MOMENTUM FOR SYSTEM OF PARTICLES

We can express Newton's second law for system of particles as

$$\sum \vec{F}_{ext} = M\,\vec{a}_{CM} \qquad \ldots(i)$$

Let us integrate Eq. (i) over a finite time interval:

$$\int \sum \vec{F}_{ext}.dt = \int M\vec{a}_{CM}\,dt$$

$$= \int M\left(\frac{d\vec{v}_{CM}}{dt}\right)dt = M\int d\vec{v}_{CM} = M\Delta\vec{v}_{CM} \qquad \ldots(ii)$$

Above equation can be written as:

$$\vec{J} = \Delta\vec{p}_{tot} \qquad \ldots(iii)$$

where $\vec{J}$ is the impulse imparted to the system by external forces and $\Delta\vec{P}_{tot}$ is the momentum of the system. Equation (iii) is the generalization of the impulse-momentum theorem for a particle to a system of many particles. It is also the mathematical representation of the non-isolated system (momentum) model for a system of many particles.

When a net average force $\vec{F}$ acts on a system of particles during a time interval Δt, we can write impulse -momentum equation as

$$\vec{J} = \underbrace{\vec{F}.\Delta t}_{\text{Impulse}} = \underbrace{m\vec{v}_f}_{\substack{\text{Final}\\ \text{momentum}}} - \underbrace{m\vec{v}_0}_{\substack{\text{Initial}\\ \text{momentum}}}$$

Impulse = Change in momentum

If the net external force on a system is zero so that the system is isolated, it follows from Eq. (i) that

$$M\vec{a}_{CM} = M\frac{d\vec{v}_{CM}}{dt} = 0 \qquad \ldots(iv)$$

Therefore, the isolated system model for a system of many particles is described by

$$\Delta\vec{p}_{tot} = 0 \qquad \ldots(v)$$

which can be rewritten as:

$$M\vec{v}_{CM} = \vec{p}_{tot} = \text{constant} \quad \left(\text{when } \Sigma\vec{F}_{ext} = 0\right) \qquad \ldots(vi)$$

It means, the total linear momentum of a system of particles is conserved if no net external force is acting on the system. Hence for an isolated system of particles, both the total momentum and the velocity of the centre of mass are constant in time. This statement is a generalization of the isolated system (momentum) model for a many-particle system.

Note that we have made no statement concerning the nature of the forces acting between members of the system. The only requirement is that the forces must be internal to the system.

Therefore, momentum is conserved for an isolated system regardless of the nature of the internal forces, even if the forces are non-conservative.

Conservation of linear momentum is universal, i.e., it applies to both macroscopic and microscopic systems. It holds good even in atomic and nuclear physics where classical mechanics fails. Further it is more generally applicable than the law of 'conservation of mechanical energy' because 'internal forces' are often non-conservative and so mechanical energy is not conserved but momentum is conserved (if $F_{ext} = 0$). Principal applications of conservation of linear momentum are in the field of collisions.

Note: Remember that the linear momentum of an isolated system is conserved. The linear momentum of one particle within an isolated system is not necessarily conserved because other particles in the system may be interacting with it. Always apply conservation of momentum to an isolated system.

Some examples:

- A bomb at rest suddenly explodes into various smaller fragments, all moving in different directions. Since the explosive forces are internal and there is no external force on the system for explosion, the centre of mass of the bomb will remain at the original position and the fragments fly such that their net momentum remains zero.
- Two men standing on a frictionless platform push each other, then also their net momentum remains zero because the push forces are internal for the two-men system.
- A boat floating in a lake also has the net momentum zero if the people on it change their positions, because the friction force required to move the people is internal of the boat system.
- Objects, initially at rest, if moving under mutual forces (electrostatic or gravitation) also have net momentum zero.
- A light spring of spring constant k is kept compressed between two blocks of masses m_1 and m_2 on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions, such that the net momentum is zero.

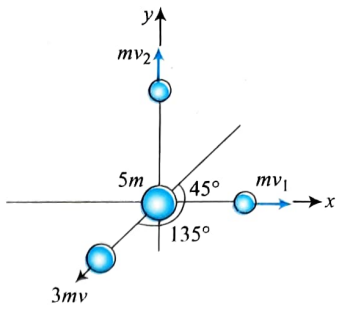
Single particle vs many particle system

Item	Single particle	Many-particle system
Impulse	$\vec{J} = \Delta\vec{p}$	$\vec{J}_{ext} = \Delta\vec{p}$
Momentum conservation	$\vec{F} = 0$ $\Rightarrow$ $\vec{p}$ = constant	$\vec{F}_{ext} = 0$ $\Rightarrow$ $\vec{p}$ = constant

ILLUSTRATION 1.21

A body of mass $5m$ explodes at rest break up into three parts. If two parts having equal masses fly off perpendicularly to each other with a velocity of 18 m/s, then calculate the velocity of the third part which has a mass 3 times the mass of equal parts.

Sol. Initially the body is at rest. Initial linear momentum of the body is zero. Let due to explosion the body break up into three parts. There is no external force acting on the body. Hence, linear momentum just after collision also be zero.



Let total mass of the body be $5m$, hence the masses of th broken parts will be m, m and $3m$ respectively.

$\vec{p}_{initial} = 0$

Linear momentum just after explosion

$\vec{p}_{final} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = mv_1\hat{i} + mv_1\hat{j} + \vec{p}_3$

As $\vec{p}_{initial} = \vec{p}_{final} \Rightarrow \vec{p}_3 = -(mv_1\hat{i} + mv_1\hat{j})$

$\Rightarrow |\vec{p}_3| = \sqrt{2}mv_1 \Rightarrow 3mv = \sqrt{2}mv_1$

$v = \frac{\sqrt{2}v_1}{3} = \frac{\sqrt{2}\times 18}{3} = 6\sqrt{2}$ m s^{-1}

$\Rightarrow 6\sqrt{2}$ m s^{-1} at an angle 135° from the either ball.

Note: During explosion there is no change in gravitational potential energy of the body. But kinetic energy of the broken parts of the body will increase due to change in internal energy. This increase in kinetic energy will be equal to energy released during explosion.

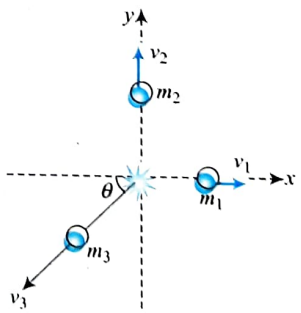
ILLUSTRATION 1.22

An explosion blows a rock into three parts. Two pieces go off at right angles to each other; 1.0 kg piece with a velocity of 12 m/s and other 2.0 kg piece with a velocity of 8 m/s. If the third piece flies off with a velocity of 40 m/s, compute the mass of the third piece.

Sol. Let m_1, m_2 and m_3 be the masses of the three pieces.

$m_1 = 1.0$ kg, $m_2 = 2.0$ kg

Let v_1 = 12 m/s, v_2 = 8 m/s, v_3 = 40 m/s and v_1 and v_2 ar directed along x- and y-axes, respectively, and v_3 be directed a shown.



By the principle of conservation of momentum, initial momentum is zero. Hence,

along x-axis: $0 = m_1v_1 - m_3v_3\cos\theta$

along y-axis: $0 = m_2v_2 - m_3v_3\sin\theta$

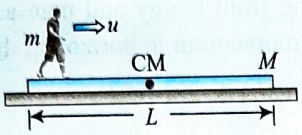
$\Rightarrow m_1v_1 = m_3v_3\cos\theta$ and $m_2v_2 = m_3v_3\sin\theta$

By squaring and adding, we get

$$\Rightarrow m_3^2 = \frac{1^2(12)^2 + (2)^2(8)^2}{(40)^2} \Rightarrow m_3 = 0.5 \text{ kg}$$

ILLUSTRATION 1.23

A man of mass m moves on a plank of mass M with a constant velocity u with respect to the plank, as shown in figure.



(a) If the plank rests on a smooth horizontal surface, determine the velocity of the plank.

(b) If the man travels a distance L with respect to the plank, find the distance travelled by the plank with respect to the ground.

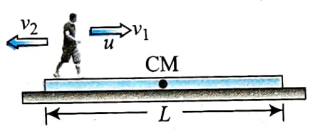
Sol. As no external forces are acting on the system in horizontal direction, so its linear momentum remains constant in that direction. Also there will be no shift in position of centre of mass, i.e., $\Delta x_{CM} = 0$.

(a) Let v_1 and v_2 be the velocities of the man and the plank w.r.t. ground. Then, we have

$$[\vec{v}_{man}]_{plank} = [\vec{v}_{man}]_{ground} - [\vec{v}_{plank}]_{ground}$$

Then, $[\vec{v}_{man}]_{ground} = [\vec{v}_{man}]_{plank} + [\vec{v}_{plank}]_{ground}$

or $v_1 = u - v_2$



Initially, the system is at rest. Therefore,

$0 = mv_1 - mv_2$

or $0 = m(u - v_2) - Mv_2$

or $v_2 = \frac{mu}{m+M}$

(b) Let x be the displacement of plank in backward direction, the displacement of mass is $L - x$ in forward direction.

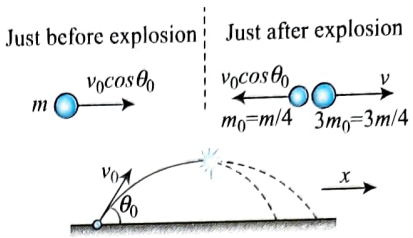
$$\Delta x_{CM} = \frac{M(-x) + m(L-x)}{M+m}$$

$$\Rightarrow 0 = -Mx + m(L-x) \Rightarrow x = \frac{mL}{M+m}$$

ILLUSTRATION 1.24

A projectile is launched at an angle θ_0 with velocity v_0. At the highest point of its path, it explodes into two parts of masses in the ratio 1:3. The lighter mass retraces its path. Find the velocity of the heavier part just after the explosion.

Sol. Let mass of the projectile be m. The projectile just before explosion is at the highest point of its path and moving with horizontal velocity of $v_0\cos\theta_0$. The lighter mass (m_0) formed after explosion retraces its path hence its velocity at the top is just reversed due to explosion. Thus, m_0 moves leftward at $v_0\cos\theta_0$. At this moment heavier part of mass $3m_0$ is moving rightward with velocity v.



The linear momentum just before explosion:

$$\vec{p}_{initial} = mv_0\cos\theta_0$$

The linear momentum just after explosion:

$$\vec{p}_{final} = -m_0v_0\cos\theta_0 + 3m_0v$$

Applying conservation of linear momentum along horizontal x-axis,

$$mv_0\cos\theta_0 = -m_0v_0\cos\theta_0 + 3m_0v$$

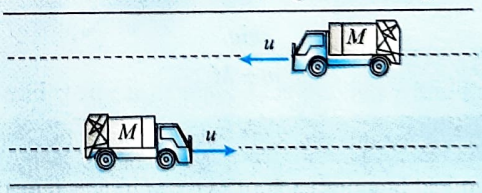
or $$mv_0\cos\theta_0 = -\frac{m}{4}v_0\cos\theta_0 + \frac{3m}{4}v$$

Hence the velocity of the heavier part just after the explosion,

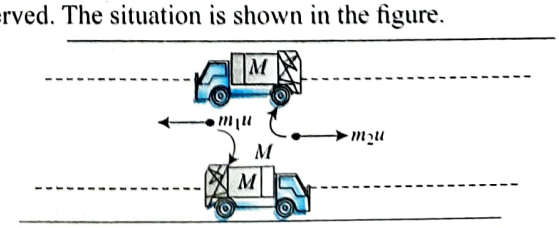
$$v = \frac{5}{3}v_0\cos\theta_0$$

ILLUSTRATION 1.25

Two trucks each of mass M, are moving without friction in opposite direction on adjacent parallel tracks with same velocity u. One is carrying potatoes and other is carrying onions, a bag of potatoes has a mass m_1 and a bag of onions has a mass m_2 (included in the mass of truck M). When trucks get close to each other while passing, the drivers exchange a bag with the other one by throwing the other one. Find the final velocities of the trucks after exchange of the bags.



Sol. Both the trucks move without friction. There is no external force acting on any of the truck in horizontal direction, hence linear momentum of any of the truck will be conserved always in horizontal direction. Here at the time of exchange of bags, the linear momentum in the direction of individual motion remains conserved. The situation is shown in the figure.



At the time when driver of the first truck carrying potatoes, throws a bag of mass m_1, in a direction perpendicular to the motion direction, towards the other truck, as shown in the figure. During throw, the bag has a velocity u in the direction of motion of the first truck. Similarly, when the second truck driver throws the onion bag conserve momentum for both the trucks independently as

First truck carrying potatoes

$$Mu - m_1u - m_2u = (M - m_1 + m_2)v_1$$

On solving, velocity of the first truck after exchange is

$$v_1 = \frac{Mu - m_1u - m_2u}{M - m_1 + m_2}$$

Similarly for second truck carrying onions

$$Mu - m_2u - m_1u = (M - m_2 + m_1)v_2$$

On solving, velocity of the first truck after exchange is

$$v_2 = \frac{Mu - m_1u - m_2u}{M - m_2 + m_1}$$

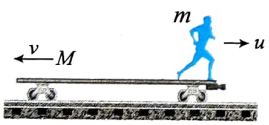
ILLUSTRATION 1.26

A railroad car of mass M is at rest on a frictionless track with a man of mass m standing at its edge. If the man jumps off from the car towards right with an initial velocity u, with respect to the car, find the velocity of the car after its jump.

Sol. Let the velocity of the car just after jumping of man be v. The net velocity of the man with respect to the ground will be $u - v$, as u is its velocity with respect to the car.

If we take man and car as system. There is no external force act on it in horizontal direction. It means the linear momentum of the system should be conserved in horizontal direction.

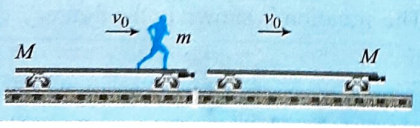
Initially, the system was at rest, thus according to momentum conservation, momentum after jump must be zero. Hence,



$$m(u - v) = Mv \Rightarrow v = \frac{mu}{m + M}$$

ILLUSTRATION 1.27

Two identical buggies each of mass M moves one after due to inertia (without friction) with some velocity. A man of mass m rides the rear buggy. At a certain moment, the man jumps into the front buggy with velocity relative to this buggy. Knowing that the mass of each buggy is equal to M. Find the velocity with which the buggies will move after that.



Sol. If we take rear 'buggy + man' as system. There is no external force acting on system in horizontal direction. Hence linear momentum of the system should be conserved. Let velocity of rear buggy just after jumping be $\vec{V}_r$, then velocity of man just after jumping will be $(\vec{V}_r + \vec{u})$.

$$\vec{p}_i = (M + m)\vec{V}_0$$

$$\vec{p}_f = M\vec{V}_r + m(\vec{V}_r + \vec{u}) \text{ and } \vec{p}_i = \vec{p}_f$$

$$\Rightarrow (M + m)\vec{V}_0 = M\vec{V}_r + m(\vec{V}_r + \vec{u})$$

$$\vec{V}_r = \vec{V}_0 - \frac{m\vec{u}}{(M + m)} \quad \ldots(i)$$

Now considering front buggy and man as system, and again conserving linear momentum in horizontal direction

$$M\vec{V}_0 + m(\vec{u} + \vec{V}_r) = (m + M)\vec{V}_f$$

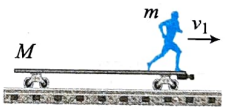
$$M\vec{V}_0 + m\left(\vec{u} + \vec{V}_0 - \frac{m\vec{u}}{(M + m)}\right) = (m + M)\vec{V}_f$$

$$\Rightarrow \vec{V}_f = \vec{V}_0 + \frac{Mm\vec{u}}{(M + m)^2}$$

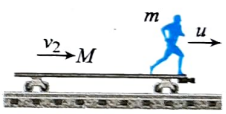
ILLUSTRATION 1.28

A railroad car of mass M with a man of mass m is moving with a velocity v_1. The man jumps in the direction of motion of car with a velocity u with respect to the car. Find the final velocities of the man and that of the car after jump.

Sol. If we take 'man and car' as system, there is no external force acting on the system in horizontal direction. The linear momentum of the system will be conserved in horizontal direction. After the jump, car attains a velocity v_2 in the same direction, which is less than v_1, due to backward push of the man for jumping. After the jump, child attains a velocity $(u + v_2)$ in the direction of motion of car with respect to ground.



Just before jumping



Just after jumping

According to conservation of linear momentum

$$(M + m)v_1 = Mv_2 + m(u + v_2)$$

Velocity of car after jump is $v_2 = \frac{(M + m)v_1 - mu}{M + m}$

Velocity of man after jump is $u + v_2 = \frac{(M + m)v_1 + Mu}{M + m}$

ILLUSTRATION 1.29

A light spring of spring constant k is kept compressed between two blocks of masses m and M on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions. The spring loses contact with the blocks when it acquires natural length. If the spring was initially compressed through a distance x, find the final speeds of the two blocks.

Sol. Consider the two blocks plus the spring to be the system. No external force acts on this system in horizontal direction. Hence, the linear momentum will remain constant. Suppose, the block of mass M moves with a speed V and the other

block with a speed v after losing contact with the spring. From conservation of linear momentum in horizontal direction, we have

$$MV - mv = 0 \quad \text{or} \quad V = \frac{mv}{M} \qquad \ldots(i)$$

Initially, the mechanical energy of the system $= \frac{1}{2}kx^2$

Finally, the mechanical energy of the system $= \frac{1}{2}mv^2 + \frac{1}{2}MV^2$

As there is no friction, mechanical energy will remain conserved. Therefore,

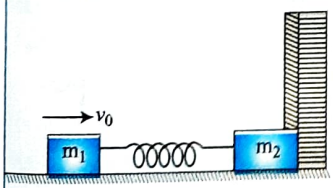
$$\frac{1}{2}mv^2 + \frac{1}{2}MV^2 = \frac{1}{2}kx^2 \qquad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

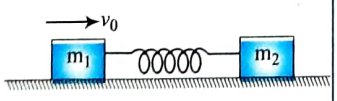
$$v = \left[\frac{kM}{m(M+m)}\right]^{1/2} x \text{ and } V = \left[\frac{km}{M(M+m)}\right]^{1/2} x$$

CONCEPT APPLICATION EXERCISE 1.3

1. As the text discusses, the conservation of linear momentum is applicable only when the system of objects is an isolated system. Which of the systems listed below are isolated systems?
   - **(i)** A ball is dropped from the top of a building. The system is the ball.
   - **(ii)** A ball is dropped from the top of a building. The system is the ball and the earth.
   - **(iii)** A billiard ball collides with a stationary billiard ball on a frictionless pool table. The system is the moving ball.
   - **(iv)** A car slides to a halt in an emergency. The system is the car.
   - **(v)** A space probe is moving in deep space where gravitational and other forces are negligible. The system is the space probe.
2. Consider two situations, given figures. Case (I), the spring-mass system is constrained; in case (II), the spring-mass system is free to move in the smooth horizontal surface. If m_1 is projected with a velocity v_0, can you conserve the momentum of the system in each case? Explain.

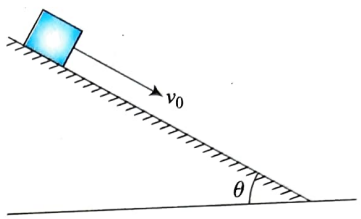


Case I

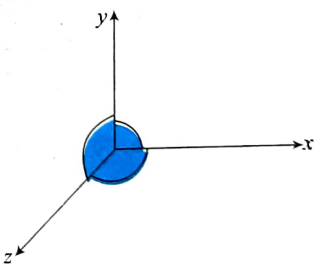


Case II

3. Two blocks of masses m_1 and m_2, interconnected with a spring of stiffness K, are kept on a smooth horizontal surface. Find out the ratio of velocity, displacement, kinetic energy and acceleration of block with mass m_1 to block with mass m_2.
4. A bomb, initially at rest, explodes into several pieces. **(a)** Is linear momentum of the system conserved? **(b)** Is kinetic energy of the system conserved? Explain.
5. You are standing perfectly still and then you take a step forward. Before the step your momentum was zero, but afterward you have some momentum. Is the principle of conservation of momentum violated in this case?
6. Does the centre of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.
7. Discuss the possibility of conservation of linear momentum of a block moving on a rough inclined plane if $\mu = \tan\theta$.



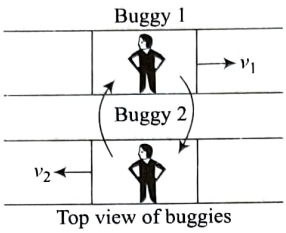
8. A shell is fired from a cannon with a speed of 100 m/s at an angle 30° with the vertical (y-direction). At the highest point of its trajectory, the shell explodes into two fragments of masses in the ratio 1:2. The lighter fragment moves vertically upwards with an initial speed of 200 m/s. What is the speed of the heavier fragment at the time of explosion?
9. A mud ball at rest explodes into three fragments of masses in the ratio 1:2:1. The two equal masses move with velocities $2\hat{i} + 5\hat{j} - 6\hat{k}$ and $-4\hat{i} + 3\hat{j} + 2\hat{k}$. Find the velocity of the third mass.



10. **(a)** A rail road car of mass M is moving without friction on a straight horizontal track with a velocity u. A man of mass m lands on it normally from a helicopter. What will be the new velocity of the car?

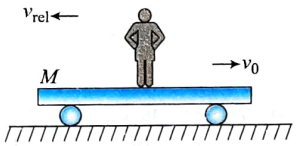
    (b) If now the man begins to run on it with speed v_{rel} with respect to car in a direction opposite to motion of the car, what will be the new velocity of the car?
11. Two trolleys A and B are free to move on a level frictionless track, and are initially stationary. A man on trolley A throws a bag of mass 10 kg with a horizontal velocity of 4 ms^{-1} with respect to himself on to trolley B of mass 100 kg. The combined mass of trolley A (excluding bag) and the man is 140 kg. Find the ratio of velocities of trolleys A and B, just after the bag lands on trolley B.

12. Two identical buggies 1 and 2 with one man in each move along parallel rails. When the buggies are opposite to each other, the men jump in a direction perpendicular to the direction of motion of buggies, so as to exchange their places. As a consequence, buggy 1 stops and buggy 2 keeps moving in the same direction with its final velocity v. Find the initial velocities v_1 and v_2 of buggies. Mass of each buggy (without man) equals M, mass of each man is m; ignore frictional effects anywhere and the buggies are constrained to move along the rails only.



Top view of buggies

13. (a) A rail road flat car of mass M can roll without friction along a straight horizontal track (figure). Initially, a man of mass m is standing on the car which is moving to the right with speed v_0. What is the change in velocity of the car if the man runs to the left so that his speed relative to the car is v_{rel} just before he jumps off at the left end?



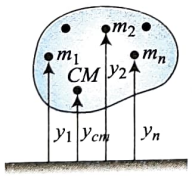
(b) If there are n men each of mass m on the car, should they all run and jump off together or should they run and jump one by one in order to give a greater velocity to the car?

ANSWERS

8. 125 m/s **11.** 11 : 14 **12.** $\frac{mv}{M-m}, \frac{Mv}{M-m}$

GRAVITATIONAL POTENTIAL ENERGY OF A SYSTEM OF PARTICLES

For any system of discrete distribution of particles, let us take the i^{th} particle of mass m_i which is situated at a height y_i (say), gravitational potential energy of m_i is given as:



$$U_i = m_i g y_i$$

Summing up the potential energy of all particles, we have the total potential energy

$$U = \sum U_i = \sum m_i g y_i = g \sum m_i y_i \quad \ldots(i)$$

The height of centre of mass of the system

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{\sum m_i y_i}{M}$$

$$\Rightarrow \sum m_i y_i = M y_{cm} \quad \ldots(ii)$$

M is the mass of the system.

From (i) and (ii), we have $U = Mgy_{cm}$

MECHANICAL ENERGY AND NON-CONSERVATIVE FORCE

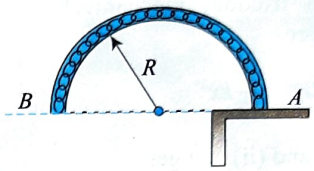
The conservation of mechanical energy may be applied to a system only when there is no work done by any non-conservative force.

$$\Delta E = E_f - E_i = W_{nc} = \Delta K + \Delta U$$

The above equation is the modified conservation of mechanical energy when work is done by non-conservative force.

ILLUSTRATION 1.30

A heavy, flexible, uniform chain lies in a smooth semicircular tube AB of radius R. The cross-sectional radius of the tube is much smaller than the radius of the tube (R). Assuming a slight disturbance to start the chain in motion, find the velocity with which it will emerge from the end B of the tube.



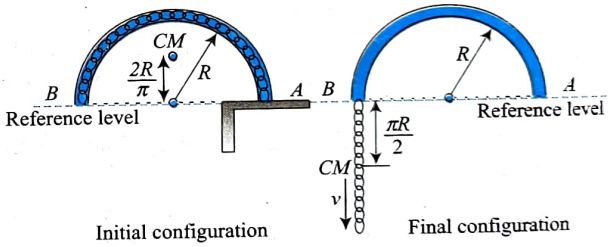
Sol. Let the linear mass density of the chain be λ. The centre of mass of the semicircular arc will be at a distance $(2R/\pi)$ from centre. And the mass of the chain will be $\lambda\pi R$.

Let us consider the horizontal line passing through points A and B is the reference level for calculating gravitational potential energy.

$$\text{Initial potential energy of chain} = (\lambda\pi R)\, g\left(\frac{2R}{\pi}\right)$$

$$\text{Final potential energy of chain} = (\lambda\pi R)\, g\left(\frac{-\pi R}{2}\right)$$

When the chain is completely slipped off the tube, all the links of the chain move with common velocity v.



Initial configuration Final configuration

$$\text{Hence the kinetic energy of chain} = \frac{1}{2}(\lambda\pi R)v^2$$

As all the surfaces are frictionless, there will not be any loss of mechanical energy hence, we can apply the law of conservation of mechanical energy.

$$\Delta K + \Delta U = 0$$

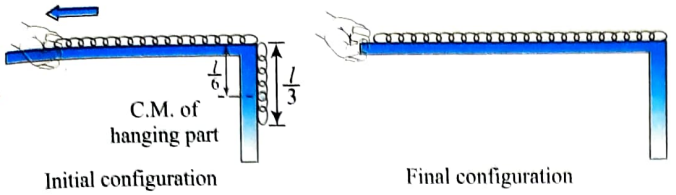
$$\left(\frac{1}{2}(\lambda\pi R)v^2 - 0\right) + \left((\lambda\pi R)g\left(\frac{-\pi R}{2}\right) - \lambda\pi R g\left(\frac{2\pi}{v}\right)\right) = 0$$

$$v = \sqrt{2gR\left(\frac{2}{\pi} + \frac{\pi}{2}\right)}$$

ILLUSTRATION 1.31

A uniform chain is held on a frictionless table with one third of its length hanging over the edge. If the chain has a length l and a mass m, how much work is required to pull the hanging part back slowly on the table?

Sol. The initial and final situation is shown in figure.



The mass of chain is distributed uniformly along the length of the chain and as the surface of the table is smooth. It can be assumed that the mass of the hanging part is at its centre of mass and also that of the resting part is on the table. As the hanging is pulled up slowly, work is required against gravity in displacing the centre of mass of the hanging chain up by $\frac{l}{6}$. Hence the work done by external agent is

$$W = \frac{m}{3} \times g \times \frac{l}{6} = \frac{mgl}{18}$$

CENTRE OF MASS FRAME OF REFERENCE OR CENTROIDAL FRAME

Centre of mass frame of reference or centroidal frame is reference frame assume attached with the centre of mass of the system at its origin. It moves together with the centre of mass.

It is a special frame and presents simple interpretations and solutions to several phenomena. Let us first discuss some of its fundamental properties.

In centroidal frame centre of mass is assumed at the origin, therefore position vector, velocity and acceleration of the centre of mass in centroidal frame all become zero.

- Sum of mass moments in centroidal frame vanishes.
  Mass moment of a particle is product of mass of the particle and its position vector.

$$\Sigma m_i \vec{r}_{i/c} = \vec{0}$$

or $m_1\vec{r}_{1/c} + m_2\vec{r}_{2/c} + ... + m_i\vec{r}_{i/c} + ... + m_n\vec{r}_{n/c} = \vec{0}$ ...(i)

- Total linear momentum of the system in centroidal frame vanishes.

$$\Sigma m_i \vec{v}_{i/c} = \vec{0}$$

or $m_1\vec{v}_{1/c} + m_2\vec{v}_{2/c} + ... + m_i\vec{v}_{i/c} + ... + m_n\vec{v}_{n/c} = \vec{0}$ ...(ii)

LINEAR MOMENTUM OF TWO PARTICLES SYSTEM RELATIVE TO CENTRE OF MASS FRAME

Consider two particles of masses m_1 and m_2 having velocities $\vec{v}_1$ and $\vec{v}_2$.

Velocity of centre of mass $\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{(m_1 + m_2)}$

Velocity of m_1 with respect to centre of mass

$$\vec{v}_{1,cm} = \vec{v}_1 - \vec{v}_{cm} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{(m_1 + m_2)}$$

$$\vec{v}_{1,cm} = \frac{m_2(\vec{v}_1 - \vec{v}_2)}{(m_1 + m_2)} = \frac{m_2\vec{v}_{1,2}}{(m_1 + m_2)} \quad ...(i)$$

Linear momentum of m_1 with respect to centre of mass

$$\vec{p}_{1,cm} = \frac{m_1 m_2}{(m_1 + m_2)}(\vec{v}_1 - \vec{v}_2)$$

$$\Rightarrow p_{1,cm} = \mu v_{1,2} \quad \text{or} \quad |\vec{p}_{1,cm}| = \mu v_{rel} \quad ...(ii)$$

Similarly, $\vec{p}_{2,cm} = \mu\vec{v}_{2,1}$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ (called reduced mass)

Total linear momentum of two particle system with respect to centre of mass

$$\vec{p}_{system,cm} = \mu\vec{v}_{12} + \mu\vec{v}_{21} = \mu(\vec{v}_{12} - \vec{v}_{21}) = 0 \quad ...(iii)$$

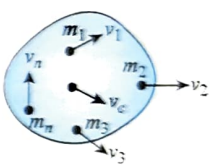
As $\vec{v}_{12} = -\vec{v}_{21}$ or $|\vec{v}_{12}| = |\vec{v}_{21}| = |\vec{v}_{rel}|$

Important Points:

- Momentum of the system relative to CM is zero. That means, neither internal forces nor external forces can change the momentum of the system relative to CM.
- Linear momentum of a system is equal to the linear momentum of CM of the system.
- For a two (or many) particle system $\Sigma m_i\vec{r}_{iC} = 0$, $\Sigma m_i\vec{v}_{iC} = 0$ and $\Sigma m_i\vec{a}_{iC} = 0$. That means, sum of (i) moment of masses, (ii) linear momenta, (iii) product of mass and acceleration of the particles relative to CM is zero individually.
- For a two particle system, momenta of the particles are equal and opposite relative to CM. Internal forces cannot change the momentum of CM of the system.

KINETIC ENERGY OF A SYSTEM OF PARTICLES

Total kinetic energy of a system of n particles (say) is equal to sum of kinetic energy of all particles relative to ground.



The velocity of the i^{th} particle is

$$\vec{v}_i = \vec{v}_{iC} + \vec{v}_C$$

$\vec{v}_i$ = velocity of m_i, $\vec{v}_{iC}$ = velocity of m_i relative to CM and $\vec{v}_C$ = velocity of CM

The kinetic energy of the i^{th} particle is $K = \sum \frac{1}{2} m_i v_i^2$

This gives $K = \frac{1}{2}\sum m_i |\vec{v}_{iC} + \vec{v}_C|^2$

$$= \frac{1}{2}\sum m_i (v_{iC}^2 + v_C^2 + 2\vec{v}_{iC} \cdot \vec{v}_C)$$

$$= \sum \frac{1}{2} m_i v_{iC}^2 + \frac{1}{2}(\sum m_i) v_C^2 + \sum m_i \vec{v}_{iC} \cdot \vec{v}_C$$

Since $\sum m_i\vec{v}_{iC} = 0$ (sum of momenta of all particles relative to the CM of the system is zero), substituting

$$\sum m_i \vec{v}_{iC} \cdot \vec{v}_C = \vec{v}_C \cdot (\sum m_i \vec{v}_{iC}) = 0$$

We have $K = \sum \frac{1}{2} m_i v_{iC}^2 + \frac{1}{2}(\Sigma m_i) v_C^2$

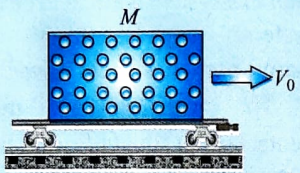
where $\Sigma m_i = M$ = total mass of the system and $\sum \frac{1}{2} m_i v_{iC}^2$ = sum of kinetic energies of the particle relative to CM of the system, which is termed as internal kinetic energy or kinetic energy relative to CM frame (= K', say)

Then, we have $K = K' + \frac{Mv_C^2}{2}$ or $K = K_{rel} + K_{cm}$

The above expression tells us that:
KE of a system of particle is equal to the KE of the system relative to CM plus KE of the CM $\left(\text{i.e., } \frac{1}{2}Mv_C^2\right)$. Hence, KE of a system is minimum relative to the frame attached with CM which is equal to K'. K' is called internal KE which does not depend on any reference frame. Thus, K' is an internal property of the system.

ILLUSTRATION 1.32

An insulated gas jar of mass M having an ideal gas of non-interacting N number of gas molecules, each of mass m, moves with a velocity $\vec{v}_0$. If the random velocity of the gas (called rms velocity) of the molecules in the jar is $\vec{v}$ then, Find the
(i) linear momentum,
(ii) kinetic energy of the system(gas +gas jar).



(gas + gas jar)

Sol.

(i) The linear momentum of a system (gas +gas jar) is equal to the linear momentum of its CM,

$\vec{p}_{system} = (M + m')\vec{v}_0$ where $m' = Nm$

$\vec{p}_{system} = (M + Nm)\vec{v}_0$

(ii) We can express total kinetic energy of system (gas +gas jar) as: $K = K_{rel} + K_{cm}$.

K_{rel} is the kinetic energy of the gas molecules relative to centre of mass, it is also called internal kinetic energy. And K_{cm} is the kinetic energy of the centre of mass, it is also called internal kinetic energy.

$$K = \frac{1}{2}Nmv^2 + \frac{1}{2}(M + Nm)v_0^2$$

Kinetic Energy of Two Particles System Relative to Centre of Mass Frame

Kinetic energy of m_1 with respect to centre of mass

$$K_{1,cm} = \frac{1}{2}m_1\left[\left(\frac{m_2\vec{v}_{1,2}}{m_1 + m_2}\right)\right]^2 = \frac{1}{2}\frac{m_1m_2^2}{(m_1 + m_2)^2}|\vec{v}_{1,2}|^2$$

Similarly, kinetic energy of m_2 with respect to centre of mass

$$K_{2,cm} = \frac{1}{2}\frac{m_2m_1^2}{(m_1 + m_2)^2}|\vec{v}_{2,1}|^2$$

Total kinetic energy of system of particles with respect to centre of mass

$$K_{system,cm} = K_{1,cm} + K_{2,cm}$$

$$= \frac{1}{2}\frac{m_1m_2^2}{(m_1 + m_2)^2}|\vec{v}_{rel}|^2 + \frac{1}{2}\frac{m_2m_1^2}{(m_1 + m_2)^2}|\vec{v}_{rel}|^2$$

$$= \frac{1}{2}\frac{m_1m_2}{(m_1 + m_2)}|\vec{v}_{rel}|^2 = \frac{1}{2}\mu|\vec{v}_{rel}|^2$$

Hence $K_{system,cm} = \frac{1}{2}\mu v_{rel}^2$

v_{rel} is the relative velocity of the particle. Always remembe that v_{rel} is independent of reference frame.

Important Point:

The kinetic energy of a system of particles (K) is equal to the kinetic energy of the system relative to centre of mass (K' plus kinetic energy of the centre of mass)

K_{total} = KE of system w.r.t. centre of mass + KE of centre of mass

or $K_{total} = K' + \frac{1}{2}Mv_{cm}^2$

M = total mass of system of particles

The kinetic energy of a system is minimum relative to th frame attached with centre of mass which is equal to K' is calle internal kinetic energy which does not depend on any referenc frame. Thus, K' is an internal property of system.

Hence total kinetic energy of two particles system relative t ground.

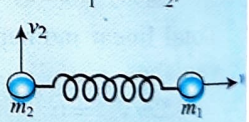
$$K_{total} = K' + \frac{1}{2}Mv_{cm}^2$$

$$K_{total} = \frac{1}{2}\mu v_{rel}^2 + \frac{1}{2}Mv_{cm}^2$$

$v_{rel} = |\vec{v}_1 - \vec{v}_2|$ and $M = (m_1 + m_2)$

ILLUSTRATION 1.33

Two small discs of masses m_1 and m_2 are connected by a weightless spring resting on a smooth horizontal plane. The discs are set in motion with initial velocities v_1 and v_2, whose directions are mutually perpendicular and in the same horizontal plane. Find the total energy E of the system with reference to the frame fixed to the centre of mass.



Sol. Let $\vec{v}_{cm}$ be the velocity of the centre of mass. Velocity of m_1 relative to the centre of mass is $(\vec{v}_1 - \vec{v}_{cm})$ and that of m_2 relative to the centre of mass is $(\vec{v}_2 - \vec{v}_{cm})$

Velocity of centre of mass $\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$

$$(\vec{v}_1 - \vec{v}_{cm}) = \frac{m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2)$$

Hence, $|\vec{v}_1 - \vec{v}_{cm}| = \frac{m_2}{m_1 + m_2}(v_1^2 + v_2^2)^{1/2}$

and $|\vec{v}_2 - \vec{v}_{cm}| = \frac{m_1}{(m_1 + m_2)}(v_1^2 + v_2^2)^{1/2}$

Total energy with respect to the centre of mass,

$$E = \frac{1}{2}m_1|\vec{v}_1 - \vec{v}_{cm}|^2 + \frac{1}{2}m_2|\vec{v}_2 - \vec{v}_{cm}|^2$$

$$= \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(v_1^2 + v_2^2) = \frac{1}{2}\mu(v_1^2 + v_2^2)$$

where $\mu = \frac{m_1m_2}{m_1+m_2}$ is called reduced mass.

WORK–ENERGY THEOREM FOR A SYSTEM OF PARTICLES

The work–energy theorem for a system must include both external and internal works:

$$W_{ext} + W_{int} = \Delta K_{cm} + \Delta K_{rel}$$

Since all the basic interactions are conservative, one can always express internal work by the change of internal potential energy.

$$W_{int} = -\Delta U_{int}$$

$$W_{ext} = \Delta K_{cm} + (\Delta K_{rel} + \Delta U_{int}) = \Delta K_{cm} + \Delta E_{int}$$

The above equation says that the external work on a system can change the translational kinetic energy of the CM and whatever forms of internal energy the system possesses.

This internal energy includes: elastic potential energy, gravitational potential energy, electromagnetic energy, chemical energy, nuclear energy, thermal energy and so on.

WORK ENERGY THEOREM RELATIVE TO CENTRE OF MASS

Work energy theorem is a very important tool in mechanics. Work energy theorem can be applied from any frame of reference. If we apply WE theorem from centre of mass frame, we can say the following:

Work done by total forces may be internal or external is equal to change in kinetic energy of the system with respect to centre of mass.

$$W_{total} = \Delta K_{cm}$$

$$W_{internal} + W_{external} = \Delta K_{cm}$$

$$W_{internal} + W_{external} = K_{cm,final} - K_{cm,initial}$$

$$W_{internal} + W_{external} = \frac{1}{2}\mu(v_{rel,f}^2 - v_{rel,i}^2)$$

TOTAL WORK OF PSEUDO FORCES IN CENTROIDAL FRAME

Let acceleration of centre of mass relative to an inertial frame is $\vec{a}_c$. Pseudo force on i^{th} particle of mass mi in centroidal frame is $(-m_i\vec{a}_c)$. Let displacement of i^{th} particle in a time interval is $\Delta\vec{r}_i$ relative to the centroidal frame.

Total work of pseudo forces on all the particles in centroidal frame can now be expressed as

$$\Sigma(-m_i\vec{a}_c)\cdot\Delta\vec{r}_i = -\vec{a}_c\cdot\Sigma(m_i\Delta\vec{r}_i) = -\vec{a}_c\cdot\vec{0} = 0$$

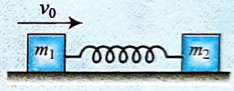
Hence the total work done in centroidal frame on all the particles of a system by pseudo forces due to acceleration of centre of mass is zero.

Note: If no external force acts on system of two particles system then $W_{ext} = 0$

- We can write $W_{internal} = \frac{1}{2}\mu(v_{rel,f}^2 - v_{rel,i}^2)$
- The total work done by an internal force pair does not depend on the frame of reference.
- Kinetic energy of a system is minimum relative to CM.
- The work done by pseudo (internal) force in CM frame is zero for a translating system.
- The work done by gravity is zero relative to CM (for the bodies having size negligible compared to the radius of the earth).
- Kinetic energy of a system is (generally) greater than KE of the system about the CM of the system. However, both are equal in CM frame.
- For a two-particle system $K_{int} = K_{system} = \frac{1}{2}\mu v_{rel}^2$ relative to CM, where μ = reduced mass of the system and v_{rel} = relative velocities between the particles.
- Internal forces as a whole cannot change the KE of the CM of the system whereas external force can change the KE of CM of the system.
- When the net work done by the internal forces is non-zero, the KE of the system about the CM changes due to the internal forces.

ILLUSTRATION 1.34

Two smooth blocks of masses m_1 and m_2 attached with an ideal spring of stiffness k and kept on a horizontal surface. If m_1 is projected with a horizontal velocity v_0, find the maximum compression of the spring.



Sol. **Method 1: (Using ground frame method)**

- Decide system: Blocks and spring
- Observation: No external force acting on system in horizontal direction
- Conclusion: Linear momentum as well as mechanical energy of system will be conserved. At the time of maximum compression both the blocks will move with common velocity.
- Problem solving: Using conservation of linear momentum at initial time and at time of maximum compression of spring.

Let the common velocity of system be V.

$$p_i = m_1v_0 + 0 \text{ and } p_f = (m_1+m_2)V$$

As $p_i = p_f \Rightarrow m_1v_0 = (m_1+m_2)V$

or $V = \frac{m_1v_0}{(m_1+m_2)}$ ...(i)

Now using conservation of mechanical energy.

$$\Delta K + \Delta U = 0$$

$$(K_f - K_i) + \Delta U_{spring} = 0$$

$$\left[\frac{1}{2}(m_1+m_2)V^2 - \frac{1}{2}m_1v_0^2\right] + \frac{1}{2}kx^2 = 0 \quad \text{...(ii)}$$

From (i) and (ii), we get

$$x=\left[\sqrt{\frac{m_1m_2}{(m_1+m_2)k}}\right]v_0$$

Method 2: (Work–energy theorem from centre of mass frame)
The linear momentum of system from CM frame is zero and no external force is acting on the system. We can apply work–energy theorem from CM frame

$$W_{ext}+W_{int}=\Delta K_{cm}=(K_f-K_i)_{cm} \quad ...(iii)$$

Initial kinetic energy of system w.r.t. CM

$$(K_i)_{cm}=\frac{1}{2}\mu(v_{rel})^2_{initial}=\frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(v_0-0)^2$$

$$\Rightarrow (K_i)_{cm}=\frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)v_0^2$$

Final kinetic energy of system w.r.t. CM

$$(K_f)_{cm}=\frac{1}{2}\mu(v_{rel})^2_{final}=0$$

As both the blocks move with common velocity hence, $(v_{rel})_{final}=0$ which gives $(K_f)_{cm}=0$

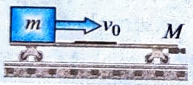
Hence from Eq. (iii)

$$0+\left(-\frac{1}{2}kx^2\right)=0-\frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)v_0^2$$

$$\Rightarrow x=\left[\sqrt{\frac{m_1m_2}{(m_1+m_2)k}}\right]v_0$$

ILLUSTRATION 1.35

A block of mass m is pushed with a velocity v_0 along the surface of a trolley car of mass M. If the horizontal ground is smooth and the coefficient of kinetic friction between the block and plank is μ, find the minimum distance of relative sliding between the block and plank.



Sol. Method 1: (Using ground frame method)

- Decide system: Block and plank
- Observation: No external force acting on system in horizontal direction
- Conclusion: Linear momentum of the system will be conserved.
  As friction is kinetic in nature and system.
  When relative sliding between block and trolley stops, velocity (V).
- Problem solving: Using conservation of linear momentum in horizontal direction.

$$mv_0+0=(m+M)V\Rightarrow V=\frac{mv_0}{(m+M)} \quad ...(i)$$

Now using WE theorem from ground frame.

$$W_{ext}=W_{int}=\Delta K\Rightarrow W_{ext}=0$$

$$W_{int}=W_{friction}=-fx=-\mu mgx$$

$$0+(-\mu mgx)=\left(\frac{1}{2}(m+M)V^2-\frac{1}{2}mv_0^2\right) \quad ...(ii)$$

From (i) and (ii), relative distance of sliding

$$x=\frac{v_0^2}{2\left(1+\frac{m}{M}\right)\mu g}$$

Method 2: (Work–energy theorem from centre of mass frame)
Considering the system from CM frame

$$W_{ext}+W_{int}=(\Delta K)_{cm}=(K_f-K_i)_{cm}$$

Initial kinetic energy of system w.r.t. CM

$$(K_i)_{cm}=\frac{1}{2}\mu(v_{rel})^2_{in}=\frac{1}{2}\frac{mM}{(m+M)}(v_0-0)^2$$

$$\Rightarrow (K_i)_{cm}=\frac{1}{2}\frac{mM}{(m+M)}v_0^2$$

Final kinetic energy of system w.r.t. CM

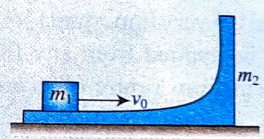
$$(K_f)_{cm}=0 \text{ or } |(v_{rel})_f|=0$$

Hence, $0+(-\mu mgx)=0-\frac{1}{2}\frac{mM}{(m+M)}v_0^2$

$$x=\frac{v_0^2}{2\left(1+\frac{m}{M}\right)\mu g}$$

ILLUSTRATION 1.36

A block of mass m_1 is projected with a velocity v_0 so that it climbs onto the smooth wedge of mass m_2. If the block does not leave the wedge, find the maximum height attained by the block.



Sol. Method 1: (Ground frame method)
At the maximum height, both m_1 and m_2 have same velocity (say v) as the relative sliding between them stops. Since, no external force acts horizontally, linear momentum is conserved in horizontal direction.

$$(\vec{p}_x)_f=(\vec{p}_x)_i$$

where $\vec{p}_{xi}=m_1v_0\hat{i}$ and $\vec{p}_{xf}=(m_1+m_2)v\hat{i}$

Then, we have $(m_1+m_2)v=m_1v_0$

WE theorem: $W_{ext}+W_{int}=\Delta K$ ...(i)

where $W_{ext}=-m_1gh$, $W_{int}=W_{constant}=0$

and $\Delta K=\left[\frac{1}{2}(m_1+m_2)v^2-\frac{m_1v_0^2}{2}\right]$

This gives $-m_1gh=\frac{1}{2}(m_1+m_2)v^2-\frac{m_1v_0^2}{2}$ ...(ii)

Substituting v from Eq. (i) in Eq. (ii), we have

$$h=\frac{v_0^2}{2\left(1+\frac{m_1}{m_2}\right)g}$$

Method 2: (Work–energy theorem from centre of mass frame)
The block attains a maximum height when the final relative velocity v_{rel}, between block and wedge becomes zero. That means $(v_{rel})_f = 0$.

The initial relative velocity between block and wedge is given as:

$$(v_{rel})_i = v_0.$$

Substituting $(v_{rel})_f = 0$ and $(v_{rel})_i = v_0$ in the formula,

$$\Delta K' = \frac{1}{2}\mu[(v_{rel})_f^2 - (v_{rel})_i^2]$$

We have $\Delta K' = -\frac{1}{2}\mu v_0^2$

While ascending through a height h, (say) the external force, that is, the earth's gravity does work on the block m_1 which can be given as:

$$W_{ext}(= W_{gravity}) = -m_1 gh$$

The internal (constraint) forces (normal reactions) between m_1 and m_2 as a whole do not perform work; $W_{int} = 0$

Finally, putting $\Delta K' = -\frac{1}{2}\mu v_0^2$, $W_{ext} = -m_1 gh$ and $W_{int} = 0$ in WE theorem $W_{ext} + W_{int} = \Delta K'$

We have $-m_1 gh + 0 = -\frac{1}{2}\mu v_0^2$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$

This gives $h = \frac{v_0^2}{2\left(1 + \frac{m_1}{m_2}\right)g}$

ILLUSTRATION 1.37

A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane and are at rest. The spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.

Sol. In this case we will analyze the situation in the reference frame of centre of mass.

As only F is the external force acting on the system, due to this force the centre of mass of the system will accelerate. Thus, with respect to centre of mass, there is a pseudo-force on the two masses in the opposite direction of the acceleration of the system. We know the total work done by pseudo forces in centroidal frame on both the blocks should be zero.

Let at the time of maximum deformation of the spring the displacements of the blocks m_1 and m_2 with respect to CM are Δx_1 (towards left) and Δx_2 (towards right) respectively. Hence the deformation of spring will be

$$\Delta x_1 + \Delta x_2 = \Delta x_{12}(= x) \quad \ldots(i)$$

Here x is the deformation of the spring at the time of maximum deformation

Since from centre of mass frame we can write,

$$-m\Delta x_1 + M\Delta x_2 = 0 \text{ or } m\Delta x_1 = M\Delta x_2 \quad \ldots(ii)$$

From Eq. (i) and Eq. (ii), we have

$$\Delta x_1 = \frac{Mx}{m+M} \text{ and } \Delta x_2 = \frac{mx}{m+M}$$

Now applying WE theorem in **CM frame**

$$W_{int} + W_{ext} = \frac{1}{2}\mu(v_{rel_f}^2 - v_{rel_i}^2) \quad \ldots(iii)$$

$v_{rel_i} = 0$ as both the particles m_1 and m_2 start from rest.

$v_{rel_f} = 0$ as both the particles will move with same velocity at the time of maximum separation between them which corresponds to maximum elongation of the spring.

Hence, we have $W_{int} + W_{ext} = 0$

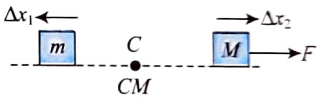
Putting $W_{int} = W_{spring} + W_{pseudo} = -\frac{kx^2}{2}$ $\quad \ldots(iv)$

The work done by pseudo (internal) force in CM frame is zero. $W_{pseudo} = 0$. As F is a constant forces, hence the work done by external forces

$$W_{ext} = F.\Delta x_2 \quad \ldots(v)$$

From (iii), (iv) and (v), we have

$$F.\Delta x_2 = \frac{k}{2}x^2 \quad \ldots(vi)$$



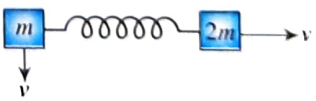
Substituting Δx_2 in Eq. (vi), we have

$$F.\left(\frac{mx}{m+M}\right) = \frac{kx^2}{2}$$

This gives $x = \frac{2mF}{k(m+M)}$

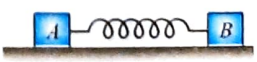
CONCEPT APPLICATION EXERCISE 1.4

1. Two blocks of masses m and $2m$ are connected by an ideal spring. The system is placed on smooth horizontal surface. The blocks are given impulse so that the blocks start moving as shown in figure. In this spring-blocks system, find the

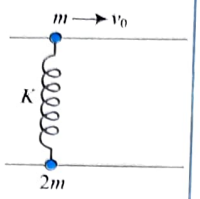


(a) KE of centre of mass
(b) KE of system w.r.t. centre of mass

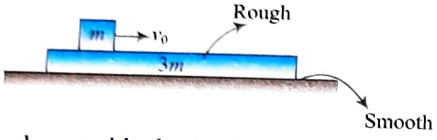
2. Two blocks A and B of mass m and $2m$ respectively are connected by a light spring of force constant k. They are placed on a smooth horizontal surface. Spring is stretched by a length x and then released. Find the relative velocity of the blocks when the spring comes to its natural length.



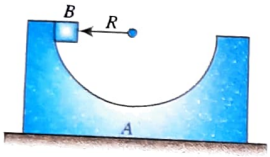
3. Two ring of mass m and $2m$ are connected with a light spring and can slide over two frictionless parallel horizontal rails as shown in figure. Ring of mass m is given velocity 'v_0' in horizontal direction as shown. Calculate the maximum stretch in spring during subsequent motion.



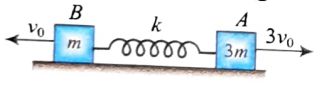
4. A small block of mass m is projected on a long plank of mass $3m$ with velocity v_0 as shown in figure. Find the work done by friction when slipping stops.



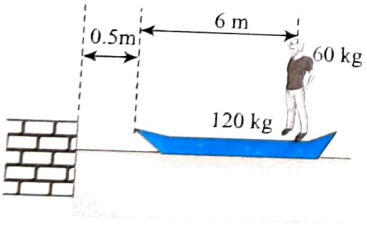
5. Figure show a block A of mass $9m$ having a smooth semicircular groove of radius R placed on a smooth horizontal surface. A block B of mass m is released from a position in groove where its radius is horizontal. Find the speed of bigger block when smaller block reaches its bottommost position.



6. The block 'A' & 'B' of masses m and $3m$ are connected by an ideal spring of force constant k. Now both the blocks are given impulse in opposite direction so that they started moving with velocities v_0 and $3v_0$ as shown in the figure. Find the maximum stretch of spring.



7. In figure a man stands on a boat floating in still water. The mass of the man and the boat is 60 kg and 120 kg, respectively.

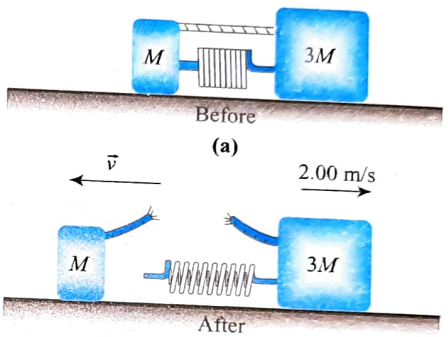


(a) If the man walks to the front of the boat and stops, what is the separation between the boat and the pier now?

(b) If the man moves at a constant speed of 3 m/s relative to the boat, what is the total kinetic energy of the system (boat + man)? Compare this energy with the kinetic energy of the system if the boat was tied to the pier.

8. An 80 kg boy and his 40 kg sister, both wearing roller blades, face each other at rest. The girl pushes the boy hard, sending him backward with velocity 3.0 m/s towards the west. Ignore friction. **(a)** Describe the subsequent motion of the girl. **(b)** How much chemical energy is converted into mechanical energy in the girl's muscles? **(c)** Is the momentum of the boy–girl system conserved in the pushing apart process? How can it be with no motion beforehand and plenty of motion afterward?

9. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one of them and the blocks are pushed together with the spring between them. A cord initially holding the blocks together is burned; after that, the block of mass $3M$ moves to the right with a speed of 2.00 m/s. **(a)** What is the velocity of the block of mass M. **(b)** Find the system's original elastic potential energy, taking $M = 0.350$ kg. **(c)** Is the original energy in the spring or in the cord? Explain your answer. **(d)** Is momentum of the system conserved in the bursting apart process? How can it be with large forces acting? How can it be with no motion beforehand and plenty of motion afterward?



10. A rifle man, who together with his rifle has a mass of 100 kg, stands on a smooth surface and fires 10 shots horizontally. Each bullet has a mass 10 g and a muzzle velocity of 800 m/s.
   (a) What velocity does the rifle man acquire at the end of 10 shots?
   (b) If the shots are fired in 10 s, what will be the average force exerted on him?
   (c) Compare his kinetic energy with that of 10 bullets.

11. A hemisphere of radius R and of mass $4m$ is free to slide with its base on a smooth horizontal table. A particle of mass m is placed on the top of the hemisphere. Find the angular velocity of the particle relative to hemisphere at an angular displacement θ when velocity of hemisphere has become v.

ANSWERS

1. (a) $\frac{5}{6}mv^2$ (b) $\frac{2}{3}mv^2$ **2.** $\sqrt{\frac{3k}{2m}}x$ **3.** $\sqrt{\frac{2m}{3k}}v_0$ **4.** $-\frac{3}{8}mv_0^2$

5. $\sqrt{\frac{5}{4}gR}$ **6.** $2\sqrt{\frac{3m}{k}}v_0$ **7.** (a) 2.5 m (b) 270 J

8. (a) 6.0 m/s east (b) 1080 J

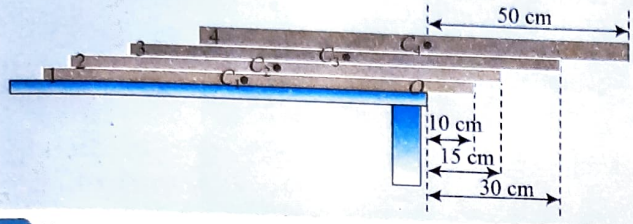
9. (a) 6.0 m/s left (b) 8.40 J **10.** (a) 0.8 m/s (b) 8 N

11. $\frac{5v}{R\cos\theta}$

Solved Examples

EXAMPLE 1.1

If four uniform and identical metre sticks were stacked on a table with 10 cm, 15 cm, 30 cm and 50 cm segments over the edge as shown in figure, would the metre sticks remain on the table or topple down?



Sol. Since the metre sticks are identical, the centre of mass of each stick is located at its midpoint. Let us take the origin to be at the edge of the table and find out the position of the centre of mass of the system of the four meter sticks.

The system of the four meter sticks will not topple down, if the position of the centre of mass of the system remains on the table.

If $x_1 (= OC_1), x_2 (= OC_2), x_3 (= OC_3)$ and $x_4 (= OC_4)$ are distances of the centre of mass of the sticks 1, 2, 3 and 4 from the edge of the table, it follows that

$x_1 = -40$ cm, $x_2 = -35$ cm, $x_3 = -20$ cm and $x_4 = 0$

Note that distances to the left of the edge of the table has been taken as negative.

Let M be the mass of each stick. If X is the distance of the centre of mass of the system of sticks from the edge of the table, then

$$\frac{Mx_1 + Mx_2 + Mx_3 + Mx_4}{M + M + M + M} = \frac{x_1 + x_2 + x_3 + x_4}{4}$$

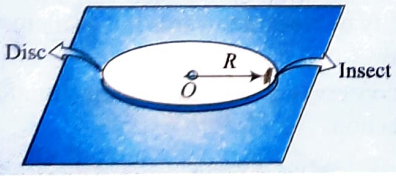
$$= \frac{-40 + (-35) + (-20) + 0}{4} = -\frac{95}{4}\text{ cm} = -23.75\text{ cm}$$

Since X is negative, the centre of mass of the system of the four sticks remains on the table.

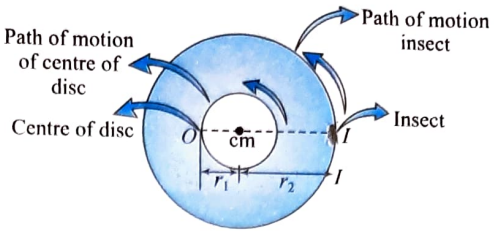
Hence, the meter sticks will remain on the table and will not topple down.

EXAMPLE 1.2

A disc of mass M and radius R is kept flat on a smooth horizontal table. An insect of mass m alights on the periphery of the disc and begins to crawl along the edge. Describe the path of the centre of the disc.



Sol. If we take 'disc +insect' as a system, there is no net external force acting on the system. Hence the centre of mass of the system will not move, and both the insect and the centre (O) of the disc move in circle of different radius. The angular speed of both is same so that the centre (O) of the disc, the insect and CM of the system always remain on a straight line.



Radius of rotation of disc centre = Distance of CM of the system (disc + insect) from the centre (O) of the disc,

$$r_1 = \frac{mR}{M+m}$$

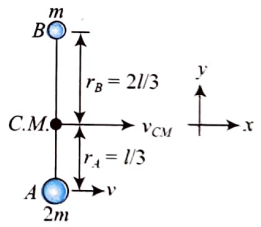
Radius of rotation of insect = Distance of insect (I) from CM,

$$r_2 = \frac{MR}{M+m}$$

EXAMPLE 1.3

Two small spheres 'A' and 'B' of masses '$2m$' and 'm' respectively, are connected with an inextensible string of length l lie on a smooth horizontal plane. The sphere 'A' is given a velocity of v m/s along the ground perpendicular to line AB as shown in figure. Find the tension in string during their subsequent motion.

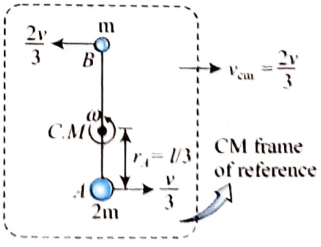
Sol. As no external force acts on the system of the two spheres hence velocity of centre of mass will remain unchanged. The centre of mass of the system will move along x-direction and both the spheres will move in circular path about centre of mass of the system. Let us solve the problem in CM frame. In CM frame particles A and B move along circular path with same angular velocity (say ω).



Velocity of CM, $v_{CM} = \frac{(m \times 0) + (2m \times v)}{3m} = \frac{2v}{3}$

Distance of CM from 'A' will be radius of rotation of the sphere 'A', $r_A = \frac{2m \times 0 + m \times l}{(2m+m)} = \frac{l}{3}$

In CM frame,



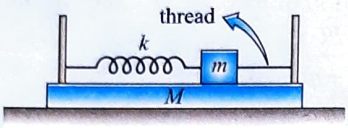
Velocity of $2m$ mass $= v - \frac{2v}{3} = \frac{v}{3}$

Angular velocity of particle A, $\omega = \frac{v/3}{l/3} = \frac{v}{l}$

$$T = M\omega^2 r = 2m \cdot \frac{v^2}{l^2} \cdot \frac{l}{3} \Rightarrow T = \frac{2mv^2}{3l}$$

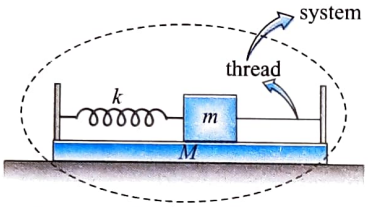
EXAMPLE 1.4

A spring is connected with plank and other end of spring is connected with a block of mass m. Initially spring is stretched by a distance x_0 and block is connected with a thread which is connected to other end of the plank as shown. If thread is cut, what will be maximum speed of the plank.



Sol. Take system: spring + plank + block

Observation for the system: No external force acts on the system in horizontal direction.



No external force acts on the system in horizontal direction

Conclusion: Both linear momentum and mechanical energy will be conserved for the system.

Analysis of problem: The speed of the plank will be maximum when all the potential energy of spring converted into kinetic energy of system.

The initial linear momentum of system is zero and no external forces are acting on system, the linear momentum of the system be zero always.

The motion of the block and plank will be such that they will always move in opposite direction or both will be at rest to justify conservation of linear momentum.

Let speed of block and plank (at maximum case) be v and V respectively. Using conservation of linear momentum at this situation

$$mv = MV \qquad ...(i)$$

Now applying conservation of mechanical energy

$$\Delta K + \Delta U = 0$$

$$\left[\left(\frac{1}{2}MV^2 + \frac{1}{2}mv^2\right) - 0\right] + \left[0 - \frac{1}{2}kx_0^2\right] = 0 \qquad ...(ii)$$

From Eq. (i), $v = \frac{MV}{m}$

Substituting this value in Eq. (ii), we get

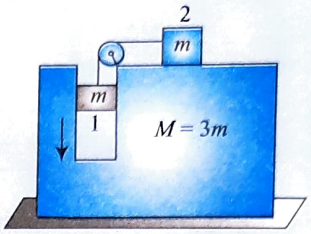
$$\left[\frac{1}{2}MV^2 + \frac{1}{2}m\left(\frac{MV}{m}\right)^2\right] = \frac{1}{2}kx_0^2$$

$$\Rightarrow \quad \frac{1}{2}\left[M + \frac{M^2}{m}\right]V^2 = \frac{1}{2}kx_0^2$$

$$V^2 = \frac{k m x_0^2}{(M^2 + Mm)} \Rightarrow V = \sqrt{\frac{k m}{M(M+m)}} \cdot x_0$$

EXAMPLE 1.5

A wedge having a vertical slot in it is placed on smoo horizontal surface as shown in the figure. Two blocks ar arranged as shown in the figure. The system is released fro rest calculate the speed of the wedge when block 1 come down a distance h.



Sol. We define system: wedge + block 1 and block 2

Observation: No external force acts on the system in horizont direction.

Conclusion: Both the linear momentum and mechanical energ will be conserved for the system.

Analysis of problem: If block 1 moves down a distance h wit respect to wedge, the block 2 will also slide towards left by th same distance with respect to wedge.

As wedge is not moving in vertical direction, the displacemen of block 1 with respect to wedge will be same as with respect t the ground.

The linear momentum of the system in horizontal directio is conserved. Initial linear momentum of the system is zero. A block 2 moves towards left, to conserve linear momentum th wedge will start moving towards right.

As block 1 is moving in the slot in the wedge, the velocity o block 1 in horizontal direction will be same as the velocity o wedge.

Block 1 will also move in vertical direction with respect t wedge as well as with respect to ground.

The velocity of block 2 with respect to wedge will be sam as velocity of block 1 in vertical direction as both are connecte with same string.

Final step: Let velocity of block 1 in vertical direction be v.

Hence velocity of block 2 with respect to wedge will be towards left.

Let velocity of wedge towards right be V.

Hence velocity of block 2 towards left (with respect to ground will be $(v - V)$.

Now using conservation of linear momentum for the system i horizontal direction.

$$(M+m)V - m(v - V) = 0$$

as $M = 3m \Rightarrow v = 5V$ ...(i

Now using conservation of mechanical energy for the system

$$\Delta K + \Delta U = 0$$

$$\left[\left(\frac{1}{2}(M+m)V^2 + \frac{1}{2}m(v-V)^2 + \frac{1}{2}mv^2\right) - 0\right] + [-mgh] = 0$$

$$\frac{1}{2}(3m+m)V^2+\frac{1}{2}m(5V-V)^2+\frac{1}{2}m(5V)^2=mgh$$

$$\Rightarrow \quad V=\sqrt{\frac{2gh}{45}}$$

EXAMPLE 1.6

A rope thrown over a pulley has a ladder with a man A on one of its ends and a counterbalancing mass M on its other end. The man whose mass is m, climbs upwards by $\Delta\vec{r}$ relative to the ladder and then stops. Ignoring the masses of the pulley and the rope, as well as the friction in the pulley axis, find the displacement of the centre of mass of this system.

Sol. **Method 1:** Let displacement of counterweight be Δy therefore the displacement of ladder will be $(-\Delta y)$.

Displacement of man $\Delta y_m = \Delta y_{m,l} + \Delta y_l$

$y_{m,l}$ = displacement of man w.r.t. ladder = $\Delta\vec{r}$

$y_m = (\Delta r - \Delta y)$

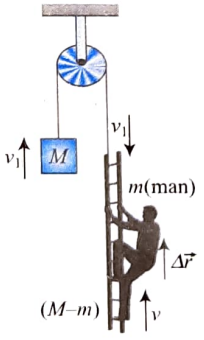
Hence displacement of centre of mass of the system

$$\Delta\vec{r}_c=\frac{M_{\text{mass}}\Delta\vec{r}_{\text{mass}}+M_{\text{ladder}}\Delta\vec{r}_{\text{ladder}}+M_{\text{man}}\Delta\vec{r}_{\text{man}}}{M_{\text{mass}}+M_{\text{ladder}}+M_{\text{man}}}$$

$$\Delta r_c=\frac{M\Delta y+(M-m)(-\Delta y)+m(\Delta r-\Delta y)}{M+(M-m)+m}$$

$$\Rightarrow \Delta r_c=\frac{m\Delta r}{2M}\ \text{(upward)}$$

Method 2: All the bodies of the system are initially at rest. The rope tension is the same both as on the left and the right hand side, at every instant and consequently the momenta of the counterbalancing mass ($\vec{p}_1$) and the ladder with the man ($\vec{p}_2$) are equal at any moment of time. i.e., $\vec{p}_1=\vec{p}_2$



or $M\vec{v}_1=m\vec{v}+(M-m)\vec{v}_1$ ...(i)

Here $\vec{v}_1$ and $\vec{v}$ are the velocities of the mass and the man respectively.

Velocity of man,

$$\vec{v}_{\text{man}}=\vec{v}_{\text{man, ladder}}+\vec{v}_{\text{ladder}}=\vec{v}'-\vec{v}_1 \qquad \ldots(\text{ii})$$

where $\vec{v}'$ is the man's velocity relative to the ladder.

From Eqs. (i) and (ii), we obtain $\vec{v}_1=\left(\frac{m}{2M}\right)\vec{v}'$

On the other hand, the momentum of the centre of mass is

$$\vec{p}=\vec{p}_1+\vec{p}_2=2\vec{p}_1$$

or $2M\vec{v}_C=2M\vec{v}_1$ or $\vec{v}_C=\vec{v}_1=\left(\frac{m}{2M}\right)\vec{v}'$

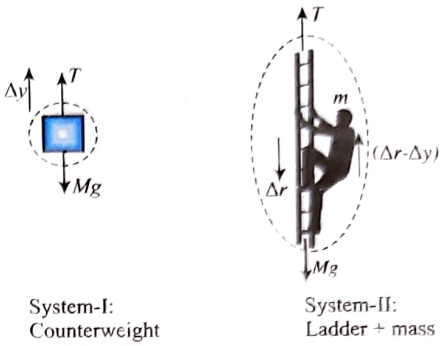
and finally the desired displacement is

$$\Delta\vec{r}_C=\int\vec{v}_C\cdot dt=\frac{m}{2M}\int\vec{v}'\cdot dt=\frac{m}{2M}\Delta\vec{r}$$

or displacement of centre of mass of system is

$$\Delta\vec{r}_C=\frac{m}{2M}\Delta\vec{r}$$

As system I and system II both are having same mass and experiencing equal force. Both starts from rest. Hence displacement of centre of masses of systems should be equal. As the system I has only one object, the displacement of the object will be same as the displacement of its centre of mass which is Δy upward.



System-I: Counterweight

System-II: Ladder + mass

Hence the displacement of centre of mass of system II will also be Δy upward.

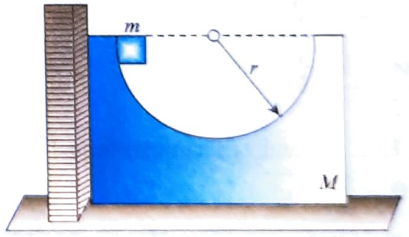
$$(\Delta\vec{y}_{\text{cm}})_{\text{system I}}=(\Delta\vec{y}_{\text{cm}})_{\text{system II}}=\Delta y\ \text{(upward)}$$

$$\Rightarrow \Delta y=\frac{m(\Delta r-\Delta y)+(M-m)(-\Delta y)}{m+(M-m)}$$

$$\Rightarrow \Delta y=\frac{m\Delta r}{2M}\ \text{(upward)}$$

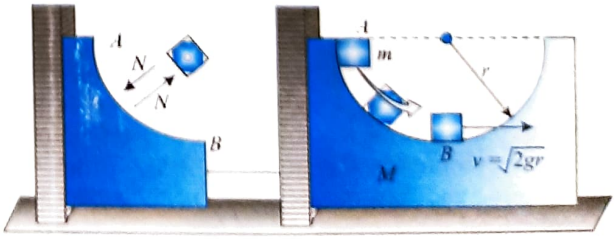
EXAMPLE 1.7

A wedge of mass M with its upper surface has hemispherical groove, rests on a smooth horizontal surface against a wall as shown in figure. A small block of mass m slides without friction on the hemispherical groove of the wedge. What is the maximum velocity attained by the wedge?



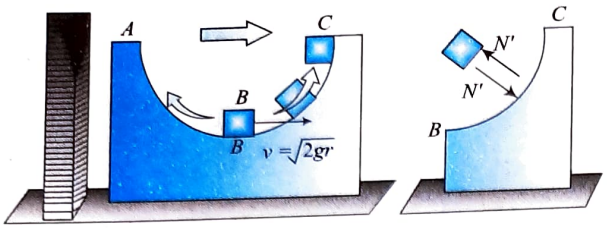
Sol. **Situation-I:** When block moves from A to B. The wedge does not move.

When the block slides from A to B, the reaction on the wedge presses it to the wall i.e., towards left. When the block reaches the lowermost position of the track, its velocity from conservation of mechanical energy will be $v=\sqrt{2gr}$



Situation-II: When block moves from B to C. The wedge starts moving.

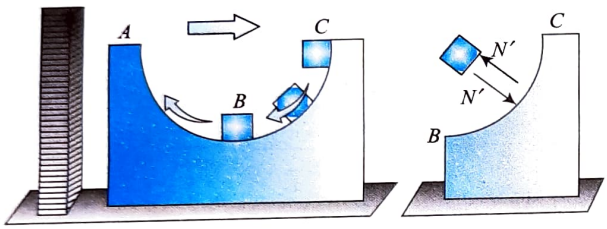
When the block moves along the right half of the groove, during its upward journey as well as downward journey the reaction of the block on the wedge is towards right as shown in Fig. (b).



Situation-III: When block moves from C to B. The wedge keeps on moving.

It means during the entire motion of the block from B to C and C to B, the wedge is accelerated towards right.

Hence maximum velocity is attained by the wedge at the instant when the block passes point B during the return journey.



From conservation of linear momentum, at the instant when the wedge is separated from the wall,

$$\Rightarrow m\sqrt{2gr} = Mv_1 + mv_2 \qquad \ldots(i)$$

From conservation of mechanical energy,

$$\Delta K + \Delta U = 0$$

$$\left(\left(\frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2\right) - 0\right) + (-mgr) = 0$$

or $$mgr = \frac{Mv_1^2}{2} + \frac{mv_2^2}{2} \qquad \ldots(ii)$$

On solving Eqs. (i) and (ii) simultaneously, we obtain two solutions $v_1 = 0,\ v_2 = \sqrt{2gr}$

and $$v_1 = \frac{2m}{M+m}\sqrt{2gr},\quad v_2 = \frac{m-M}{m+M}\sqrt{2gr}$$

The first solution corresponds to the instant when the block reaches for the first time at point B. At this instant the block moves with velocity v_2 and the wedge is at rest.

The second solution corresponds to the instant when the block has the maximum velocity $(v_1)_{max} = \dfrac{2m\sqrt{2gr}}{M+m}$

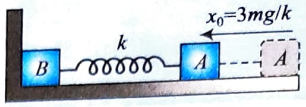
EXAMPLE 1.8

A system consists of block A and B each of mass m connected by a light spring as shown with block B in contact with a wall. The block A compresses the spring by $3mg/k$ from natural length of spring and then released from rest. Neglect friction everywhere. Find:

(a) acceleration of centre of mass of system comprising A and B just after A is released.

(b) velocity of centre of mass of system comprising A and B when block B just loose contact with the wall.

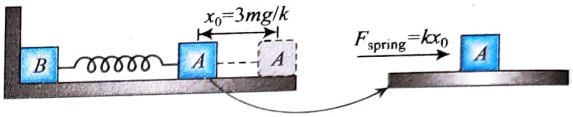
(c) maximum extension in the spring after system loses contact with wall.



Sol.

(a) The normal reaction provided by wall provide external force to the system of blocks A and B.

The block B will not move but the block A will starts moving due to initial compression of spring.



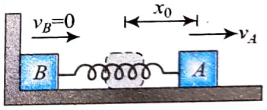
Acceleration of block A just after release

$$a_A = \frac{F_{sp}}{m} = \frac{k\left(\frac{3mg}{k}\right)}{m} \Rightarrow a_A = 3g \text{ (towards right)}$$

and acceleration of block B is zero. Hence acceleration of the system

$$a_{cm} = \frac{m_A a_A + m_B a_B}{m_A + m_B} = \frac{m(3g) + m(0)}{(m+m)} = \frac{3}{2}g$$

(b) The block B will lose contact with wall when block is moving towards right and spring reaches to its natural length.



For velocity of block A, using $\Delta K + \Delta U = 0$

$$\left(\frac{1}{2}mv_A^2 - 0\right) + \left(0 - \frac{1}{2}kx_0^2\right) = 0 \Rightarrow v_A^2 = \frac{k}{m}x_0^2$$

$$\Rightarrow v_A = \sqrt{\frac{k}{m}} \cdot x_0 = \sqrt{\frac{k}{m}} \times \frac{3mg}{k}$$

$$\Rightarrow v_A = 3g\sqrt{\frac{m}{k}}$$

The velocity of block B just at the time of losing contact with wall will be zero.

$$\Rightarrow v_{cm} = \frac{m_A v_A + m_B v_B}{m_A + m_B} = \frac{m\left(3g\sqrt{\frac{m}{k}}\right)}{m+m} = \frac{3g}{2}\sqrt{\frac{m}{k}}$$

(c) When both the blocks move with same velocity the extension spring will be maximum. The common velocity of the blocks is equal to the velocity of CM Applying C.O.M.E initial and the state when extension is maximum.

$$\Delta K + \Delta U = 0$$

$$\left[\frac{1}{2}(m+m)V_0^2-0\right]+\left[\frac{1}{2}kx^2-\frac{1}{2}kx_0^2\right]=0$$

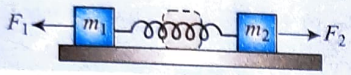
or $$\frac{1}{2}(2m)\left(\frac{9}{4}g^2\left(\frac{m}{k}\right)\right)+\left(\frac{1}{2}kx^2-\frac{1}{2}k\left(\frac{3mg}{k}\right)^2\right)=0$$

or $$\frac{9}{4}\frac{m^2g^2}{k}+\left[\frac{1}{2}kx^2-\frac{9}{2}\frac{m^2g^2}{k}\right]=0$$

or $$kx^2=\frac{9}{2}\frac{m^2g^2}{k} \Rightarrow x=\frac{3mg}{\sqrt{2}k}$$

EXAMPLE 1.9

Two constant forces F_1 and F_2 acts on two smooth particles (blocks) of masses m_1 and m_2 interconnected by a light spring of stiffness k. Find the maximum elongation of the spring using WE theorem.



Sol. In this case we will analyze the situation from CM frame of reference. Let at the time of maximum deformation of the spring the displacements of the blocks m_1 and m_2 with respect to CM are Δx_1 (towards left) and Δx_2 (towards right) respectively. Hence the deformation of spring will be

$$\Delta x_1+\Delta x_2=\Delta x_{12}(=x) \quad \ldots(i)$$

Here x is the deformation of the spring at the time of maximum deformation

Since from centre of mass frame we can write,

$$-m_1\Delta x_1+m_2\Delta x_2=0 \text{ or } m_1\Delta x_1=m_2\Delta x_2 \quad \ldots(ii)$$

From Eq. (i) and Eq. (ii), we have

$$\Delta x_1=\frac{m_2x}{m_1+m_2} \quad \text{and} \quad \Delta x_2=\frac{m_1x}{m_1+m_2}$$

Now applying WE theorem in **CM frame**

$$W_{int}+W_{ext}=\frac{1}{2}\mu(v_{rel_f}^2-v_{rel_i}^2) \quad \ldots(iii)$$

$v_{rel_i}=0$ as both the particles m_1 and m_2 start from rest.

$v_{rel_f}=0$ as both the particles will move with same velocity at the time of maximum separation between them which corresponds to maximum elongation of the spring.

Hence, we have $W_{int}+W_{ext}=0$

Putting $W_{int}=W_{spring}+W_{pseudo}$,

Work done by spring force

$$W_{int}=W_{spring}=-\frac{kx^2}{2} \quad \ldots(iv)$$

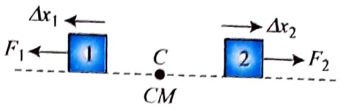
The work done by pseudo (internal) force in CM frame is zero. $W_{pseudo}=0$

As F_1 and F_2 are constant forces, hence the work done by external forces

$$W_{ext}=F_1\Delta x_1+F_2\Delta x_2 \quad \ldots(v)$$

From (iii), (iv) and (v),we have

$$F_1\Delta x_1+F_2\Delta x_2=\frac{k}{2}x^2 \quad \ldots(vi)$$



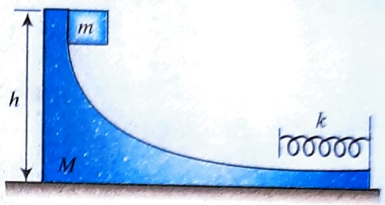
Substituting Δx_1 and Δx_2 in Eq. (vi), we have

$$F_1\left(\frac{m_2x}{m_1+m_2}\right)+F_2\left(\frac{m_1x}{m_1+m_2}\right)=\frac{kx^2}{2}$$

This gives $$x=\frac{2(m_1F_2+m_2F_1)}{k(m_1+m_2)}$$

EXAMPLE 1.10

A block of mass m is released from rest from a height h onto a smooth sledge of mass M fitted with an ideal spring of stiffness k.



(a) Find the velocity of the block and sledge just before the block touches the spring.

(b) Find maximum compression of the spring.

Sol. **Method 1:**

(a) As no external force in horizontal direction, the linear momentum will be conserved in horizontal direction. Let the velocity of block and wedge be v and V respectively up to the case when the block reaches on horizontal surface on the wedge.

$$mv=MV \quad \ldots(i)$$

Mechanical energy is also conserved in this case

$$\Delta K+\Delta U=0$$

$$\left(\frac{1}{2}mv^2+\frac{1}{2}MV^2-0\right)+(-mgh)=0 \quad \ldots(ii)$$

$$\frac{1}{2}mv^2+\frac{1}{2}M\left(\frac{m}{M}v\right)^2=mgh$$

$$v=\sqrt{\frac{2Mgh}{M+m}} \text{ and } V=\sqrt{\frac{2m^2gh}{M(M+m)}}$$

Method 2: Applying work–energy theorem from centre of mass frame

$$W_{gravity}=(\Delta K)_{CM}$$

$$mgh=\frac{1}{2}\left(\frac{mM}{m+M}\right)(v+V)^2$$

Also $$mv=MV \Rightarrow v=\frac{MV}{m}$$

$$\Rightarrow mgh=\frac{1}{2}\left(\frac{mM}{m+M}\right)V^2\left(\frac{M}{m}+1\right)^2$$

$$\Rightarrow V^2=\sqrt{\frac{2m^2gh}{M(m+M)}}$$

(b) Initial linear momentum of the system is zero and there is no external force on the system, hence at the time of maximum compression of spring the linear momentum of the system should be zero. Hence final kinetic energy should be zero.

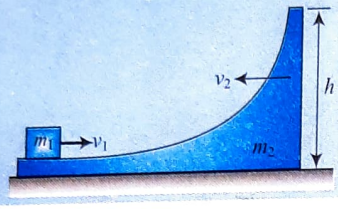
$$\Delta U + \Delta K = 0 \Rightarrow \Delta K = K_f - K_i = 0$$

Hence, $\Delta U_{\text{gravity}} + \Delta U_{\text{spring}} = 0$

$$-mgh + \frac{1}{2}kx^2 = 0 \Rightarrow x = \sqrt{\frac{2mgh}{k}}$$

EXAMPLE 1.11

A particle of mass m_1 is projected to the right with a speed v_1 onto a smooth wedge of mass m_2 which is simultaneously projected due to the left with a speed v_2. If the particle attains the highest point of the wedge, find h.



Sol. Applying work–energy theorem

$$W_{\text{total}} = \Delta K_{\text{blocks, cm}} + \Delta K_{\text{cm}}$$

As no external forces acting, the velocity of centre of mass will be constant. Hence $\Delta K_{\text{cm}} = 0$.

Hence $W_{\text{total}} = \Delta K_{\text{blocks, cm}}$

$$W_{\text{gravity}} = \left[\frac{1}{2}\mu v_{rel}^2\right]_{\text{final}} - \left[\frac{1}{2}\mu v_{rel}^2\right]_{\text{initial}}$$

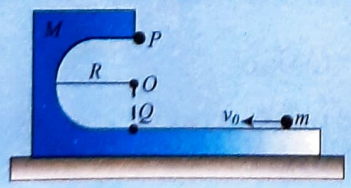
$$-m_1 gh = 0 - \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(v_1 + v_2)^2$$

$$m_1 gh = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(v_1 + v_2)^2$$

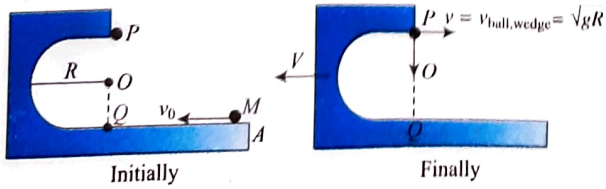
$$h = \frac{1}{2}\frac{m_2(v_1 + v_2)^2}{g(m_1 + m_2)}$$

EXAMPLE 1.12

A small ball of mass m is projected with a minimum horizontal velocity v_0 on a smooth wedge of mass M so that it will reach the highest point P of the wedge. Find the value of v_0



Sol.



Initially Finally

If the ball reaches at point P, the velocity of the ball with respect to wedge should be $\sqrt{gR}$.

Using work–energy theorem from centre of mass frame at A and the highest point P

$$W_{\text{ext}} + W_{\text{int}} = W_{\text{ext}} + 0 = (\Delta K)_{\text{cm}}$$

$$W_{\text{gravity}} = (\Delta K)_{\text{CM}}$$

$$-mg(2R) = \left[\frac{1}{2}\mu v_{rel}^2\right]_{\text{final}} - \left[\frac{1}{2}\mu v_{\text{rel}}^2\right]_{\text{initial}}$$

$$-mg2R = \frac{1}{2}\mu(V + v)^2 - \frac{1}{2}\mu v_0^2 \quad \ldots(i)$$

$$\mu = \left(\frac{mM}{m + M}\right) \text{ and } v = \sqrt{gR}$$

As there is no external forces acting on the system in horizontal direction. The linear momentum of the system should be conserved.

i.e., $mv_0 = m(V - v) + MV$

$$\Rightarrow V = \frac{m(v_0 + v)}{(m + M)} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\Rightarrow v_0 = \sqrt{\left(5 + \frac{4m}{M}\right)gR}$$

EXAMPLE 1.13

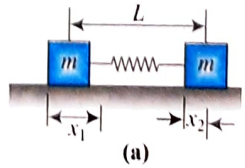
Two blocks are at rest on a frictionless, level table. Both blocks have the same mass m, and they are connected by a spring of negligible mass. The separation distance of the blocks when the spring is relaxed is L. During a time interval Δt, a constant force of magnitude F is applied horizontally to the left block, moving it through a distance x_1. During this time interval, the right block moves through a distance x_2. At the end of this time interval, the force F is removed.

(a) Find the resulting speed $\vec{v}_{\text{CM}}$ of the centre of mass of the system.

(b) Find the total energy of the system associated with vibration relative to its centre of mass after the force F is removed.

Sol.

(a) If we take both the blocks and springs as a system. As we push on the left block. It begins to move to the right and the spring begins to compress. As a result, the spring pushes to the right on the right block, which begins to move to the right. At any given time, the blocks are generally moving with different velocities. As the centre of mass of the system moves to the right with a constant speed after the force is removed, the two blocks oscillate back and forth with respect to the centre of mass of the system.



(a)

(b)

Let us apply the impulse–momentum theorem to the system of two blocks, recognizing that the force F is constant during the time interval Δt while the force is applied.

$$\Delta p_x = I_x \Rightarrow (2m)(v_{CM} - 0) = F\Delta t$$

or $2mv_{CM} = F\Delta t$ ...(i)

Displacement of centre of mass of the system

$$\Delta x_{cm} = \frac{m_1x_1 + m_2x_2}{(m_1 + m_2)} = \frac{mx_1 + mx_2}{(m+m)} \quad \text{or} \quad \Delta x_{cm} = \frac{x_1 + x_2}{2}$$

During the time interval Δt, the centre of mass of the system moves a distance $\frac{1}{2}(x_1 + x_2)$.

Now we can find the time interval in terms of $(v_{cm})_{av}$

$$\Delta t = \frac{\frac{1}{2}(x_1 + x_2)}{(v_{cm,})_{av}}$$

Because the centre of mass is modeled as a particle under constant acceleration, the average velocity of the centre of mass is the average of the initial velocity, which is zero and the final velocity v_{CM}:

$$\Delta t = \frac{\frac{1}{2}(x_1 + x_2)}{\frac{1}{2}(0 + v_{CM})} = \frac{(x_1 + x_2)}{v_{CM}} \quad \text{...(ii)}$$

From (i) and (ii), we get $2mv_{CM} = F\frac{(x_1 + x_2)}{v_{CM}}$

Solve for v_{CM}: $v_{CM} = \sqrt{F\frac{(x_1 + x_2)}{2m}}$

(b) The vibrational energy is all the energy of the system other than the kinetic energy associated with translational motion of the centre of mass. The kinetic energy of the system can be expressed as $K = K_{CM} + K_{vib}$, where K_{vib} is the kinetic energy of the blocks relative to the centre of mass due to their vibration. The potential energy of the system is U_{vib}, which is the potential energy stored in the spring when the separation of the blocks is some value other than L.

From the conservation of mechanical energy for the system

$$\Delta K_{CM} + \Delta K_{vib} + \Delta U_{vib} = W$$

or $\Delta K_{CM} + \Delta E_{vib} = W$

When $\Delta K_{vib} + \Delta U_{vib} = E_{vib}$

The initial values of the kinetic energy of the centre of mass and the vibrational energy of the system are zero. Use this fact and substitute for the work done on the system by the force F:

$$K_{CM} + E_{vib} = W = Fx_1$$

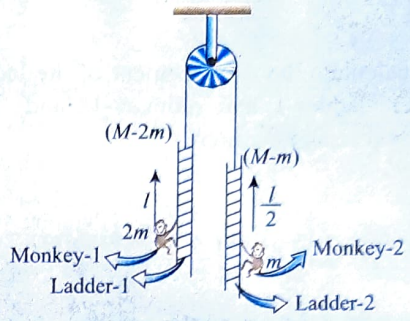
Solve for the vibrational energy and use the result from part (a):

$$E_{vib} = Fx_1 - K_{CM} = Fx_1 - \frac{1}{2}(2m)v_{CM}^2 = F\frac{(x_1 - x_2)}{2}$$

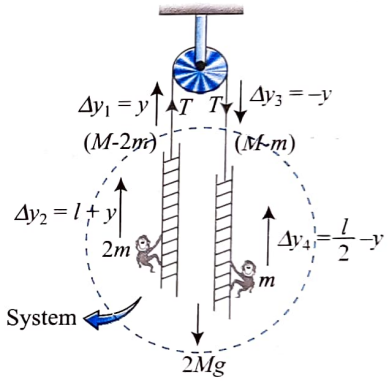
EXAMPLE 1.14

Two ladders are hanging from ends of a light rope passing over a light and smooth pulley. A monkey of mass $2m$ hangs near the bottom of one ladder whose mass is M–$2m$. Another monkey of mass m hangs near the bottom of the other ladder whose mass is M–m.

The monkey of mass $2m$ moves up a distance l with respect to the ladder. The monkey of mass m moves up a distance $l/2$ with respect to the ladder. Does the centre of mass of the system change, if so then calculate its value? Calculate the displacement of the ladder also?



Sol. Here we are considering 'both the monkeys + ladders' as a system. The external forces acting on the system are tension force in the string and gravitational force. When the monkeys are at rest, the tension force in the string is balanced by the weight of the system. But when the monkey starts moving up, they pull the ladders in downward direction. It means the tension in string increases, and now there will be net external force starts acting on the system in upward direction and there will be displacement of centre of mass of the system in upward direction.



Let the displacement of ladder-1 in this process by y upward, hence the displacement of ladder-2 will be y-downward as both the ladders are connected together by a single string. During this time the displacement of monkey-1 will be $(l+y)$ and that of monkey-2 will be $\left(\frac{l}{2} - y\right)$.

$$\Delta y_{cm} = \frac{m_1\Delta y_1 + m_2\Delta y_2 + m_3\Delta y_3 + m_4\Delta y_4}{m_1 + m_2 + m_3 + m_4}$$

Here, m_1 = mass of the ladder-1

Δy_1 = displacement of the ladder-1 = y

m_2 = mass of the ladder-1

Δy_2 = displacement of the ladder-1 = $(l + y)$

m_3 = mass of the ladder-2

Δy_3 = displacement of the ladder-2 = $-y$

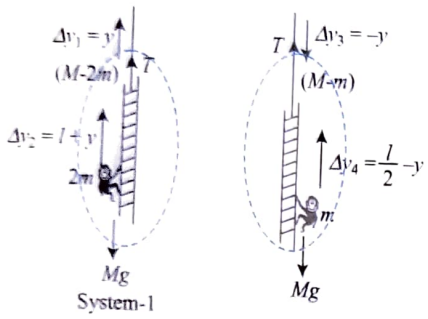
m_4 = mass of the ladder-2

Δy_4 = displacement of the ladder-2 = $\left(\frac{l}{2} - y\right)$

$$\Delta y_{cm} = \frac{(M-2m)(y)+2m\cdot(l+y)+(M+m)(-y)+m\left(\frac{l}{2}-y\right)}{(M-2m)+2m+(M-m)+m}$$

or $\Delta y_{cm} = \frac{5ml}{4M}$

Now let us calculate the displacement of the ladder. For this let us consider 'ladder-1 and monkey-1' and 'ladder-2 and monkey-2' as two separate systems.



As both the systems has equal masses and net external forces it means the displacements of centre of mass of both the systems should be equal

$$(\Delta y_{cm})_1 = (\Delta y_{cm})_2$$

$$\frac{m_1\Delta y_1 + m_2\Delta y_2}{m_1+m_2} = \frac{m_3\Delta y_3 + m_4\Delta y_4}{m_3+m_4}$$

$$\frac{(M-2m)(y)+2m(l+y)}{(M-2m)+2m} = \frac{(M-m)(-y)+m\left(\frac{l}{2}-y\right)}{(M-m)+m}$$

$$(M-2m)y+2m(l+y) = -(M-m)y+m\left(\frac{l}{2}-y\right)$$

$$My-2my+2ml+2my = -My+my+\frac{ml}{2}-my$$

$$My+2ml = -My+\frac{ml}{2}$$

$$2My = -\frac{3ml}{2}$$

$$y = -\frac{3ml}{4M}$$

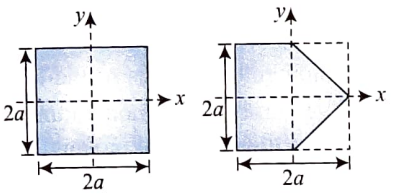
Exercises

Single Correct Answer Type

1. A man of mass m is standing on a plank of mass M kept on a rough surface. When the man walks from left to right on the plank, the centre of mass (man + plank) of the system:
   (1) remains stationary
   (2) shift towards left
   (3) shift towards right
   (4) shifts toward right if $M > m$ and toward left at $M < m$

2. A body falling vertically downwards under gravity breaks in two parts of unequal masses. The centre of mass of the two parts taken together shifts horizontally towards:
   (1) heavier piece
   (2) lighter piece
   (3) does not shift horizontally
   (4) depends on the vertical velocity at the time of breaking

3. Two corners of a square plate as shown in Fig. (a) are folded to make it appear like Fig. (b). Both corners are put together at centre of square 'O'. If O is taken to be origin (0,0), the centre of mass of Fig. (b) will be at

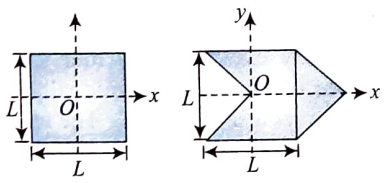


(a) (b)

(1) $\left(-\frac{a}{8}, 0\right)$ (2) $\left(-\frac{a}{6}, 0\right)$

(3) $\left(\frac{a}{12}, 0\right)$ (4) $\left(-\frac{a}{12}, 0\right)$

4. A piece of paper Fig. (a) is in form of a square. Now an arrow sign is made by cutting and rejoining a quarter part of the square of side 'L' as shown. If O is taken to be origin (0, 0), the centre of mass of Fig. (b) will be at

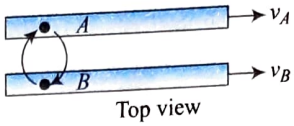


(a) (b)

(1) $\left(\frac{L}{3}, 0\right)$ (2) $\left(\frac{L}{4}, 0\right)$

(3) $\left(\frac{3L}{8}, 0\right)$ (4) None of these

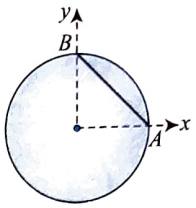
5. Two planks A and B are sliding in the same direction on smooth horizontal surface such that A is faster than B. Packets of equal weight are transferred from A to B. Due to this:



Top view

(1) A will be accelerated and B will be retarded
(2) B will be accelerated and A will be retarded
(3) There will be no change in A but B will be accelerated
(4) There will be no change in B but A will be accelerated

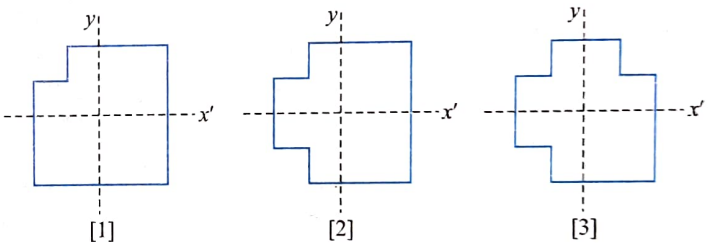
6. An object comprises of a uniform ring of radius R and its uniform chord AB (not necessarily made of the same material) as shown. Which of the following can not be the centre of mass of the object?



(1) $\left(\frac{R}{3}, \frac{R}{3}\right)$ (2) $\left(\frac{R}{3}, \frac{R}{2}\right)$

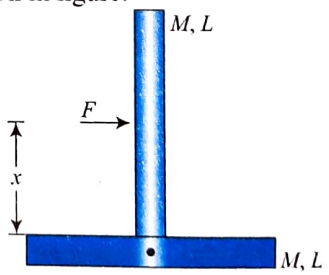
(3) $\left(\frac{R}{4}, \frac{R}{4}\right)$ (4) None of these

7. A machinist starts with three identical square plates but cuts one corner from first, two corners from the second and three corners from the third. Rank the three according to the x-coordinate of their centre of mass, from smallest to largest.



(1) 3, 1, 2 (2) 1, 3, 2
(3) 3, 2, 1 (4) 1 and 3 tie, then 2

8. An inverted T-shaped object is placed on a smooth horizontal floor as shown in figure.

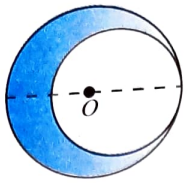


A force F is applied on the system as shown in figure. The value of x so that the system performs pure translational motion is

(1) $\frac{L}{4}$ (2) $\frac{3L}{4}$

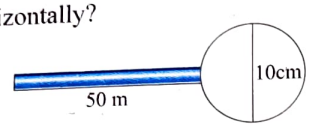
(3) $\frac{L}{2}$ (4) $\frac{3L}{2}$

9. A circular plate of uniform thickness has a diameter of 28 cm. A circular portion of diameter 21 cm is removed from the plate as shown. O is the centre of mass of complete plate. The position of centre of mass of remaining portion will shift towards left from 'O' by



(1) 5 cm (2) 9 cm
(3) 4.5 cm (4) 5.5 cm

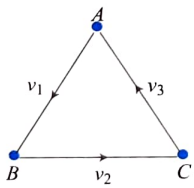
10. Figure shows a thin uniform rod 50 cm long and has a mass of 100 g. A hollow metal ball is filled with air and has a diameter 10 cm and total mass 50 g is fixed to one end of the rod. At what point along its length will the ball and rod balance horizontally?



(1) 20 cm from the centre of the rod
(2) 10 cm from the centre of the rod
(3) at the centre of the rod
(4) where the ball is attached to the rod

11. Two blocks of masses 6 kg and 4 kg are attached to the two ends of a massless string passing over a smooth fixed pulley. If the system is released, the acceleration of the centre of mass of the system will be
(1) g, vertically downwards
(2) $g/5$, vertically downwards
(3) $g/25$, vertically downwards
(4) zero

12. Three particles of equal masses are placed at the corners of an equilateral triangle as shown in the figure. Now particle A starts with a velocity v_1 towards line AB, particle B starts with a velocity v_2 towards line BC and particle C starts with velocity v_3 towards line CA. The displacement of CM of three particle A, B and C after time t will be (given if $v_1 = v_2 = v_3$)

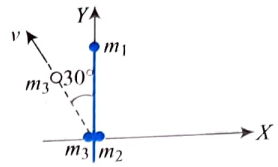


(1) zero (2) $\dfrac{v_1 + v_2 + v_3}{3} t$

(3) $\dfrac{v_1 + \frac{\sqrt{3}}{2} v_2 + \frac{v_3}{2}}{3} t$ (4) $\dfrac{v_1 + v_2 + v_3}{4} t$

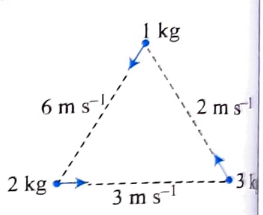
13. Two particles A and B initially at rest, move towards each other by mutual force of attraction. At the instant when the speed of A is n and the speed of B is $3n$, the speed of the centre of mass of the system is
(1) $3n$ (2) $2n$
(3) $1.5n$ (4) 0

14. Three point like equal masses m_1, m_2 and m_3 are connected to the ends of a massless rod of length L which lies at rest on a smooth horizontal plane (see figure). At $t = 0$, an explosion occurs between m_2 and m_3, and as a result, mass m_3 is detached from the rod, and moves with a known velocity v at an angle of 30° with the y-axis. Assume that the masses m_2 and m_3 are unchanged during the explosion. What is the velocity of the centre of mass of the system consisting of three masses, after the explosion?



(1) $\dfrac{v}{4}(\hat{i} - 3\hat{j})$ (2) $\dfrac{v}{4}(-\hat{i} + \sqrt{3}\,\hat{j})$
(3) $-v$ (4) None of these

15. Three particles of masses 1 kg, 2 kg and 3 kg are situated at the corners of an equilateral triangle move at speed 6 ms^{-1}, 3 ms^{-1} and 2 ms^{-1} respectively. Each particle maintains a direction towards the particle at the next corner symmetrically. Find velocity of CM of the system at this instant.



(1) 3 ms^{-1} (2) 5 ms^{-1}
(3) 6 ms^{-1} (4) zero

16. An object initially at rest explodes into three fragments A, B and C. The momentum of A is $p\hat{i}$ and that of B is $\sqrt{3}p\hat{j}$ where p is a + ve number. The momentum of C will be
(1) $(1 + \sqrt{3})\,p$ in a direction making angle 120° with that of A
(2) $(1 + \sqrt{3})\,p$ in a direction making angle 150° with that of B
(3) $2p$ in a direction making angle 150° with that of A
(4) $2p$ in a direction making angle 150° with that of B

17. A vessel at rest explodes breaking it into three pieces. Two pieces having equal mass fly off perpendicular to one another with the same speed of 30 m/s. The third piece has three times the mass of each of the other two pieces. What is the direction (w.r.t. the pieces having equal masses) and magnitude of its velocity immediately after the explosion?
(1) $10\sqrt{2}$, 135° (2) $10\sqrt{2}$, 90°
(3) $10\sqrt{2}$, 60° (4) $10\sqrt{2}$, 30°

18. A stationary body explodes into four identical fragments such that three of them fly off mutually perpendicular to each other, each with same KE, E_0. The energy of explosion will be
(1) $6E_0$ (2) $\dfrac{4E_0}{3}$
(3) $4E_0$ (4) $8E_0$

19. A stationary body explodes into two fragments of masses m_1 and m_2. If momentum of one fragment is p, the energy of explosion is
(1) $\dfrac{p^2}{2(m_1 + m_2)}$ (2) $\dfrac{p^2}{2\sqrt{m_1 m_2}}$
(3) $\dfrac{p^2 (m_1 + m_2)}{2m_1 m_2}$ (4) $\dfrac{p^2}{2(m_1 - m_2)}$

20. A gun of mass M, fires a shell of mass m horizontally and the energy of explosion is such as would be sufficient to project the shell vertically to a height 'h'. The recoil velocity of the gun is

(1) $\left(\frac{2m^2gh}{M(m+M)}\right)^{\frac{1}{2}}$ (2) $\left(\frac{2m^2gh}{M(m-M)}\right)^{\frac{1}{2}}$

(3) $\left(\frac{2m^2gh}{2M(m-M)}\right)^{\frac{1}{2}}$ (4) $\left(\frac{2m^2gh}{2M(m+M)}\right)^{\frac{1}{2}}$

21. A body of mass 1 kg initially at rest, explodes and breaks into three fragments of masses in the ratio 1 : 1 : 3. The two pieces of equal mass fly off perpendicular to each other with a speed of 15 ms^{-1} each. What is the velocity of the heavier fragment?

(1) $10\sqrt{2}\ ms^{-1}$ (2) $5\sqrt{3}\ ms^{-1}$

(3) $10\sqrt{3}\ ms^{-1}$ (4) $5\sqrt{2}\ ms^{-1}$

22. A particle of mass $2m$ is projected at an angle of 45° with the horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s of explosion, the particle breaks into two equal pieces. As a result of this one part comes to rest. The maximum height from the ground attained by the other part is ($g = 10\ m/s^2$)

(1) 50 m (2) 25 m

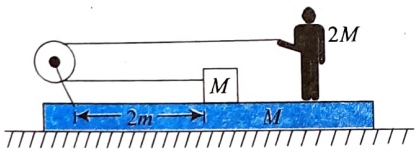
(3) 40 m (4) 35 m

23. A particle of mass $4m$ is projected from the ground at some angle with horizontal. Its horizontal range is R. At the highest point of its path it breaks into two pieces of masses m and $3m$, respectively, such that the smaller mass comes to rest. The larger mass finally falls at a distance x from the point of projection, where x is equal to

(1) $\frac{2R}{3}$ (2) $\frac{7R}{6}$

(3) $\frac{5R}{4}$ (4) none of these

24. A block of mass M is tied to one end of a massless rope. The other end of the rope is in the hands of a man of mass $2M$ as shown in figure. The block and the man are resting on a rough wedge of mass M. The whole system is resting on a smooth horizontal surface. The man starts walking towards right while holding the rope in his hands. Pulley is massless and frictionless. Find the displacement of the wedge when the block meets the pulley. Assume wedge is sufficiently long so that man does not fall down.



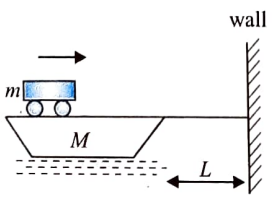
(1) 1/2 m towards right

(2) 1/2 m towards left

(3) The wedge does not move at all

(4) 1 m towards left

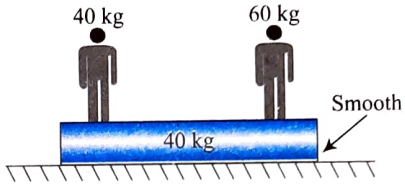
25. A car of mass m is initially at rest on the boat of mass M tied to the wall of dock through a massless, inextensible string. The car accelerates from rest to velocity v_0 in times t_0. At $t = t_0$ the car applies brake and comes to rest relative to the boat in negligible time. Neglect friction between the boat and water; the time 't' at which boat will strike the wall is



(1) $\frac{L(M+m)}{mv_0}$ (2) $t_0 + \frac{L(M+m)}{mv_0}$

(3) $t_0 + \frac{L(M+m)}{Mv_0}$ (4) none of these

26. Two men 'A' and 'B' are standing on a plank. 'B' is at the middle of the plank and 'A' is at the left end of the plank of length 1.2 m. Lower surface of the plank is smooth. System is initially at rest and masses are as shown in figure. 'A' and 'B' start moving such that the position of 'B' remains fixed with respect to ground, then 'A' meets 'B'. Then the point where A meets B is located at



(1) the middle of the plank

(2) 30 cm from the left end of the plank

(3) the right end of the plank

(4) none of these

27. A man stands at one end of the open truck which can run on frictionless horizontal rails. Initially, the man and the truck are at rest. Man now walks to the other end and stops. Then which of the following is true?

(1) The truck moves opposite to direction of motion of the man even after the man ceases to walk.

(2) The centre of mass of the man and the truck remains at the same point throughout the man's walk.

(3) The kinetic energy of the man and the truck are exactly equal throughout the man's walk.

(4) The truck does not move at all during the man's walk.

28. A cracker is thrown into air with a velocity of 10 m/s at an angle of 45° with the vertical. When it is at a height of 0.5 m from the ground, it explodes into a number of pieces which follow different parabolic paths. What is the velocity of centre of mass, when it is at a height of 1 m from the ground? ($g = 10\ m/s^2$)

(1) $4\sqrt{5}$ m/s (2) $2\sqrt{5}$ m/s

(3) $5\sqrt{4}$ m/s (4) 10 m/s

29. A man stands at one end of a boat which is stationary in water. Neglect water resistance. The man now moves to the other end of the boat and again becomes stationary. The centre of mass of the 'man plus boat' system will remain stationary with respect to water

(1) only when the man is stationary initially and finally

(2) only if the man moves without acceleration on the boat

(3) only if the man and the boat have equal masses

(4) in all cases

30. Two particles are shown in figure At $t = 0$, a constant force $F = 6$ N starts acting on the 3 kg man. Find the velocity of the centre of mass of these particles at $t = 5$ s.

(1) 5 m/s (2) 4 m/s
(3) 6 m/s (4) 3 m/s

31. Two blocks of masses 5 kg and 2 kg are placed on a frictionless surface and connected by a spring. An external kick gives a velocity of 14 m/s to the heavier block in the direction of lighter one. The magnitudes of velocities of two blocks in the centre of mass frame after the kick are, respectively,
(1) 4 m/s, 4 m/s (2) 10 m/s, 4 m/s
(3) 4 m/s, 10 m/s (4) 10 m/s, 10 m/s

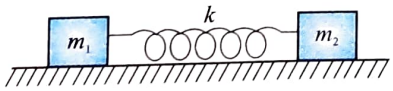
32. In a system of particles 8 kg mass is subjected to a force of 16 N along +ve x-axis and another 8 kg mass is subjected to a force of 8 N along +ve y-axis. The magnitude of acceleration of centre of mass and the angle made by it with x-axis are given, respectively, by
(1) $\frac{\sqrt{5}}{2}$ ms^{-2}, $\theta = 45°$ (2) $3\sqrt{5}$ ms^{-2}, $\theta = \tan^{-2}\left(\frac{2}{3}\right)$
(3) $\frac{\sqrt{5}}{2}$ ms^{-2}, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ (4) 1 ms^{-2}, $\theta = \tan^{-2}\sqrt{3}$

33. A ball kept in a closed box moves in the box making collisions with the walls. The box is kept on a smooth surface. The velocity of the centre of mass
(1) of the box remains constant
(2) of the (box + ball) system remains constant
(3) of the ball remains constant
(4) of the ball relative to the box remains constant

34. A ball of mass m is projected with a speed v into the barrel of a spring gun of mass M initially at rest lying on a frictionless surface. The mass sticks in the barrel at the point of maximum compression in the spring. The fraction of kinetic energy of the ball stored in the spring is
(1) $\frac{m}{M}$ (2) $\frac{M}{m+M}$
(3) $\frac{m}{M+M}$ (4) none of these

35. A railway flat car has an artillery gun installed on it. The combined system has a mass M and moves with a velocity V. The barrel of the gun makes an angle α with the horizontal. A shell of mass m leaves the barrel at a speed v relative to the barrel. The speed of the flat car so that it may stop after the firing is
(1) $\frac{mv}{M+m}$ (2) $\left(\frac{Mv}{M+m}\right)\cos\alpha$
(3) $\left(\frac{mv}{M+m}\right)\cos\alpha$ (4) $(M+m)v\cos\alpha$

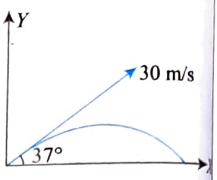
36. Two blocks m_1 and m_2 are pulled on a smooth horizontal surface, and are joined together with a spring of stiffness k as shown in figure. Suddenly, block m_2 receives a horizontal velocity v_0, then the maximum extension x_m in the spring is



(1) $v_0\sqrt{\frac{m_1m_2}{m_1+m_2}}$ (2) $v_0\sqrt{\frac{2m_1m_2}{(m_1+m_2)k}}$
(3) $v_0\sqrt{\frac{m_1m_2}{2(m_1+m_2)k}}$ (4) $v_0\sqrt{\frac{m_1m_2}{(m_1+m_2)k}}$

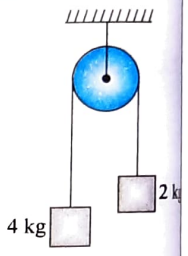
37. A 3000 kg space probe is moving in a gravity free space at a constant velocity of 300 m/s. To change the direction of space probe, rockets have been fired in a direction perpendicular to the direction of initial motion of the space probe, the rocket firing exerts a thrust of 4000 N for 225 s. The space probe will turn by an angle of (neglect the mass of the rockets fired)
(1) 30° (2) 60°
(3) 45° (4) 37°

38. An object of mass 10 kg is launched from the ground at $t = 0$, at an angle of 37° above the horizontal with a speed of 30 m/s. At some time after its launch, an explosion splits the projectile into two pieces. One piece of mass 4 kg is observed at (105 m, 43 m) at $t = 2$ s. Find the location of second piece at $t = 2$ s.



(1) (10, 2) (2) (48, 16)
(3) (10, – 2) (4) Information insufficient

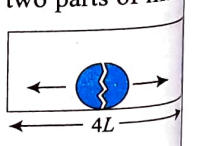
39. For the system shown in figure, the string is light and pulley is frictionless. The 4 kg block is given an upward velocity of 1 m/s.



The centre of mass of the two blocks will [neglect the impulse duration]
(1) accelerate down with $g/3$.
(2) initially accelerate downwards with g and then after some time accelerate down with $g/3$.
(3) initially accelerate with $\vec{g}$ and then the acceleration is 0
(4) initially accelerate with $\vec{g}$ and then with accelerate with $g/9$.

40. Two identical carts constrained to move on a straight line on which sit two twins of same mass, are moving with equal velocity. At some time snow begins to drop uniformly. Ram sitting on one of the carts, picks the snow from cart and throws off the falling snow sideways and in the second cart Shyam is asleep.
(1) Cart carrying Ram will have more speed finally than that carrying Shyam.
(2) Cart carrying Ram will have less speed finally than that carrying Shyam.
(3) Cart carrying Ram will have same speed finally than that carrying Shyam.
(4) depends on the amount of snow thrown.

41. A bomb of mass $3m$ is kept inside a closed box of mass $3m$ and length $4L$ at its centre. It explodes in two parts of mass m and $2m$. The two parts move in opposite directions and stick to the opposite sides of the walls of box. The box is kept on a smooth horizontal

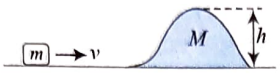


surface. What is the distance moved by the box during this time interval?

(1) 0 (2) $\frac{L}{6}$

(3) $\frac{L}{12}$ (4) $\frac{L}{3}$

42. For the system shown in the figure, a small block of mass m and smooth irregular shaped block of mass M, both free to move are placed on a smooth horizontal plane. The minimum velocity v imparted to block so that it will overcome the highest point of M is



(1) $\sqrt{2gh}$ (2) $\sqrt{2\left(1+\frac{m}{M}\right)gh}$

(3) $\sqrt{\frac{2m}{M}gh}$ (4) $\sqrt{2\left(1+\frac{M}{m}\right)gh}$

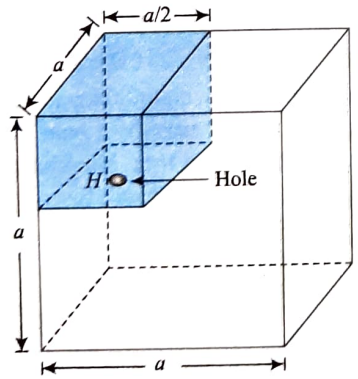
43. Two objects that are moving along an xy-plane on a frictionless floor collide. Assume that they form a closed, isolated system. The following table gives some of the momentum components (in kilogram meters per second) before and after the collision. What are the missing values (a, b)?

	Before collision		After collision	
Object	p_X	p_Y	p_X	p_Y
A	–4	5	3	a
B	b	–2	4	2

(1) 10, 11 (2) 1, 11

(3) 5, 7 (4) 6, 4

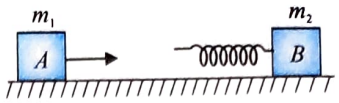
44. Figure shows a hollow cube of side 'a' and volume 'V'. There is a small chamber of volume $V/4$ in the cube as shown. The chamber is completely filled by m kg of water. Water leaks through a hole H and spreads in the whole cube. Then the work done by gravity in this process assuming that the complete water finally lies at the bottom of the cube is



(1) $\frac{1}{2}mga$ (2) $\frac{3}{8}mga$

(3) $\frac{5}{8}mga$ (4) $\frac{1}{8}mga$

45. A block 'A' of mass m_1 hits horizontally the rear side of a spring (ideal) attached to a block B of mass m_2 resting on a smooth horizontal surface. After hitting, 'A' gets attached to the spring.



Some statements are given at any moment of time:

i. If velocity of A is greater than B, then kinetic energy of the system will be decreasing.

ii. If velocity of A is greater than B, then kinetic energy of the system will be increasing.

iii. If velocity of A is greater than B, then momentum of the system will be decreasing.

iv. If velocity of A is greater than B, then momentum of the system will be increasing.

Now select correct alternative:

(1) only iv (2) only i

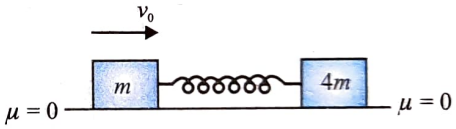
(3) ii and iv (4) i and iii

46. A strip of wood of mass M and length l is placed on a smooth horizontal surface. An insect of mass m starts at one end of the strip and walks to the other end in time t, moving with a constant speed. The speed of the insect as seen from the ground is

(1) $\frac{l}{t}\left(\frac{M}{M+m}\right)$ (2) $\frac{l}{t}\left(\frac{m}{M+m}\right)$

(3) $\frac{l}{t}\left(\frac{M}{m}\right)$ (4) $\frac{l}{t}\left(\frac{m}{M}\right)$

47. Two blocks of masses m and $4m$ lie on a smooth horizontal surface connected with a spring in its natural length. Mass m is given velocity v_0 through an impulse as shown in figure. Which of the following is *not true* about subsequent motion?



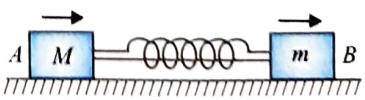
(1) Kinetic energy is maximum in ground frame and centre of mass (CM) frame simultaneously.

(2) Value of maximum and minimum kinetic energy is same in CM and ground frame.

(3) Minimum kinetic energy is zero in CM frame but non-zero in ground frame.

(4) Maximum and minimum kinetic energy of m in ground frame is, respectively, $\frac{1}{2}mv_0^2$ and zero.

48. Block A of mass $M = 2$ kg is connected to another block B of mass $m = 1$ kg with a string and a spring of force constant $k = 600$ N/m as shown in figure. Initially, spring is compressed to 10 cm and whole system is moving on a smooth surface with a velocity $v = 1$ m/s. At any time, thread is burnt, the velocity of block A, when B is having maximum velocity w.r.t. ground, is

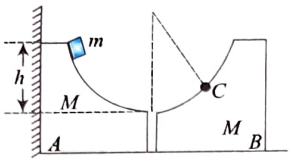


(1) zero (2) 1 m/s

(3) 3 m/s (4) none of these

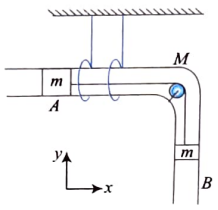
49. Two identical blocks, having mass M each, are conjugated and placed on a smooth floor as shown in figure. A small

block of mass m is released from position as shown. Velocity of block B is maximum



(1) when m is at the highest position on B.
(2) when m is at the lowest position and moving left.
(3) when m is at C.
(4) when m is at lowest position and moving right.

50. In figure, a hollow tube of mass M is free in horizontal direction. The system is released from rest. There is no friction present. The tube and blocks are taken as system.



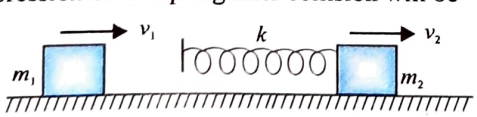
i. Momentum of the system is conserved in x-direction.
ii. Speed of A w.r.t. M = speed of B w.r.t. M.
iii. Trajectory of centre of mass is X-constant.
iv. Centre of mass has finite acceleration.

Evaluate the above statements and choose the correct option from the following:
(1) Statements i, ii are true and iii, iv are false.
(2) Statements i, ii are false and iii, iv are true.
(3) All statements are true.
(4) All statements are false.

51. A trolley is moving horizontally with a velocity of v m/s w.r.t. earth. A man starts running in the direction of motion of trolley from one end of the trolley with a velocity 1.5 v m/s w.r.t. the trolley. After reaching the opposite end, the man turns back and continues running with a velocity of 1.5 v m/s w.r.t. trolley in the backward direction. If the length of the trolley is L, then the displacement of the man with respect to earth, measured as a function of time, will attain a maximum value of

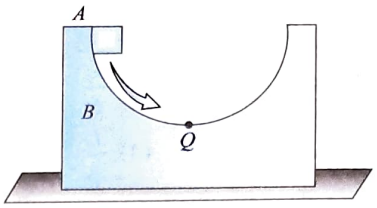
(1) $\frac{4}{3}L$ (2) $\frac{2}{3}L$
(3) $\frac{5L}{3}$ (4) $1.5L$

52. Two blocks of masses $m_1 = 2$ kg and $m_2 = 4$ kg are moving in the same direction with speeds $v_1 = 6$ m/s and $v_2 = 3$ m/s, respectively on a frictionless surface as shown in the figure. An ideal spring with spring constant $k = 30000$ N/m is attached to the back side of m_2. Then the maximum compression of the spring after collision will be



(1) 0.06 m (2) 0.04 m
(3) 0.02 m (4) none of these

53. Block A is released on the circular track of bigger block B from rest from the position as shown in figure. Friction is absent everywhere. Choose the correct statement.

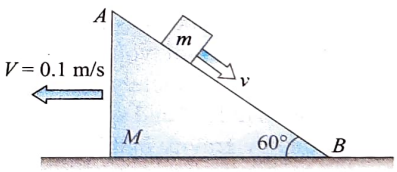


(1) Acceleration of center of mass of A and B is zero initial
(2) When A reaches Q acceleration of center of mass of and B is zero
(3) Normal force of the ground on the wedge is greater th combined weight of A and B, when A reaches at point
(4) Acceleration of B is maximum when A reaches Q

54. Two identical particles A and B of mass m each are connecte together by a light and inextensible string of length l. Th particle are held at rest in air in same horizontal level at separation l. Both particles are released simultaneously an one of them (say A) is given speed v_0 vertically upwar The maximum height attained by the centre of mass of th system of A and B from initial level is (Ignore air resistance

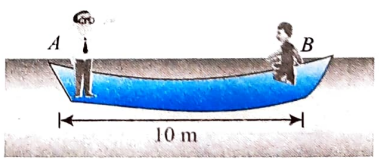
(1) $\frac{v_0^2}{2g}$ (2) $\frac{v_0^2}{8g}$
(3) $\frac{v_0^2}{4g}$ (4) $\frac{v_0^2}{12g}$

55. A particle of mass $m = 0.2$ kg is released from rest from point A of a triangular wedge of mass $M = 4.8$ kg free to slide on a frictionless horizontal plane. The particle slid down the smooth face AB of the wedge. When the velocit of the wedge is 0.1 m/s, the velocity of the particle in m relative to the wedge is



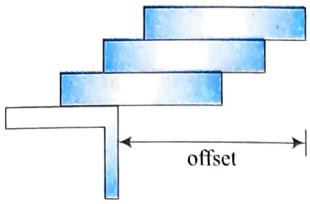
(1) 4.8 (2) 5
(3) 7.5 (4) 10

56. Two persons A and B of weight 80 kg and 50 kg respectively are standing at opposite ends of a boat of mass 70 kg and length 10 m at rest. When they interchange their position then displacement of the boat will be



(1) 1.5 m towards right (2) 1.0 m towards right
(3) 1.5 m towards left (4) stationary

57. Three identical rods of mass M and length L are placed o one another on the table so as to produce the maximu overhang as shown in figure. The maximum total overhan will be



(1) $\frac{2}{3}L$ (2) $\frac{4}{3}L$

(3) $\frac{5}{6}L$ (4) $\frac{11}{12}L$

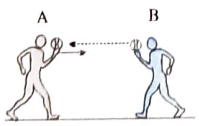
Multiple Correct Answers Type

1. Which of the following is/are correct?
   (1) If centre of mass of three particles is at rest and it is known that two of them are moving along different lines, then the third particle must also be moving.
   (2) If centre of mass remains at rest, then net work done by the forces acting on the system must be zero.
   (3) If centre of mass remains at rest, then the net external force must be zero.
   (4) If speed of centre of mass is changing, then there must be some net work being done on the system from outside.

2. Choose the correct statements from the following:
   (1) The general form of Newton's second law of motion is $\vec{F}_{ext} = m\vec{a}$.
   (2) A body can have energy and get no momentum.
   (3) A body having momentum must necessarily have kinetic energy.
   (4) The relative velocity of two bodies in a head-on elastic collision remains unchanged in magnitude and direction.

3. A man standing on the edge of the terrace of a high rise building throws a stone, vertically up with a speed of 20 m/s. Two seconds later, an identical stone is thrown vertically downwards with the same speed of 20 m/s. Then
   (1) the relative velocity between the two stones remains constant till one hits the ground
   (2) both will have the same kinetic energy, when they hit the ground
   (3) the time interval between their hitting the ground is 2 s
   (4) if the collision on the ground is perfectly elastic, both will rise to the same height above the ground

4. Consider two skaters A and B initially at rest on ice (friction is negligible) with A holding a ball. A has greater mass than B and the ball has some significant mass. A throws the ball to B. B catches it and throws it back to A who catches it again. The magnitudes of the skater's (excluding ball) final velocities, momentum and kinetic energies (denoted below as v, p and K respectively) are related as

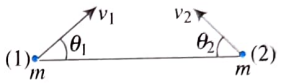


   (1) $v_A = v_B$ (2) $v_A < v_B, p_A < p_B$
   (3) $p_A = p_B, K_A < K_B$ (4) $p_A < p_B, K_A < K_B$

5. A man is standing on a plank which is placed on smooth horizontal surface. There is sufficient friction between feet of man and plank. Now man starts running over plank, correct statement is/are

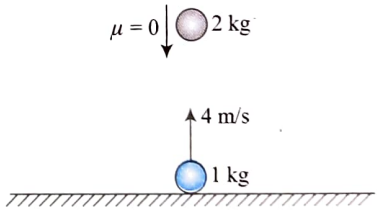
   (1) Work done by friction on man with respect to ground is negative.
   (2) Work done by friction on man with respect to ground is positive
   (3) Work done by friction on plank with respect to ground is positive.
   (4) Work done by friction on man with respect to plank is zero.

6. Two particles of equal mass m are projected from the ground with speeds v_1 and v_2 at angles θ_1 and θ_2 as shown in the figure. Given $\theta_2 > \theta_1$ and $v_1 \cos\theta_1 = v_2 \cos\theta_2$. Which statement/s is/are correct?



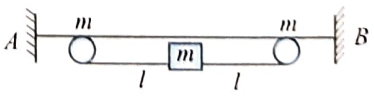
   (1) Centre of mass of particles will move along a vertical line.
   (2) Centre of mass of particles will move along a line inclined at some angle with vertical.
   (3) Particle '1' will be above centre of mass level when both particles are in air.
   (4) Particle '2' will be above centre of mass level when both particles are in air.

7. A ball of mass 1 kg is thrown up with an initial speed of 4 m/s. A second ball of mass 2 kg is released from rest from some height as shown in figure. Choose the correct statement(s).



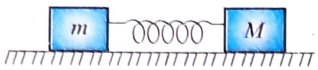
   (1) The centre of mass of the two balls comes down with acceleration $g/3$.
   (2) The centre of mass first moves up and then comes down.
   (3) The acceleration of the centre of mass is g downwards.
   (4) The centre of mass of the two balls remains stationary.

8. Two small rings, each of mass 'm', are connected to the block of same mass 'm' through an inextensible massless string of length 'l'. Rings are constrained to move over smooth rod AB. Initially, the system is held at rest as shown in figure. Let u and v be the velocities of ring and block, respectively when string makes an angle 60° with the vertical.



   (1) $u = \sqrt{\frac{gl}{5}}$ (2) $u = \sqrt{\frac{8gl}{5}}$
   (3) $v = \sqrt{3gl}$ (4) $v = \sqrt{\frac{3gl}{5}}$

9. An ideal spring is permanently connected between two blocks of masses M and m. The blocks–spring system can move over a smooth horizontal table along a straight line along the length of the spring as shown in figure. The blocks are brought nearer to compress the spring and then released. In the subsequent motion,



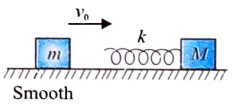
(1) initially they move in opposite directions with velocities inversely proportional to their masses

(2) the ratio of their velocities remains constant

(3) linear momentum and energy of the system remain conserved

(4) the two blocks will oscillate about their centre of mass, which remains stationary

10. A block of mass m moving with a velocity v_0 collides with a stationary block of mass M to which a spring of stiffness k is attached, as shown in figure. Choose the correct alternative(s).



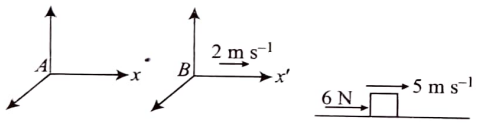
(1) The velocity of the centre of mass is v_0.

(2) The initial kinetic energy of the system in the centre of mass frame is $\frac{1}{4}\left(\frac{mM}{M+m}\right)v_0^2$.

(3) The maximum compression in the spring is $v_0\sqrt{\left(\frac{mM}{m+M}\frac{1}{k}\right)}$.

(4) When the spring is in the state of maximum compression, the kinetic energy in the centre of mass frame is zero.

11. Consider a block of mass 10 kg. which rests on a smooth surface and is subjected to a horizontal force of 6 N. If observer A is in a fixed frame x.



(1) The final speed of the block in 4 s is 7.4 ms^{-1}, if it has initial speed of 5 ms^{-1} measured from fixed frame.

(2) Same speed will be observed by an observer B, attached to the x' axis that moves at a constant velocity of 2 ms^{-1} relative to A.

(3) Principle of impulse and momentum is valid for observers in any inertial reference frame.

(4) Momentum of a body is reference frame dependent.

12. For a two-body system in absence of external forces, the kinetic energy as measured from ground frame is K_0 and from centre of mass frame is K_{cm}. Pick up the correct statements.

(1) The kinetic energy as measured from centre of mass frame is least.

(2) Only the portion of energy K_{cm} can be transformed from one form to another due to internal changes in the system.

(3) The system always retains at least $K_0 - K_{cm}$ amount of kinetic energy as measured from ground frame irrespective of any kind of internal changes in the system.

(4) The system always retains at least K_{cm} amount of kinetic energy as measured from ground frame irrespective of any kind of internal changes in the system

13. Two particles of masses m and $2m$ move with accelerations $2a\hat{i}$ and $-a\hat{i}$ respectively:

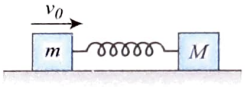
(1) The acceleration of CM is zero

(2) The net force acting on $(m + 2m)$ is zero

(3) The velocity of CM must be zero

(4) The bodies must interact with each other

14. Two blocks of masses m and M are interconnected by an ideal spring of stiffness k. If m is pushed with a velocity v_0.



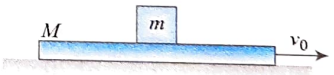
(1) Velocity of centre of the system $(M + m)$; $v_C = \frac{mv_0}{M+m}$

(2) Kinetic energy of the system $(M + m)$ with respect to CM is equal to $\frac{Mmv_0^2}{2(M+m)}$

(3) The maximum work done by the spring is equal to $-\frac{Mmv_0^2}{2(M+m)}$

(4) Maximum compression of spring $x_{max} = v_0\sqrt{\frac{Mm}{(M+m)k}}$

15. A block of mass m is placed gently onto a long plank of mass m moving with a velcoity v_0 on a smooth horizontal floor. If friction is present between M and m:



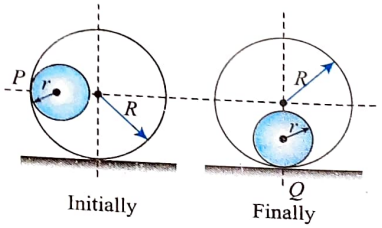
(1) Velocity of centre of mass $v_C = \frac{Mv_0}{M+m}$

(2) Work done by friction force on m is positive

(3) Work done by friction force on M is negative

(4) Total work done by friction force $W_f = \frac{Mmv_0^2}{2(M+m)}$

16. Consider a hollow sphere of mass M and radius R resting on a smooth surface. A smaller sphere of mass m and radius r is initially held at position P within the bigger sphere. If the small sphere is now released, it rolls down the inner surface of hollow sphere and finally stops at its bottom point Q. Then:



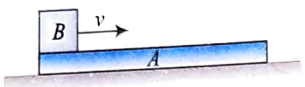
(1) horizontal displacement of the smaller sphere on smooth surface is $\frac{M(R-r)}{M+m}$

(2) horizontal displacement of the bigger sphere on smooth surface is $\frac{M(R+r)}{M+m}$

(3) horizontal displacement of the bigger sphere on smooth surface is $\frac{m(R-r)}{M+m}$

(4) none of these

17. A long plank 'A' is at rest on a smooth horizontal surface. A small block 'B', whose mass is half of 'A', is placed on the plank at one end and projected along A with some velocity u. The coefficient of friction between the plank and block is μ:



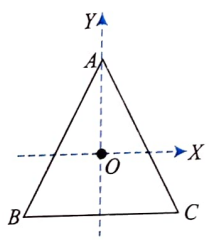
(1) The plank and block will reach the final common velocity $\frac{u}{3}$.

(2) The work done against friction is two-thirds of the initial kinetic energy of block.

(3) Before the block and plank reach a common velocity, the acceleration of A relative to B is $\frac{2}{3}\mu g$.

(4) Before the block and plank reach a common velocity the acceleration of A relative to B is $\frac{3}{2}\mu g$.

18. A uniform wire frame ABC is in the shape of an equilateral triangle in xy-plane with centroid at the origin.



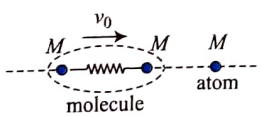
(1) If AB is removed, the centre of mass of the remaining figure is in fourth quadrant.

(2) If AC is removed, the centre of mass of the remaining figure is in first quadrant.

(3) If BC is removed, the centre of mass of the remaining figure is on the positive Y-axis.

(4) If AC is removed, the centre of mass of the remaining figure is in third quadrant.

19. A molecule capable of vibrations is modeled as two point atoms of mass M connected by a spring. Initially molecule is not vibrating when it collides elastically with a single atom of mass M as shown in figure. Mark the CORRECT option(s).



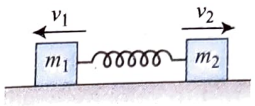
(1) Velocity of centre of mass of system consist of atom and molecules is $\frac{2v_0}{3}$.

(2) Maximum energy stored in spring is $\frac{Mv_0^2}{4}$.

(3) Velocity of centre of mass of molecule after collision is $\frac{v_0}{2}$.

(4) Momentum of molecule after collision is Mv_0.

20. Two blocks of mass m_1 and m_2 are connected by a spring in its natural length having spring constant k and are placed on a smooth surface. Block m_1 and m_2 are given velocities v_1 and v_2 as shown in figure. Choose CORRECT statements.



(1) Momentum of the system of m_1 and m_2 in the ground frame is conserved.

(2) Momentum of the system of m_1 and m_2 in the frame attached to center of mass of m_1 and m_2 is always zero.

(3) The term $\frac{1}{2}(m_1+m_2)v_C^2$ is always constant, where v_C is velocity of center of mass of m_1 and m_2.

(4) Kinetic energy in center of mass frame remains constant.

21. A man of mass m walks from end A to the other end B of a plank of mass M and length l, placed on a smooth horizontal surface. The coefficient of friction between man and plank is μ as no external force in horizontal direction.

(1) If man runs at his maximum acceleration, the acceleration of plank is $\frac{m}{M}\mu g$.

(2) Time taken by man to reach other end of the plank is $\sqrt{\frac{2Ml}{(M+m)\mu g}}$

(3) Magnitude of displacement of centre of mass of plank is $\frac{Ml}{m+M}$

(4) Velocity of centre of mass of (man + plank) is zero.

22. Two men 'A' and 'B' of mass 50 kg and 70 kg respectively are standing on the ends of a plank of mass 80 kg. The length of plank is 5m and it is kept on a smooth horizontal surface. Now man starts moving and exchange their positions on the plank. Then

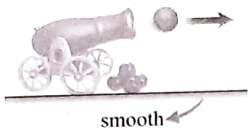
(1) The distance moved by centre of mass of the system 'A' + 'B' + plank is 50 cm

(2) The displacement of the plank is 50 cm (toward right)

(3) The distance moved by man 'A' with respect to ground is 5.5 m

(4) The distance moved by man 'B' with respect to ground is 6 m.

23. A shell of mass m is fired from a cannon of mass nm (without the shell) with a relative speed of v_0. The shell is projected horizontally. At the time of firing.



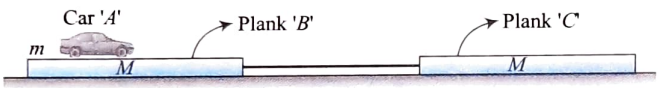
(1) The total energy released is $\frac{1}{2}\frac{mn}{(n+1)}v_0^2$

(2) The speed of shell is $\frac{mv_0}{n+1}$

(3) The speed of cannon is $\frac{v_0}{n+1}$

(4) The total energy released is $\frac{1}{2}\frac{mv_0^2}{n+1}$

24. A car 'A' of mass m is initially at rest on a plank of mass M tied to the identical plank 'C' of same mass M through a massless inextensible string as shown in the figure. The car accelerates from rest to velocity v_0 with respect to plank 'B' in time t_0 sec. At time $t = t_0$ the car applies brake and comes to rest relative to plank in negligible time. Neglect friction between planks and ground.



(1) when the car 'A' reaches to velocity v_0 with respect to the plank 'B', the velocity of the car is $\frac{2Mv_0}{(m+2M)}$

(2) when the car 'A' reaches to velocity v_0 with respect to the plank 'B', the velocity of the car is $\frac{mv_0}{(m+2M)}$

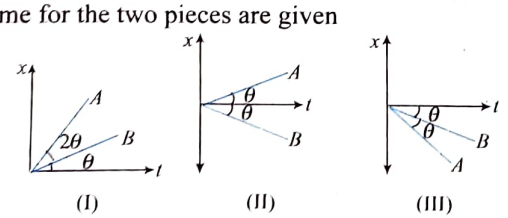
(3) the velocity of plank 'B' just after applying the break is $\frac{mMv_0}{(m+M)(m+2M)}$

(4) the velocity of plank 'B' just after applying the break is $\frac{2mMv_0}{(m+M)(m+2M)}$

Linked Comprehension Type

For Problems 1–3

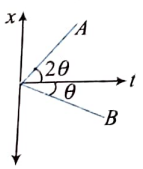
An initially stationary box on a frictionless floor explodes into two pieces, piece A with mass m_A and piece B with mass m_B. Two pieces then move across the floor along x-axis. Graph of position versus time for the two pieces are given



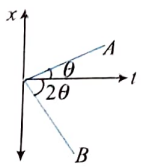
(I)

(II)

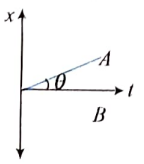
(III)



(IV)



(V)



(VI)

1. Which graphs pertain to physically possible explosions?

(1) II, IV, V (2) I, III, VI

(3) I, III, V (4) II, III, VI

2. Match column-A with the column-B.

Column-A	Column-B
a. $m_A = m_B$	p. I
b. $m_A > m_B$	q. II
c. $m_A < m_B$	r. III
	s. IV
	t. V
	u. VI

(1) a-u, b-r, c-p (2) a-q, b-t, c-s

(3) a-q, b-s, c-t (4) a-u, b-t, c-s

3. If graphs are possible then, in which of the following cases external impulse must be acting on the box?

(1) II (2) IV

(3) V (4) VI

For Problems 4–5

Two identical buggies each of mass 150 kg move one after the other without friction with same velocity 4 m/s. A man of mass m rides the rear buggy. At a certain moment, the man jumps into the front buggy with a velocity v relative to his buggy. As a result of this process, rear buggy stops. If the sum of kinetic energies of the man and the front buggy just after collision with the front buggy differs from that just before collision by 2700 J, then

4. The mass m of the man is

(1) 60 kg (2) 75 kg

(3) 50 kg (4) 90 kg

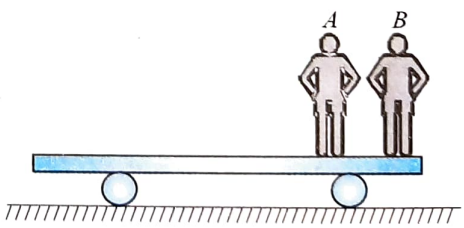
5. The velocity v of the man relative to the buggy is

(1) 16 m/s (2) 8 m/s

(3) 10 m/s (4) 15 m/s

For Problems 6–8

Two persons, A of mass 60 kg and B of mass 40 kg, are standing on a horizontal platform of mass 50 kg. The platform is supported on wheels on a horizontal frictionless surface and is initially at rest. Consider the following situations.



(i) Both A and B jump from the platform simultaneously and in the same horizontal direction.

(ii) A jumps first in a horizontal direction and after a few seconds B also jumps in the same direction.

In both the situations above, just after the jump, the person (A or B) moves away from the platform with a speed of 3 m/s relative to the platform and along the horizontal.

Answer these questions.

6. In situation (i), just after both A and B jump from the platform, velocity of centre of mass of the system (A, B and the platform) is
(1) 2 m/s (2) 6 m/s
(3) 5 m/s (4) none of these

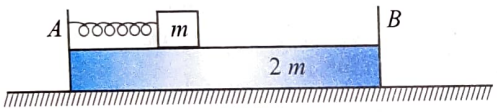
7. Final speed of the platform in situation (i), i.e., just after both A and B have jumped will be
(1) 2 m/s (2) 6 m/s
(3) 5 m/s (4) 3 m/s

8. Final speed of the platform in situation (ii), i.e., just after B has jumped, will be nearly
(1) 7.5 m/s (2) 5.5 m/s
(3) 4.5 m/s (4) 2.5 m/s

For Problems 9–11

According to the principle of conservation of linear momentum, if the external force acting on the system is zero, the linear momentum of the system will remain conserved. It means if the centre of mass of a system is initially at rest, it will remain at rest in the absence of external force, that is, the displacement of centre of mass will be zero.

Two blocks of masses 'm' and '$2m$' are placed as shown in figure. There is no friction anywhere. A spring of force constant 'k' is attached to the bigger block. Mass 'm' is kept in touch with the spring but not attached to it. 'A' and 'B' are two supports attached to '$2m$'. Now m is moved towards left so that spring is compressed by distance 'x' and then the system is released from rest.



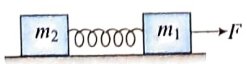
9. Find the relative velocity of the blocks after 'm' leaves contact with the spring.
(1) $\sqrt{\frac{2k}{3m}}$ (2) $3x\sqrt{\frac{k}{2m}}$
(3) $x\sqrt{\frac{3k}{2m}}$ (4) None of these

10. Now m arrives at B. Due to its inertia of motion 'm' breaks the support 'B' and due to some resistance offered by 'B', the resulting velocity of 'm' is reduced to half of its previous value. Then what you can say about the velocity of '$2m$'?
(1) It is also reduced to half of its previous value.
(2) It is reduced to less than half of its previous value.
(3) It is reduced to more than half of its previous value.
(4) It remains the same.

11. What is the loss in the energy of the system due to breaking of 'B'?
(1) $\frac{3}{8}kx^2$ (2) $\frac{3}{4}kx^2$
(3) $\frac{1}{8}kx^2$ (4) $\frac{5}{8}kx^2$

For Problems 12–14

Two blocks of masses m_1 and m_2 are connected by an ideal spring of force constant k. The blocks are placed on smooth horizontal surface. A horizontal force F acts on the block m_1. Initially spring is relaxed, both the blocks are at rest.



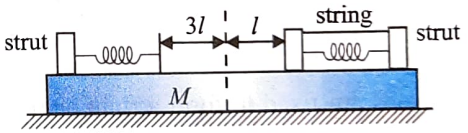
12. What is acceleration of centre of mass of system at the instant of maximum elongation of spring?
(1) Zero (2) $\frac{F(m_1+m_2)}{m_1m_2}$
(3) $\frac{F}{m_1}$ (4) $\frac{F}{m_1+m_2}$

13. Which of the following statement is not true in the context of above system.
(1) Centre of mass reference frame is an inertial frame.
(2) Kinetic energy of the system is minimum in centre of mass frame.
(3) At the instant of maximum deformation both the blocks are instantaneously at rest in centre of mass reference frame.
(4) Acceleration of centre of mass is constant in ground frame.

14. What is maximum elongation of spring.
(1) $\frac{2m_1F}{(m_1+m_2)k}$ (2) $\frac{m_1^2F}{2(m_1+m_2)^2k}$
(3) $\frac{2m_2F}{k(m_1+m_2)}$ (4) $\frac{m_2^2F}{2(m_1+m_2)^2k}$

For Problems 15–17

According to the principle of conservation of linear momentum if the external force acting on the system is zero, the linear momentum of the system will remain conserved. It means if the centre of mass of a system is initially at rest, it will remain at rest in the absence of external force, that is, the displacement of centre of mass will be zero.



A plank of mass M is placed on a smooth horizontal surface. Two light identical springs, each of stiffness K, are rigidly connected to struts at the end of the plank as shown in figure. When the springs are in their unextended position, the distance between their free ends is $3l$. A block of mass m is placed on the plank and pressed against one of the springs so that it is compressed to l. To keep the block at rest it is connected to the strut by means of a light string. Initially, the system is at rest. Now the string is burnt.

15. The maximum displacement of the plank is
(1) $\frac{5ml}{M}$ (2) $\frac{5ml}{M+m}$
(3) $\frac{3ml}{M+m}$ (4) $\frac{4ml}{M+m}$

16. The maximum velocity of the plank is
(1) $\sqrt{\frac{km}{(M+m)}}\,l$ (2) $\sqrt{\frac{k}{(M+m)}}\,l$
(3) $\sqrt{\frac{km}{M(M+m)}}\,l$ (4) $\sqrt{\frac{kM}{m(M+m)}}\,l$

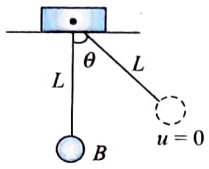
17. The maximum kinetic energy of the block m is

(1) $\frac{kml^2}{2M(M+m)}$ (2) $\frac{kml^2}{M(M+m)}$

(3) $\frac{kMl^2}{2(M+m)}$ (4) none of these

For Problems 18–21

A small ball B of mass m is suspended with light inelastic string of length L from a block A of same mass m which can move on smooth horizontal surface as shown in the figure. The ball is displaced by angle θ from equilibrium position and then released.



18. The displacement of block when ball reaches the equilibrium position is

(1) $\frac{L\sin\theta}{2}$ (2) $L\sin\theta$

(3) L (4) none of these

19. Tension in string when it is vertical, is

(1) mg (2) $mg(2-\cos\theta)$

(3) $mg(3-2\cos\theta)$ (4) none of these

20. Maximum velocity of block during subsequent motion of the system after release of ball is

(1) $[gl(1-\cos\theta)]^{1/2}$

(2) $[2gl(1-\cos\theta)]^{1/2}$

(3) $[gl\cos\theta]^{1/2}$

(4) information are insufficient to decide

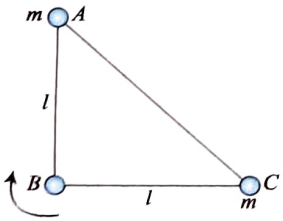
21. The displacement of centre of mass of $A+B$ system till the string becomes vertical is

(1) zero (2) $L(1-\cos\theta)$

(3) $\frac{L}{2}(1-\cos\theta)$ (4) none of these

For Problems 22–24

Three massless rods are fixed to form a right angled triangular frame such that $AB = BC = l$. Two identical small objects of mass m are fixed at A and C. The frame is hinged about B such that the frame can rotate in vertical plane about an horizontal axis without friction. Initially AB is vertical and BC is horizontal and the system is released from rest.



22. The maximum shift of centre of mass of two mass system from its initial position is

(1) $\sqrt{2}l$ (2) $1.5l$

(3) $\frac{l}{\sqrt{2}}$ (4) $2l$

23. The magnitude of acceleration of mass A when the rod AC becomes horizontal is

(1) $g(1+\sqrt{2})$ (2) $g(2+\sqrt{2})$

(3) $2g(1+\sqrt{2})$ (4) none of these

24. Tension in the rod AC when it (rod AC) becomes horizontal

(1) mg (2) $\sqrt{2}mg$

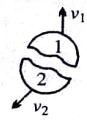
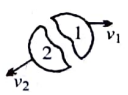
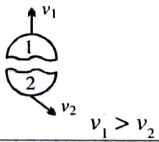
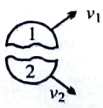
(3) $\frac{mg}{\sqrt{2}}$ (4) None of these

Matrix Match Type

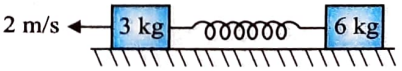
1. Centre of mass of a system is a unique point which is very helpful in solving the problems related to the motions of rigid bodies. In columns I and II, some statements regarding centre of mass are given. Match the columns:

Column I	Column II
i. In the absence of external forces, the internal forces	a. inside the material of the body
ii. Centre of mass of a body can be	b. may affect the motion of individuals
iii. The kinetic energy of the system of two blocks	c. do not affect the motion of centre of mass
iv. Centre of mass of a solid cylinder is	d. outside the material of the body

2. A body initially moving towards right explodes into two pieces 1 and 2 having masses m_1 and m_2. Direction of motion of the pieces is shown in column I and possible mass ratio are shown in column II.

Column I	Column II
i. 	a. $m_1 > m_2$
ii. 	b. $m_1 = m_2$
iii. 	c. $m_1 < m_2$
iv. 	d. Impossible for any masses

3. Two blocks of masses 3 kg and 6 kg are connected by an ideal spring and are placed on a frictionless horizontal surface. The 3 kg block is imparted a speed of 2 m/s towards left. (Consider left as positive direction.)

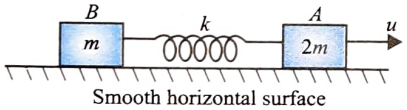


Column I	Column II
i. When the velocity of the 3 kg block is $\frac{2}{3}$ m/s	a. The velocity of centre of mass is $\frac{2}{3}$ m/s
ii. When the speed of the 3 kg block is $\frac{2}{3}$ m/s	b. deformation of the spring is zero
iii. When the speed of the 3 kg block is minimum	c. deformation of the spring is maximum
iv. When the velocity of the 6 kg block is maximum	d. both the blocks are at rest with respect to each other

4. In column I information about the force(s) acting on a body are mentioned, while in column II information about motion of body. Match the entries of column II with the entries of column I.

Column I	Column II
i. A single force through centre of mass	a. Rotational motion
ii. Equal and opposite forces separated by non-zero distance	b. Translational motion
iii. Equal and opposite forces acting at same point	c. No motion
iv. A single force not through centre of mass	d. Centre of mass performs curvilinear motion

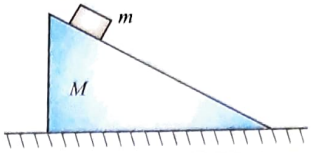
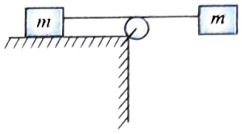
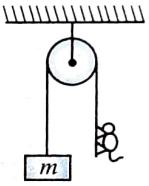
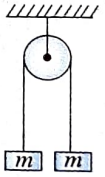
5. Two blocks A and B of masses m and $2m$, respectively, are connected by a massless spring of spring constant K. This system lies over a smooth horizontal surface. At $t = 0$, block A has velocity u towards right as shown, while the speed of block B is zero, and the length of the spring is equal to its natural length at that instant. In each situation of column I, certain statements are given and corresponding results are given in column II. Match the statements in column I with the corresponding results in column II.



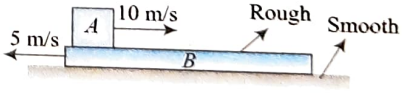
Smooth horizontal surface

Column I	Column II
i. The velocity of block A	a. can never be zero
ii. The velocity of block B	b. may be zero at certain instants of time
iii. The kinetic energy of the system of two blocks	c. is minimum at maximum compression of spring
iv. The potential energy of spring	d. is maximum at maximum extension of spring

6. In each situation of column I, a system involving two bodies is given. All strings and pulleys are light and friction is absent everywhere. Initially, each body of every system is at rest. Consider the system in all situations of column I from rest till any collision occurs. Then match the statements in column I with the corresponding results in column II.

Column I	Column II
i. The block plus wedge system is placed over smooth horizontal surface. After the system is released from rest, the centre of mass of system 	a. shifts towards right
ii. The string connecting both the blocks of mass m is horizontal. The left block is placed over smooth horizontal table as shown. After the two blocks system is released from rest, the centre of mass of system 	b. shifts downwards
iii. The block and the monkey have the same mass. The monkey starts climbing up the rope. After the monkey starts climbing up, the centre of mass of monkey + block system 	c. shifts upwards
iv. Both blocks of mass m are initially at rest. The left block is given initial velocity u downwards. Then, the centre of mass of two-block system afterwards 	d. does not shift

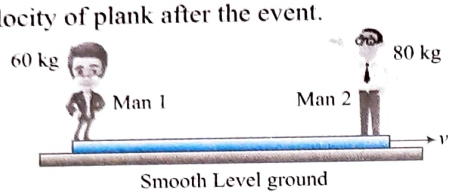
7. A small block 'A' of mass 1 kg is sliding on the rough top surface of a plank 'B' of mass 2 kg. The plank is placed on a smooth horizontal surface. Now the plank and the block starts moving with velocities 5 m/s and 10 m/s respectively as shown in the figure, match the following:



Column I	Column II
i. Velocity of center of mass	a. Keep on changing all the time
ii. Momentum of center of mass	b. First decreases then become zero
iii. Momentum of the block 'A'	c. Zero
iv. Kinetic energy of the plank 'B'	d. Constant

(1) (i – c,d); (ii – c,d); (iii – b); (iv – b)
(2) (i – d); (ii – c,d); (iii – a); (iv – b)
(3) (i – a); (ii – d); (iii – c); (iv – b)
(4) (i – c,d); (ii – a,c); (iii – b); (iv – b)

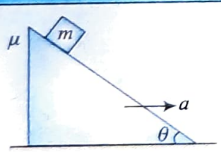
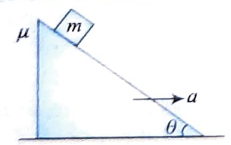
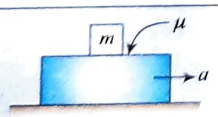
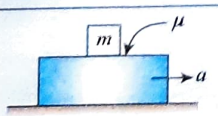
8. Two men of mass 60 kg and 80 kg stand on a plank of mass 20 kg. Both of them can jump with a velocity of 1 m/s relative to the plank. In each event shown in column I, find the velocity of plank after the event.



Column I	Column II
i. Man '1' alone jumps to the left	a. $-\frac{17}{40}$ m/s
ii. Man '2' alone jumps to the right	b. $-\frac{1}{2}$ m/s
iii. Man '1' jumps to left and Man '2' jumps to right simultaneously	c. $\frac{3}{8}$ m/s
iv. Man '1' jumps to left and after that Man '2' jumps to right	d. $-\frac{1}{8}$ m/s

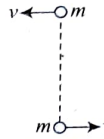
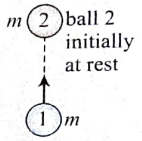
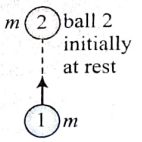
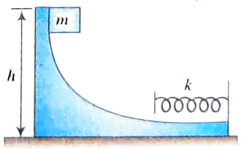
(1) (i – b); (ii – a); (iii – c); (iv – d)
(2) (i – c); (ii – b); (iii – d); (iv – a)
(3) (i – b); (ii – c); (iii – d); (iv – b)
(4) (i – d); (ii – c); (iii – b); (iv – a)

9. In the arrangement shown in column I the wedge and the plank are made to move with constant acceleration 'a'.

Column I	Column II
i.  $a = g\tan\theta$	a. Net force on block of mass m is ma
ii.  $\frac{(g\tan\theta-\mu g)}{(\mu\tan\theta+1)} < a < g\tan\theta$	b. Net force on block of mass m is more than ma
iii.  $a < \mu g$	c. Net force on block of mass m is less than ma
iv.  $a > \mu g$	d. Friction force on block of mass m is ma
	e. Contact force on block of mass m is more than ma

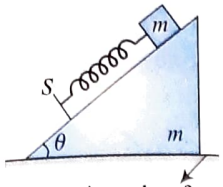
(1) (i – a,b,e); (ii – a,b,c,d); (iii – a,d,e); (iv – a,c)
(2) (i – a,e); (ii – a,c); (iii – a,b,c,d); (iv – b,c,d)
(3) (i – a,b,d); (ii – a,e); (iii – a,d,e); (iv – c)
(4) (i – a,e); (ii – a,e); (iii –a,d,e); (iv – c)

10. Column I describes the system and initial motion of its particles and column II gives characteristics associated with subsequent motion.

Column I	Column II
i. Two identical particles under influence of mutual gravitational force only. Particles are given initial velocity perpendicular to line joining as shown. 	a. Centre of mass of system does not accelerate.
ii. Two identical balls on a smooth table. Elastic head on collision takes place. 	b. In centre of mass frame momentum of system is zero.
iii. Two identical balls on a smooth table. Oblique elastic collision takes place. 	c. Momentum of system is conserved.
iv. Block is released on a wedge as shown in figure. All the surfaces are smooth. 	d. Mechanical energy of system is conserved
	e. Total kinetic energy is constant

(1) (i – a,b); (ii – b,c,d); (iii – a,b,c,d); (iv – b,d)
(2) (i – a,d); (ii – a,b,c,d); (iii – c,d); (iv – b,e)
(3) (i – a,b,c,d); (ii – a,b,c,d); (iii – a,b,c,d); (iv – b,d)
(4) (i – a,b,c,e); (ii – a,b,c,d); (iii – a,b); (iv – a,b)

11. A block of mass m is placed on wedge also of mass m. The wedge is placed on smooth horizontal fixed surface. One end of a light spring is connected to block and the other end to a light support S rigidly fixed to wedge as shown. Friction is absent everywhere. The system is initially released from rest with spring unstressed. Match statements in column I with corresponding statements in column II.

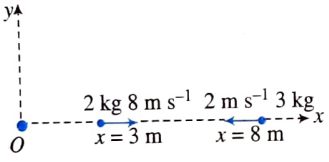


Column I	Column II
i. At the instant compression in spring is maximum	a. speed of block is zero
ii. At the instant spring has natural length that is, it is unstressed.	b. Speed of block is. non zero
iii. At the instant net force on wedge is zero	c. speed of block is maximum
iv. At the instant elastic potential energy stored in spring is least	d. speed of block is minimum

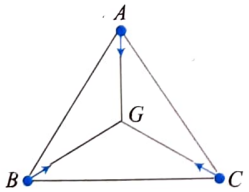
(1) (i – a); (ii – a,d); (iii – b); (iv – d)
(2) (i – a,d); (ii – a,d); (iii – b,c); (iv – a,d)
(3) (i – d); (ii – d); (iii – a,d); (iv – d)
(4) (i – a,d); (ii – b,c); (iii – a,d); (iv – b,d)

Numerical Value Type

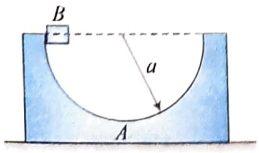
1. Figure shows position and velocities of two particles moving under mutual gravitational attraction in space at time $t = 0$. The position of centre of mass after one second is '*' m. Fill '*'.



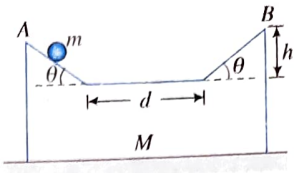
2. Three particles A, B and C of equal mass move with equal speed $V = 5\ ms^{-1}$ along the medians of an equilateral triangle as shown in the figure. They collide at the centroid G of the triangle. After the collision, A comes to rest, B retraces its path with the speed $V = 5\ ms^{-1}$. What is the speed of C?



3. Figure shows a wedge A of mass $6m$ smooth semicircular groove of radius $a = 8.4$ m placed on a smooth horizontal surface. A small block B of mass m is released from a position in groove where its radius is horizontal. Find the speed (in $m\,s^{-1}$) of bigger block when smaller block reaches its bottommost position.

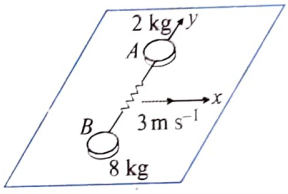


4. A ball of mass m is allowed to roll down the wedge of mass $M = 2m$ as shown in the figure. What is the displacement of wedge (in m) when the ball reaches from A to B? Take $\theta = 45°$, $h = 1$ m, $d = 4$ m.

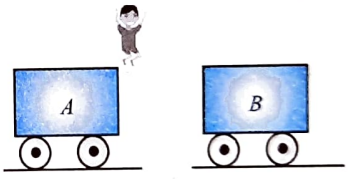


5. A frog sits on the end of a long board of length $L = 5$ m. The board rests on a frictionless horizontal table. The frog wants to jump to the opposite end of the board. What is the minimum take-off speed (in m/s), i.e., relative to ground that allows the frog to do the trick? The board and the frog have equal masses.

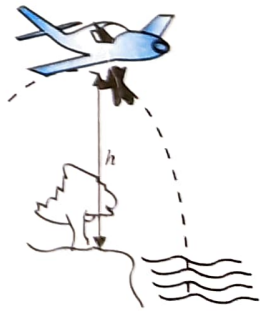
6. A massless spring of force constant $1000\ Nm^{-1}$ is compressed a distance of 20 cm between discs of 8 kg and 2 kg, spring is not attached to discs. The system is given an initial velocity $3\ ms^{-1}$ perpendicular to length of spring as shown in the figure. What is ground frame velocity of 2 kg block (in ms^{-1}) when spring regains its natural length.



7. A child of mass 4 kg jumps from cart B to cart A and then immediately back to cart B. The mass of each cart is 20 kg and they are initially at rest. In both the cases the child jumps at 6 m/s relative to the cart. If the cart moves along the same line with negligible friction with the final velocities of v_B and v_A, respectively, find the ratio of $6v_B$ and $5v_A$.



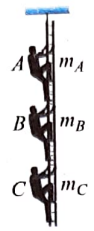
8. A man of mass $M = 58$ kg jumps from an aeroplane as shown in figure. He sees the hard ground below him and a lake at a distance $d = 1$ m from the point directly below him. He immediately puts off his jacket (mass $m = 2$ kg) and throws it in a direction directly away from the lake. If he just fails to strike the ground, find the distance (in 10^1 m) he should walk now to pick his jacket. (Neglect air resistance and take the velocity of man at the time of jump with respect to earth zero.)



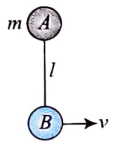
9. A man standing on a trolley pushes another identical trolley (both trolleys are at rest on a rough surface), so that they are set in motion and stop after some time. If the ratio of mass of first trolley with man to mass of second trolley is 3, then find the ratio of the stopping distances of the second trolley to that of the first trolley. (Assume coefficient of friction to be the same for both the trolleys)

10. A railway flat car, whose mass together with the artillery gun is $M = 2\ m$, moves at a speed V. The gun barrel makes an angle $\alpha = 60°$ with the horizontal. A shell of mass m leaves the barrel at a speed $v = 12\ ms^{-1}$, relative to the barrel. Find the speed of the flat car V (in ms^{-1}) in order that it may stop after the firing.

11. A ladder of mass 20 kg is hanging from ceiling as shown in figure. Three men A, B and C of masses 40 kg, 60 kg and 50 kg are climbing the ladder. Man A is climbing with upward retardation 2 m/s², B is climbing up with a constant speed of 0.5 m/s and C is climbing with upward acceleration of 1 m/s². Find the tension in the string supporting the ladder (in N).

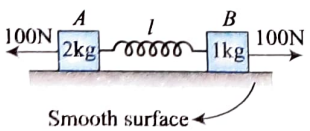


12. Two masses 'A' and 'B' having mass m and $2m$ respectively are kept on a smooth horizontal table as shown in the figure. The mass 'B' is given a velocity v perpendicular to the string parallel to the table. If $m = 0.3$ kg, $v = 5$ m/s and $l = 1$ m, then find the tension in the string (in N).

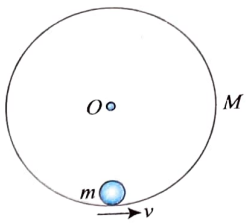


13. In the figure shown a small block B of mass m is released from the top of a smooth movable wedge A of the same mass m. The height of wedge -shown in figure is $h = 100$ cm. B ascends another movable smooth wedge C of the same m. Neglecting friction any where, find the maximum height (in cm) attained by block B on wedge C.

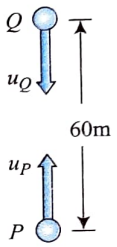
14. In the figure shown initially spring is in relaxed state and blocks are at rest. Now 100 N force is applied on block A and B as shown in figure. After some time velocity of 'A' becomes 2 m/s and that of 'B' 4 m/s and block A displaced by amount 10 cm towards left and spring is stretched by amount 30 cm. Then work done by spring force on A is W_0 joule. Find $|W_0|$ (in J).



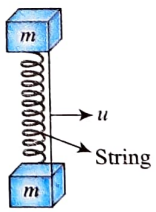
15. A ring of mass $M = 90$ gram and radius $R = 3$ meter is kept on a frictionless horizontal surface such that its plane is parallel to horizontal plane. A particle of mass $m = 10$ gram is placed in contact with the inner surface of ring as shown figure. An initial velocity $v = \sqrt{2}$ m/s is given to the particle along the tangent of the ring. Find the magnitude of the force of interaction (in milli-newton) between the particle and ring.



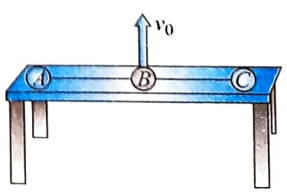
16. Two particles P and Q have mass 4.0 kg and 1 kg respectively. They are projected along a vertical line with velocity u_P = 20 m/s and $u_Q = 5$ m/s when separation between them was 100 m. P was projected vertically up while Q was projected vertically down. Calculate the maximum height attained (in m) by the centre of mass of the system of two particles measured from the initial position of P. Assume that the particles do not collide and that the ground is far below their point of projection [$g = 10\ m/s^2$].



17. Two blocks of masses $m = 2$ kg and $M = 8$ kg are connected to a spring of force constant $k = 1$ kN/m. The spring is compressed by 20 cm and the two blocks are held in this position by a string. The system is placed on a horizontal smooth surface and given a velocity $u = 3$ m/s perpendicular to the spring. The string snaps while moving. The speed (in m/s) of the block of mass m when the spring regains its natural length is



18. Three identical particles are placed on a horizontal smooth table, connected with strings as shown. The particle B is imparted a velocity $v_0 = 9$ m/s in horizontal direction perpendicular to the line ABC. The speed (in m/s) of particle A when it is about to collide with C.



Archives

JEE MAIN

Single Correct Answer Type

1. Distance of the centre of mass of a solid uniform cone from its vertex is z_0. If the radius of its base is R and its height is h then z_0 is equal to

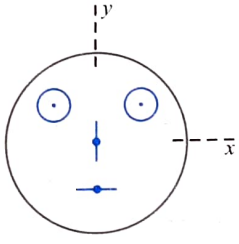
(1) $\frac{h^2}{4R}$ (2) $\frac{3h}{4}$

(3) $\frac{5h}{8}$ (4) $\frac{3h^2}{8R}$ **(JEE Main 2015)**

JEE ADVANCED

Single Correct Answer Type

1. Look at the drawing given in the figure which has been drawn with ink of uniform line-thickness. The mass of ink used to draw each of the two inner circles, and each of the two line segments is m. The mass of the ink used to draw the outer circle is $6m$.

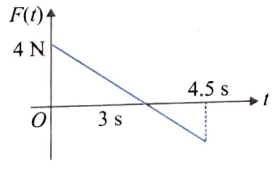


The coordinates of the centres of the different parts are: outer circle (0, 0), left inner circle $(-a, a)$, right inner circle (a, a), vertical line (0, 0) and horizontal line $(0, -a)$. The y-coordinate of the centre of mass of the ink in this drawing is

(1) $\frac{a}{10}$ (2) $\frac{a}{8}$

(3) $\frac{a}{12}$ (4) $\frac{a}{3}$ **(IIT-JEE 2009)**

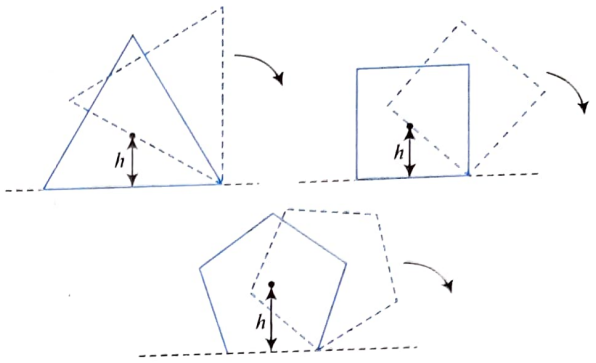
2. A block of mass 2 kg is free to move along the x-axis. It is at rest and from $t = 0$ onwards it is subjected to a time-dependent force $F(t)$ in the x direction. The force $F(t)$ varies with t as shown in the figure. The kinetic energy of the block after 4.5 seconds is



(1) 4.50 J (2) 7.50 J

(3) 5.06 J (4) 14.06 J **(IIT-JEE 2010)**

3. Consider regular polygons with number of sides $n = 3, 4, 5$ ..... as shown in the figure. The centre of mass of all the polygons is at height h from the ground. They roll on a horizontal surface about the leading vertex without slipping and sliding as depicted. The maximum increase in height of the locus of the centre of mass for each polygon is Δ. Then Δ depends on n and h as:



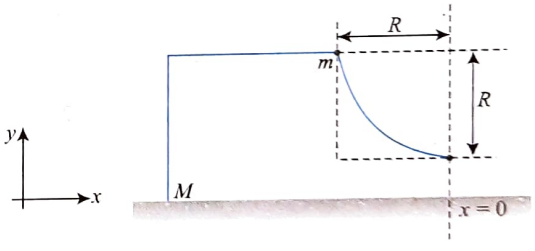
(1) $\Delta = h\sin^2\left(\frac{\pi}{n}\right)$ (2) $\Delta = h\sin\left(\frac{2\pi}{n}\right)$

(3) $\Delta = h\left(\frac{1}{\cos\left(\frac{\pi}{n}\right)} - 1\right)$ (4) $\Delta = h\tan^2\left(\frac{\pi}{2n}\right)$

(JEE Advanced 2017)

Multiple Correct Answers Type

1. A block of mass M has a circular cut with a frictionless surface as shown. The block rests on the horizontal frictionless surface of a fixed table. Initially the right edge of the block is at $x = 0$, in a coordinate system fixed to the table. A point mass m is released from rest at the topmost point of the path as shown and it slides down.



When the mass loses contact with the block, its position is x and the velocity is v. At that instant, which of the following options is/are correct?

(1) The velocity of the point mass m is $v = \sqrt{\frac{2gR}{1+\frac{m}{M}}}$

(2) The velocity of the block M is $V = -\frac{m}{M}\sqrt{2gR}$

(3) The position of the point mass is $x = -\sqrt{2}\frac{mR}{M+m}$

(4) The x component of displacement of the centre of mass of the block M is $-\frac{mR}{M+m}$ **(JEE Advanced 2017)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	2. (3)	3. (4)	4. (2)	5. (2)
6. (2)	7. (2)	8. (1)	9. (3)	10. (2)
11. (3)	12. (1)	13. (4)	14. (4)	15. (4)
16. (4)	17. (1)	18. (1)	19. (3)	20. (1)
21. (4)	22. (4)	23. (2)	24. (2)	25. (2)
26. (3)	27. (2)	28. (1)	29. (4)	30. (3)
31. (3)	32. (3)	33. (2)	34. (2)	35. (3)
36. (4)	37. (3)	38. (4)	39. (4)	40. (4)
41. (4)	42. (2)	43. (2)	44. (3)	45. (2)
46. (1)	47. (2)	48. (1)	49. (2)	50. (3)
51. (3)	52. (3)	53. (3)	54. (2)	55. (2)
56. (3)	57. (4)			

Multiple Correct Answers Type

1. (1),(3)	2. (2),(3)	3. (1),(2),(3),(4)
4. (2),(4)	5. (1),(3),(4)	6. (1),(4)
7. (2),(3)	8. (1),(4)	9. (1),(2),(3),(4)
10. (3),(4)	11. (1),(3),(4)	12. (1),(2),(3)
13. (1),(2)	14. (1),(2),(3),(4)	15. (1),(2),(3),(4)
16. (1),(3)	17. (1),(2),(4)	18. (1),(3),(4)
19. (1),(2),(3),(4)	20. (1),(2),(3)	21. (1),(2),(4)
22. (2),(3)	23. (1),(2),(3)	24. (1),(3)

Linked Comprehension Type

1. (1)	2. (2)	3. (4)	4. (3)	5. (1)
6. (4)	7. (1)	8. (4)	9. (3)	10. (1)

11. (1)	12. (4)	13. (1)	14. (3)	15. (2)
16. (3)	17. (3)	18. (1)	19. (4)	20. (1)
21. (3)	22. (1)	23. (1)	24. (1)	

Matrix Match Type

1. i. → b., c.; ii. → a., d.; iii. → c.; iv. → a.
2. i. → d.; ii. → d.; iii. → c.; iv. → a., b., c.
3. i. → a., c., d.; ii. → a., b., c., d.; iii. → a.; iv. → a., b.
4. i. → b.; ii. → a.; iii. → c.; iv. → a., b., d.
5. i. → a.; ii. → b.; iii. → a., c.; iv. → b., d.
6. i. → b.; ii. → a., b.; iii. → c.; iv. → d.

7. (1)	8. (2)	9. (4)	10. (3)	11. (2)

Numerical Value Type

1. (8)	2. (5)	3. (2)	4. (2)	5. (5)
6. (5)	7. (1)	8. (3)	9. (9)	10. (2)
11. (1670)	12. (5)	13. (25.00)	14. (6)	15. (6)
16. (38.75)	17. (5.00)	18. (6.00)		

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2)

JEE Advanced

Single Correct Answer Type

1. (1)	2. (3)	3. (3)

Multiple Correct Answers Type

1. (1),(4)

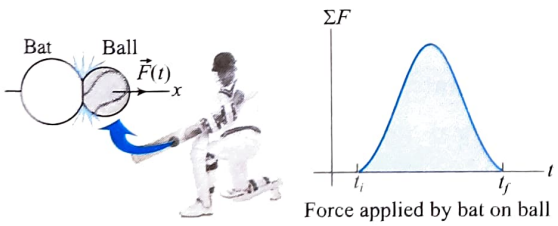
2 Impulse and Collision

INTRODUCTION

In daily life, collision is the most common phenomenon. In collision process two objects may or may not touch physically. They interact for a short time for exchanging their momentum and energy. The force that two colliding bodies exert on one another acts only for a short time, giving a brief but strong push. This type of force is called an impulsive force. In other words, impulsive force is a huge force acting for a very small time on a body. In this chapter we will learn about impulsive forces and collision.

IMPULSE

There are many situations in which the force acting on an object is not constant but varies with time. For example, when batsman hits the ball, the period of contact is very less but force is huge, Fig. (a) shows a cricket ball being hit by bat, and Fig. (b) illustrates approximately how the force applied to the ball by the bat changes during the time of contact. The magnitude of the force is zero at the instant t_i (just before the bat touches the ball). During contact, the force rises to a maximum and then returns to zero at the time t_f (when the ball leaves the bat). The time interval $\Delta t = t_f - t_i$ during which the bat and ball are in contact is quite short, being only a few thousandths of a second, although the maximum force can be very large, often exceeding thousands of newtons.



(a) **(b)**

The ball experiences a force $\vec{F}(t)$ that varies during the collision and changes the linear momentum $\vec{p}$ of the ball. That change is related to the force by Newton's second law written in the form $\vec{F} = d\vec{p}/dt$. By rearranging this second-law expression, we see that, in time interval dt, the change in the ball's momentum is

$$d\vec{p} = \vec{F}(t)\,dt \qquad \text{...(i)}$$

If we integrate both sides of Eq. (i) from a time t_i just before the collision to a time t_f just after the collision:

$$\int_{p_i}^{p_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\,dt \qquad \text{...(ii)}$$

The left side of this equation gives us the change in momentum: $\vec{p}_f - \vec{p}_i = \Delta\vec{p}$. The right side, which has a measure of both the magnitude and the duration of the collision force, is called the **impulse** $\vec{J}$ of the collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}dt \qquad \text{...(iii)}$$

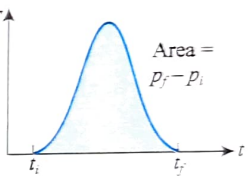
Thus, the change in an object's momentum is equal to the impulse on the object, which is also area under force-time graph. Equation (iii) can also be written in the vector form,

$$m\vec{v}_f = m\vec{v}_i + \int_{t_i}^{t_f} \vec{F}dt \text{ or } m\vec{v}_f = m\vec{v}_i + \vec{J} \qquad \text{...(iv)}$$

Equation (iv) is called Impulse–Momentum equation.

INSTANTANEOUS IMPULSE

There are many occasions when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown, the value of their product (i.e., impulse) can be known by measuring the initial and final momenta. Thus, we can write



$$\vec{J} = \int \vec{F}\,dt = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

Regarding the impulse, it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under the force–time (F–t) graph in the same time interval.

Important Points:

- Magnitude of impulse is equal to area under the $F-t$ graph. $J = \int F dt = F_{av}\Delta t$.
- It is not a property of any particle, but it is a measure of the degree to which an external force changes the momentum of the particle.

ILLUSTRATION 2.1

The hero of a stunt film fires 50 g bullets from a machine gun, each at a speed of 1.0 km/s. If he fires 20 bullets in 4 s, what average force does he exert against the machine gun during this period?

Sol. The linear momentum of each bullet

$$p = mv = 0.050 \times 1000 = 50 \text{ kg m/s}$$

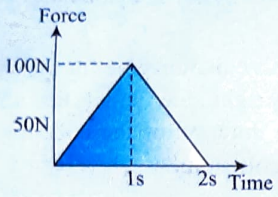
is imparted to gun. Thus, the rate of change of linear momentum of the gun is

$$\frac{dp}{dt} = \frac{50 \times 20}{4} = 250 \text{ N}$$

In order to hold the gun, the hero must exert a force of 250 N against the gun.

ILLUSTRATION 2.2

A box of mass m = 10 kg resting on a frictionless horizontal ground is acted upon by a horizontal force F, which varies as shown. Find speed of the particle when the force ceases to act.



Sol. The area under force–time graph gives the impulse. Hence impulse acting on the box

$$\vec{J} = \int_{t_i}^{t_f} \vec{F} dt = \text{Area under the graph} = \frac{1}{2} \times 100 \times 2 = 100 \text{ N-s}$$

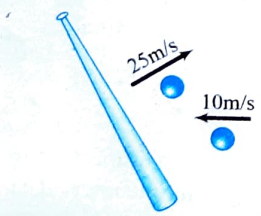
Using Impulse–Momentum equation:

$$m\vec{v}_f = m\vec{v}_i + \int_{t_i}^{t_f} \vec{F} dt \text{ or } m\vec{v}_f = m\vec{v}_i + \vec{J}$$

$$10 \times v = 10 \times 0 + 100 \Rightarrow v = 10 \text{ m/s}$$

ILLUSTRATION 2.3

A ball of mass 200 g is moving horizontally with a speed of 10 m/s. It is struck by a bat, as a result it starts moving with a speed of 25 m/s at an angle of 37° above the horizontal in the same vertical plane as shown in the figure.



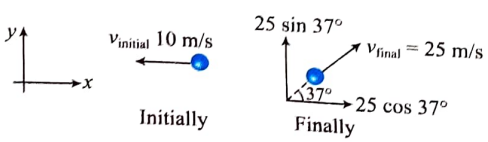
(a) Find the average force exerted by the bat if duration of impact is 0.50 s.

(b) Find the average force exerted by the bat if duration of impact is 0.05 s.

(c) Find the average force exerted by the bat if duration of impact is 0.005 s.

(d) What do you conclude for impulse of weight of the ball as duration of contact decreases?

Sol. Initially the ball is moving horizontally with 10 m/s after collision with bat, it starts moving with a speed of 25 m/s at an angle of 37° above the horizontal



Initial linear momentum of the ball,

$$\vec{p}_{initial} = (200 \times 10^{-3} \text{ kg}) \times 10 \text{ m/s } (-\hat{i}) = -2\hat{i} \text{ N-s}$$

Final linear momentum of the ball,

$$\vec{p}_{final} = (200 \times 10^{-3} \text{ kg}) \times (25 \cos 37^\circ \hat{i} + 25 \sin 37^\circ \hat{j}) \text{ m/s}$$

$$\Rightarrow \vec{p}_{final} = (4\hat{i} + 3\hat{j}) \text{N-s}$$

Now applying principle of impulse and momentum

$$\vec{p}_{final} = \vec{p}_{initial} + \vec{J}$$

or $\vec{J} = (\vec{p}_{final} - \vec{p}_{initial}) = (4\hat{i} + 3\hat{j}) - (-2\hat{i}) = (6\hat{i} + 3\hat{j}) \text{N-s}$

The average force exerted by the bat on ball,

$$\vec{F}_{av} = \frac{\vec{J}}{\Delta t} = \frac{(6\hat{i} + 3\hat{j})}{\Delta t} \text{ N} \quad \ldots(i)$$

(a) Substituting Δt = 0.50 s, in Eq. (i) we get

$$\vec{F}_{av} = \frac{\vec{J}}{\Delta t} = \frac{(6\hat{i} + 3\hat{j})}{0.5} = 12\hat{i} + 6\hat{j} \text{ N}$$

(b) Substituting Δt = 0.05 s, in Eq. (i) we get

$$\vec{F}_{av} = \frac{\vec{J}}{\Delta t} = \frac{(6\hat{i} + 3\hat{j})}{0.05} = 120\hat{i} + 60\hat{j} \text{ N}$$

(c) Substituting Δt = 0.005 s, in Eq. (i) we get

$$\vec{F}_{av} = \frac{\vec{J}}{\Delta t} = \frac{(6\hat{i} + 3\hat{j})}{0.005} = 1200\hat{i} + 600\hat{j} \text{ N}$$

(d) It is clear from the above results that as the duration of contact between the ball and the bat decreases, effect of the weight of the ball also decreases as compared with that of the force of the bat and for sufficiently short time interval, it can be neglected.

IMPULSIVE FORCE

A force of relatively higher magnitude and acting for relatively shorter time is called impulsive force. An impulsive force can change the momentum of a body in a finite magnitude in a very short time interval. Impulsive force is a relative term. There is no clear boundary between an impulsive and a non-impulsive force.

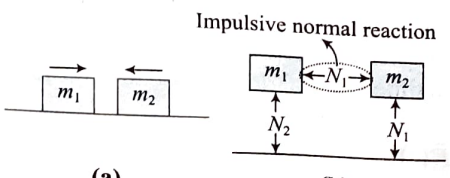
Note: Usually colliding forces are impulsive in nature.

Since the application time is very small, hence, very little motion of the particle takes place.

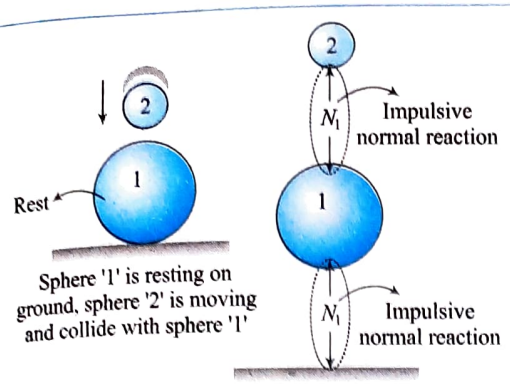
Important Points:

- Gravitational force and spring force are always non-impulsive.
- Normal, tension and friction are case dependent.
- An impulsive force can only be balanced by another impulsive force.

1. **Impulsive normal:** In case of collision, normal forces at the surface of collision are always impulsive.

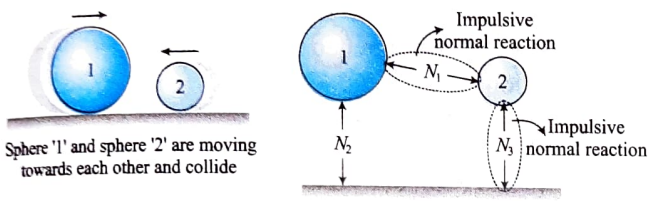


In Figs. (a) and (b), N_1 is impulsive; N_2 is non-impulsive. In Figs. (c) and (d), both the normals are impulsive.



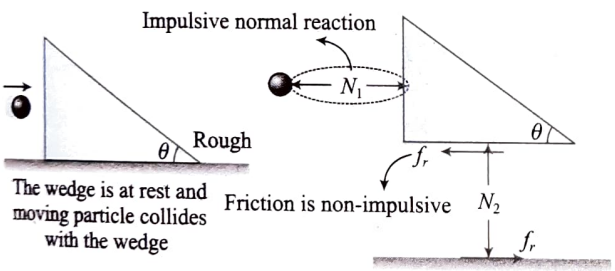
(c) (d)

In Figs. (e) and (f), the normal reaction between the spheres (N_1) is impulsive. This impulsive reaction is inclined with horizontal. Its vertical component of this impulsive reaction is in vertical up direction on the sphere '1', while the vertical component of this impulsive reaction will be in vertical downward direction of sphere '2'. Hence the normal reaction between sphere '2' and ground will be impulsive.



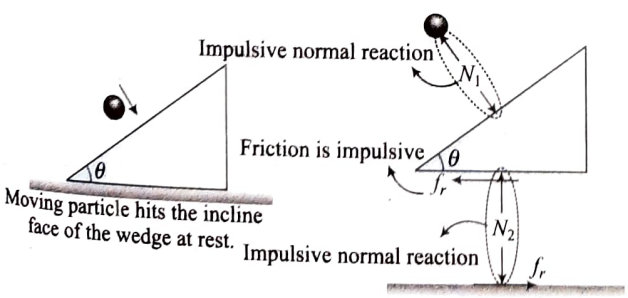
(e) (f)

2. **Impulsive friction:** If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.



(a) (b)

The impulsive normal reaction (N_1) is in horizontal direction. Hence the normal reaction between ground and wedge (N_2) is non-impulsive. It means here friction should be non-impulsive.



(c) (d)

In this case the vertical component of impulsive normal reaction (N_1) will be responsible for impulsive normal reaction between ground and wedge (N_2). Hence friction force between wedge and ground will be impulsive in nature.

3. **Impulsive tension:** When a string jerks, equal and opposite tension acts suddenly at each end. Consequently, equal and opposite impulses act on the bodies attached with the string in the direction of the string. There are two cases to be considered.

Note: In case of rod, tension is always impulsive. In case of spring, tension is always non-impulsive.

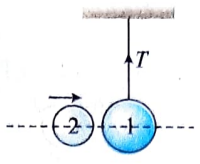
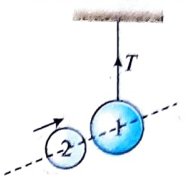
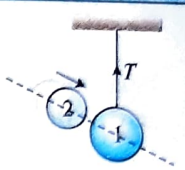
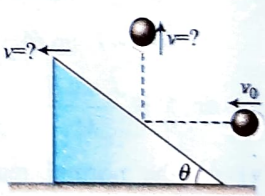
Case (a)	Case (b)	Case (c)
		
T is non-impulsive	T is non-impulsive	T is non-impulsive
The impulsive normal reaction between the spheres is in horizontal direction. It will not have any component in vertical downward direction, hence the tension in this case non-impulsive.	The impulsive normal reaction between the spheres will have component in vertical up direction on sphere '1'. The string will slack. Hence no impulsive reaction.	The impulsive normal reaction between the spheres has component in vertical downward direction on sphere '1'. Hence tension in the string should be impulsive in nature.

ILLUSTRATION 2.4

A right angled wedge of mass M and base angle θ resting over a smooth horizontal surface. A small ball of mass m, moving horizontally with velocity v_0 collide with wedge on sloping surface. If after collision the ball starts moving in vertical, upward direction, then calculate:

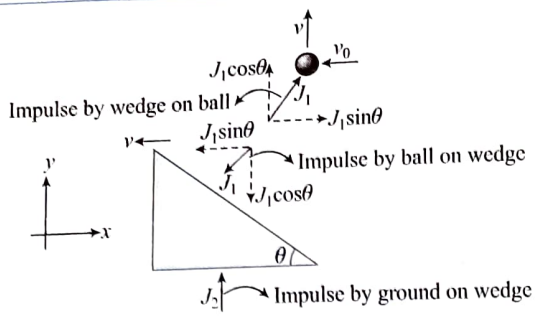


(i) impulse of normal reaction applied by wedge on the ball.
(ii) impulse of normal reaction applied by ground on wedge
(iii) velocity of the ball just after collision.
(iv) velocity of the wedge just after collision.
(v) If the coefficient of friction between ground and wedge is μ. Then calculate frictional impulse.

Sol.

(i) In this case both the normal reactions (between the ball and wedge and between wedge and ground) will be impulsive in nature. We know impulse $\vec{J} = \int_0^{\Delta t} \vec{F} \cdot dt$

Hence the direction of impulse will be same as the direction of impulsive force, here in this case it is normal reaction. Making impulsive diagram of the situation (impulsive diagram will be same as FBD in laws of motion)



Now applying impulse momentum equation for ball,

In x-direction, $(\vec{p}_{\text{initial}})_x + (\vec{J}_1)_x = (\vec{p}_{\text{final}})_x$

$$-mv_0 + J_1 \sin\theta = 0 \quad \text{...(i)}$$

Final linear momentum of the ball in x-direction will be zero as the ball starts moving in upward direction.

Hence from (i), $J_1 = \dfrac{mv_0}{\sin\theta} = mv_0 \operatorname{cosec}\theta$

(ii) As wedge is not moving in vertical direction, then from impulse diagram of wedge

$$J_2 = J_1 \cos\theta = mv_0 \cot\theta$$

(iii) The ball just after collision starts moving up. Now we can apply impulse momentum equation in y-direction

$$(\vec{p}_{\text{initial}})_y + (\vec{J}_1)_y = (\vec{p}_{\text{final}})_y$$

$$0 + J_1 \cos\theta = mv$$

or $v = \dfrac{J_1\cos\theta}{m} = \dfrac{mv_0\cot\theta}{m} \Rightarrow v = v\cot\theta$

(iv) Just after collision the wedge starts moving towards left. Now we can apply impulse–momentum equation for wedge in horizontal direction.

$$(\vec{p}_{\text{initial}})_x + (\vec{J}_1)_x = (\vec{p}_{\text{final}})_x$$

$$0 + (-J_1\sin\theta) = -MV$$

or $V = \dfrac{J_1\sin\theta}{M} = \dfrac{mv_0}{M}$

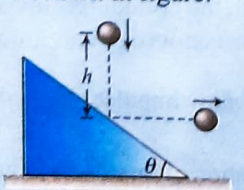
(v) As the normal reaction between ground and wedge is impulsive, it means the friction force between the wedge and ground will also be impulsive. Hence the impulse of friction force will be

$$J_{\text{friction}} = \mu J_2 = \mu(mv_0\cot\theta)$$

or frictional impulse $J_{\text{friction}} = \mu mv_0\cot\theta$

ILLUSTRATION 2.5

A particle of mass m is released from height h on an inclined plane. If after collision with the plane, the particle starts moving horizontally as shown in figure.



Find:

(i) velocity of the particle just after collision.

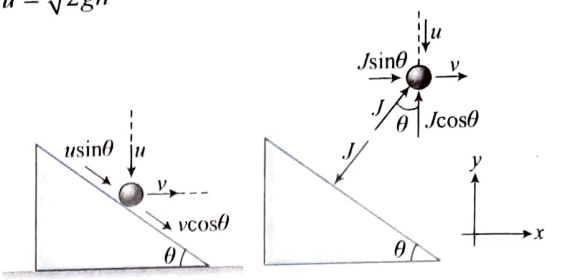
(ii) impulse provided by the plane to the particle.

(iii) loss of the kinetic energy of the particle due to impact.

Sol.

(i) Velocity of the particle just before impact with the plane

$$u = \sqrt{2gh}$$



The linear momentum of the particle just after impact, in the direction normal to impulse will not change. Hence velocity of the particle parallel to inclined plane just before collision and just after collision should be same.

$$v\cos\theta = u\sin\theta \Rightarrow v = u\tan\theta$$

or $v = \sqrt{2gh}\tan\theta$

(ii) The particle just after collision started moving in horizontal direction. Hence the component of the velocity of the ball in vertical direction after impact should be zero. Using impulse momentum equation to the particle in vertical direction.

$$(\vec{p}_{\text{initial}})_y + \vec{J} = (\vec{p}_{\text{final}})_y$$

$$\Rightarrow -mu + J\cos\theta = 0$$

or $J = \dfrac{mu}{\cos\theta} = mu\sec\theta$ or $J = m\sqrt{2gh}\sec\theta$

(iii) Loss of kinetic energy due to impact

$$(\Delta K)_{\text{loss}} = K_{\text{initial}} - K_{\text{final}} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}m(\sqrt{2gh})^2 - \frac{1}{2}m(\sqrt{2gh})^2\tan^2\theta$$

$$= \frac{1}{2}m\cdot 2gh[1-\tan^2\theta]$$

or $(\Delta K)_{\text{loss}} = mgh\,[1-\tan^2\theta]$

ILLUSTRATION 2.6

A bob of mass M is suspended by an inextensible string. A particle of mass m, moving with velocity v_0 strike the bob as shown in the figure. If the particle comes to rest just after collision, then calculate:

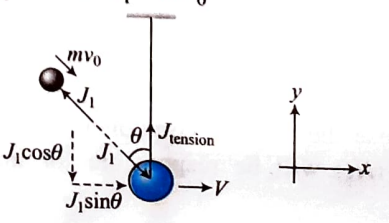
(i) impulse of tension in the string

(ii) velocity of the bob just after collision.

Sol.

(i) Here tension in the string will be impulsive in nature. Let us make the impulse diagram of the situation. Applying impulse momentum equation for the particle

$$mv_0 - J_1 = 0 \text{ or } J_1 = mv_0 \quad \text{...(i)}$$



As the string is inextensible, the bob cannot move in vertical direction. Hence, $J_{tension} = J\cos\theta$ or $J_{tension} = mv\cos\theta$.

(ii) The horizontal component of impulse ($J_1 \sin\theta$) will be responsible for motion of the bob in horizontal direction, hence

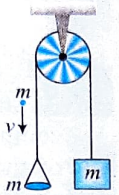
$$J\sin\theta = MV \text{ or } V = \frac{J_1\sin\theta}{M}$$

$$\Rightarrow V = \frac{mv_0\sin\theta}{M}$$

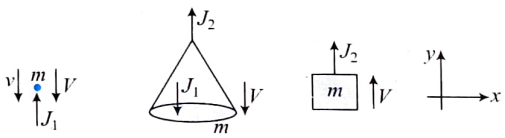
After collision the bob starts moving in the circular path.

ILLUSTRATION 2.7

A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially, the system is at rest, then a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v, find the speed with which the system moves just after the collision.



Sol. When the particle strike the pan the normal reaction between the pan and the particle is impulsive in nature. Hence tension in the string should also be impulsive. Let the speed of the system just after collision be V.



Impulse on the particle | Impulse on the pan | Impulse on the block

Writing impulse momentum equation

$$\vec{p}_{initial} + \vec{J} = \vec{p}_{final}$$

For the particle: $-mv + J_1 = -mV$ ...(i)

For the pan: $0 + J_2 - J_1 = -mV$

or $J_2 - J_1 = -mV$ ...(ii)

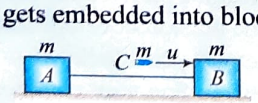
For the block: $0 + J_2 = mV$

or $J_2 = mV$ ...(iii)

From Eqs. (i), (ii) and (iii), $V = \frac{v}{3}$

ILLUSTRATION 2.8

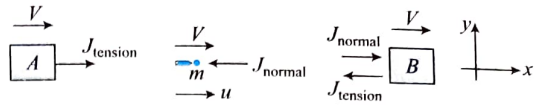
Two identical blocks A and B, connected by a massless string, are placed on a frictionless horizontal plane. A bullet having the same mass, moving with speed u, strikes block B from behind as shown. If the bullet gets embedded into block B, then find



(a) the velocity of A, B, C after collision;
(b) impulse on A due to tension in the string;
(c) impulse on C due to normal force of collision;
(d) impulse on B due to normal force of collision.

Sol.

(a) The normal reaction between bullet and block B is impulsive nature. Also the tension in string connecting the blocks 'A' and 'B' is also impulsive. Let the speed of the system after collision be V. Let us make impulse diagram of the bullet and blocks.



Impulse on the block 'A' | Impulse on the 'bullet' | Impulse on the block 'B'

Now writing impulse momentum equation

$$\vec{p}_{initial} + \vec{J} = \vec{p}_{final}$$

For block 'A': $0 + J_{tension} = mV$

or $J_{tension} = mV$ ...(i)

For bullet: $mu - J_{normal} = mV$ ...(ii)

For block 'B': $0 + J_{normal} - J_{tension} = mV$

or $J_{normal} - J_{tension} = mV$ ...(iii)

From (i), (ii) and (iii), $V = \frac{u}{3}$

Alternative: If we take both blocks and bullet as system, then both the tensile impulse and normal impulse will be internal to the system. Hence no external impulse acting on the system, then linear momentum of the system should be conserved.

$$\vec{p}_{initial} = \vec{p}_{final} \Rightarrow mu + m\times 0 + m\times 0 = (m+m+m)V$$

or $V = \frac{u}{3}$

(b) From impulse equation of the block 'A' [Eq. (i)]
Impulse on A due to tension in the string

$$J_{tension} = m\left(\frac{u}{3}\right) = \frac{mu}{3} \text{ (towards left)}$$

(c) From impulse equation of the bullet 'C' [Eq. (ii)]

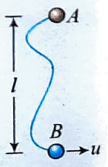
$$mu - J_{normal} = m\left(\frac{u}{3}\right)$$

or $J_{normal} = \frac{2mu}{3}$ (towards left)

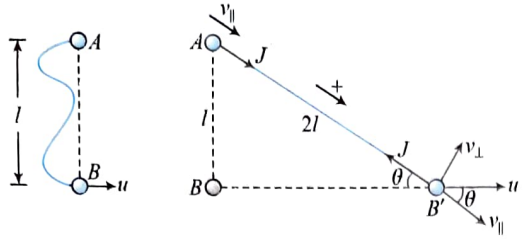
(d) The impulse on the block 'B' due to normal force of collision will be equal to the impulse on bullet i.e., $\frac{2mu}{3}$ (towards right).

ILLUSTRATION 2.9

Two particles A and B of equal mass m are attached by a string of length $2l$ and initially placed over a smooth horizontal table in the position shown in figure. Particle B is projected across the table with speed u perpendicular to AB as shown in the figure. Find the velocities of each particle after the string becomes taut and the magnitude of the impulse tension.



Sol. **1st Approach:** When the particle 'B' is projected on the table with velocity u, it moves with constant velocity till the string open to its full length. At this time the string becomes taut. The tension developed in the string at this state is impulsive in nature.



Initial configuration When string becomes taut

In $\Delta ABB'$: $\sin\theta = \frac{l}{2l} = \frac{1}{2} \Rightarrow \theta = 30°$

and $\cos\theta = \frac{\sqrt{(2l)^2 - l^2}}{2l} = \frac{\sqrt{3}}{2}$

As both the particles are connected by a string, the velocities of the particles along the string should be equal say ($v_{||}$). Now applying impulse momentum equation for the particles

For particle A: $J = mv_{||}$ ...(i)

For particle B: $mu\cos\theta - J = mv_{||}$ ...(ii)

From Eqs. (i) and (ii), $v_{||} = \frac{u\cos\theta}{2}$

or $v_{||} = \frac{\sqrt{3}u}{4}$

2nd approach: If we take both particles and string as a system. The impulse generated in the string will be internal to system. It means no external impulse on the system. Hence the linear momentum will be conserved in any direction parallel to horizontal table. Let us conserve the linear moment in the direction of length of string.

$\vec{p}_{\text{initial}} = \vec{p}_{\text{final}}$

$0 + mu\cos\theta = (m+m)v_{||}$

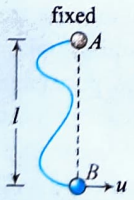
or $v_{||} = \frac{u\cos\theta}{2} = \frac{u\cos 30°}{2} = \frac{\sqrt{3}u}{4}$

The linear momentum of the ball B will not change in the direction normal to impulse. Hence the velocity of ball B will remains unchanged perpendicular to length of the string.

or $v_\perp = u\sin\theta = u\sin 30° = \frac{u}{2}$

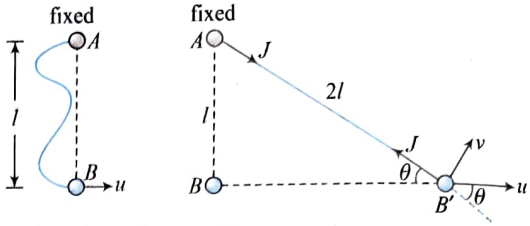
ILLUSTRATION 2.10

A string AB of length $2l$ is fixed at A to a point on smooth horizontal table. A particle of mass m attached to B is initially at a distance l from A as shown in figure. The particle is projected horizontally with speed u at right angles to AB



(a) Find the impulse of tension developed in the string.
(b) Velocity of the particle just after the string become taut.

Sol. The particle connected at B slides on the table with speed u till the string opens to its full length ($2l$). The tension developed in the string at this moment will be impulsive in nature. Let us make impulse diagram of this situation.



Initial configuration When string becomes taut

In $\Delta ABB'$, $\sin\theta = \frac{l}{2l} = \frac{1}{2} \Rightarrow \theta = 30°$

and $\cos\theta = \frac{\sqrt{(2l)^2 - l^2}}{2l} = \frac{\sqrt{3}}{2}$

(a) As string is inextensible and it is fixed at end A. Hence the component of the velocity of the particle along the length of the string should be zero. Using impulse momentum equation along the length of the string.

$mu\cos\theta - J = 0$

or $J = mu\cos\theta = \frac{mu\sqrt{3}}{2}$

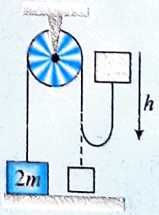
(b) The linear momentum of the particle normal to impulse direction will not change. Hence the velocity of the particle will remain unchanged perpendicular to the length of the string.

$v = u\sin\theta = u\cdot\frac{1}{2} = \frac{u}{2}$

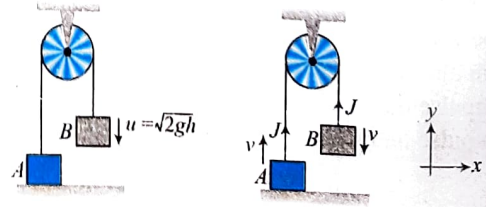
After this the ball starts moving in circular path of radius $2l$.

ILLUSTRATION 2.11

A mass $2m$ rests on a horizontal table. It is attached to a light inextensible string which passes over a smooth pulley and carries a mass m at the other end. If the mass m is raised vertically through a distance h and is then dropped, what is the speed with which the mass $2m$ begins to rise?



Sol. When the block B is released, it moves under gravity. After moving a distance h, it pull the string in downward direction. At this time tension developed in string is impulsive in nature. This impulse pulls the block 'A' in upward direction and both the blocks starts moving with common velocity.



Just before string gets taut Just after string is taut

The velocity of block B just before the string becomes taut

$$u = \sqrt{2gh} \qquad \ldots(i)$$

Now using impulse momentum equation

$$\vec{p}_{initial} + J = \vec{p}_{final}$$

For block A: $0 + J = 2mv$

or $J = mv$ $\qquad \ldots(ii)$

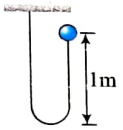
For block B: $-mu + J = -mv$ $\qquad \ldots(iii)$

From Eqs. (i), (ii) and (iii); $v = \frac{u}{3} = \frac{\sqrt{2gh}}{3}$

Hence the block A starts moving up with velocity $\frac{\sqrt{2gh}}{3}$.

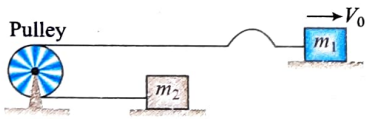
CONCEPT APPLICATION EXERCISE 2.1

1. A ball of mass 1 kg is attached to an inextensible string. The ball is released from the position shown in figure. Find the impulse imparted by the string to the ball immediately after the string becomes taut.

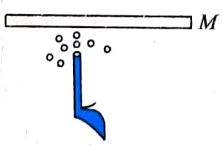


2. A sphere of mass m slides with velocity v on a frictionless surface towards a smooth inclined wall as shown in figure. If the collision with the wall is perfectly elastic, find:
   (a) the impulse imparted by the wall on the sphere,
   (b) the impulse imparted by the floor on the sphere.

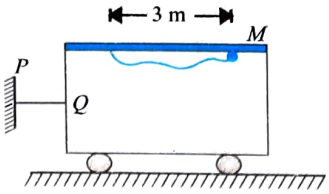
3. Two particles of masses m_1 and m_2 are connected by a light and inextensible string which passes over a fixed pulley. Initially, the particle m_1 moves with a velocity v_0 when the string is not taut. Neglecting friction in all contacting surfaces, find the velocities of the particles m_1 and m_2 just after the string is taut.



4. A 3.00 kg steel ball strikes the wall with a speed of 10.0 m/s at an angle of 60° with the surface. It bounces off with the same speed and angle. If the ball is in contact with the wall for 0.200 s, what is the average force exerted by the wall on the ball?
5. A machine gun can fire bullets of 50 grams at a speed of 2000 m s^{-1}; the man holding the gun can exert an average force of 200 N against the gun. Calculate the maximum number of bullets which he can fire per minute.
6. A plate of mass M is held at rest by firing bullets from below. Each bullet has a mass m, velocity u (up) just before hitting and stops after hitting the plate for a moment and falls. Determine the number of bullets striking the plate per unit time.



7. A particle of mass 1 kg is attached to a string of length 5 m. The string is attached to a fixed point O. It is released from the position as shown in figure. Calculate



   (a) the impulse developed in the string when it becomes taut;
   (b) the velocity of the particle just after the string becomes taut;
   (c) the impulse developed in this string PQ at this instant.

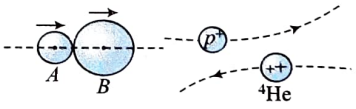
ANSWERS

1. $2\sqrt{10}$ kg m/s
2. (a) $2mv$ cosec θ (b) $2mv \cot\theta$
3. $v = \frac{m_1 v_0}{m_1 + m_2}$
4. 260 N
5. 120
6. $\frac{Mg}{mu}$
7. (a) $\frac{16}{\sqrt{5}}$ kg m/s (b) $\frac{12}{\sqrt{5}}$ m/s (c) $\frac{48}{5\sqrt{5}}$ kg m/s

COLLISION OR IMPACT

The term **collision** represents an event during which two particles come close to each other and interact by means of forces. The interaction forces are assumed to be much greater than any external forces present, so we can use the impulse approximation. The basic idea of a 'collision' is that the motion of the colliding particles (or of at least one of them) changes rather abruptly and that we can make a relatively clean separation of times that are 'before the collision' and that are 'after the collision'.

During collision, two particles may or may not touch physically, they interact for a short time exchanging their momenta and energy, e.g., when an alpha particle (He^4) 'collides' with the nucleus of gold (Au^{197}), the force acting between them being repulsive-the particles may not touch, even then it may be called a 'collision'



The collision is, in fact, a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles.

Note:

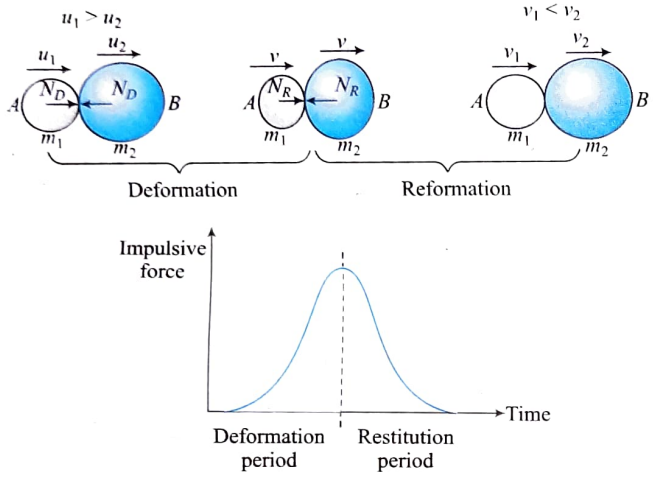
- The duration of collision Δt is negligible as compared to the usual time intervals of observation of motion.
- In a collision, the effects of external non-impulsive forces such as gravity are not taken into account as due to small duration of collision (Δt). Average impulsive force responsible for collision is much larger than external forces acting on the system.

TYPES OF COLLISION

On the Basis of Energy

Consider two bodies A and B are moving along a straight line with velocities u_1 and u_2 respectively. The velocity of body 'A' (u_1) is greater than the velocity of the body 'B' (u_2) for two bodies to collide.

After some time the two bodies will comes in contact. As the velocity of the body 'A' is greater than that of body 'B', there will be deformation in the bodies. This deformation continuous till they acquire same velocity.



When two bodies collide, the impulsive forces may vary in time in complicated way. Regardless of the complexity of the time behavior of the impulsive force, however, this force is internal to the system of two particles. Therefore, the two particles form an isolated system and the momentum of the system must be conserved in any collision.

In contrast, the total kinetic energy of the system of particles may or may not be conserved, depending on the type of collision.

If the bodies possess elastic property they will try to regain their shape that is the elastic potential energy stored in deformation period will get converted into kinetic energy. If bodies are perfectly elastic than total potential energy will converted into kinetic energy. This type of the collision is known as **perfectly elastic collision**.

Thus in a perfectly elastic collision, the particles regain their shape and size completely after collision. That is, no fraction of mechanical energy remains stored as deformation potential energy in the bodies. It means, kinetic energy of a system after collision is equal to kinetic energy of a system before collision. Thus, in addition to the linear momentum, kinetic energy also remains conserved before and after collision.

If the bodies do not possess elastic properties, the two bodies will continue to move together and the process of collision ends here at maximum deformation, this is known as **perfectly inelastic collision**.

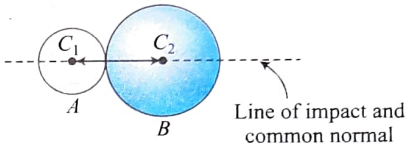
Actually, collision between all real objects is neither perfectly elastic nor perfectly inelastic, its inelastic in nature. Thus, in an inelastic collision, the particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external impulsive forces, law of conservation of linear momentum still holds good.

On the Basis of Impact

Let two bodies collide at a common point. The line passing through the common normal to the surfaces in contact during the impact is called **'line of impact'**. The impulsive force during collision acts along this line on both the bodies. Now let us categorize the collision into various types depending on the position of the centre of mass of the bodies relative to the line of impact as following.

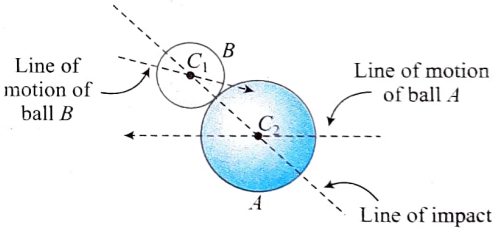
Central Impact (Collision)

When the line joining the CM of the bodies lies on the line of impact, we call the impact "central impact".

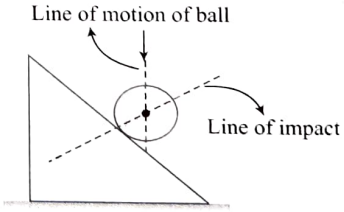


Oblique Impact (Collision)

When the line of motion of the bodies does not coincide with the line of impact, we call it "oblique impact".



(a)



(b)

Head-on Impact (Collision)

When the both lines of motion and lines of impact of the colliding bodies coincide, the impact is called direction-central impact or head-on collision. For instance, collision of two identical balls moving in same line can have head-on collision.

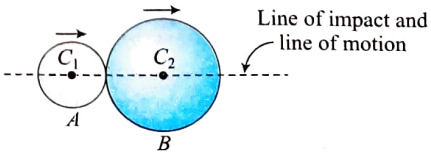
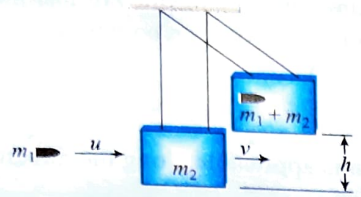


ILLUSTRATION 2.12

A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h. How can we determine the speed of the projectile from a measurement of h?



Sol. The bullet and the block form an isolated system in terms of linear momentum if we can conserve the linear momentum immediately before and immediately after the collision. Because the bullet imbeds in the block, we can categorize the collision between them as perfectly inelastic.

$$m_1u + m_2 \times 0 = (m_1 + m_2)v \Rightarrow v = \frac{m_1u}{(m_1 + m_2)} \quad ...(i)$$

After collision the impulsive forces disappear, now we can conserve the mechanical energy of the system (bullet + block).

$$\Delta K = \Delta U$$

$$\frac{1}{2}(m_1 + m_2)v^2 = (m_1 + m_2)gh \quad ...(ii)$$

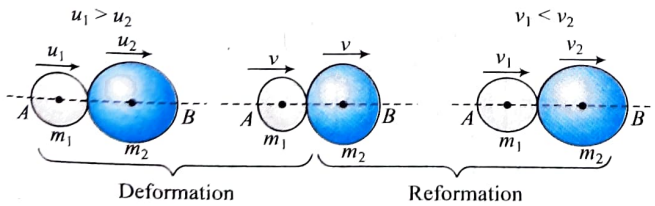
From Eqs. (i) and (ii), $u = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{2gh}$

COEFFICIENT OF RESTITUTION (e)

It is the measurement of degree of elasticity of a collision. The coefficient of restitution is defined as the ratio of the impulses of recovery and deformation of either body.

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r\,dt}{\int F_d\,dt}$$

Two smooth balls A and B approach each other such that their centers are moving along the line CD in the absence of external impulsive force. Let the velocities of A and B just before collision be u_1 and u_2, respectively, and the velocities of A and B just after collision be v_1 and v_2, respectively.



Since, linear momentum is conserved for the system.

$$\Rightarrow m_1u_1 + m_2u_2 = (m_1 + m_2)v = m_1v_1 + m_2v_2$$

$$\Rightarrow v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \quad ...(i)$$

Impulse of deformation:

J_D = change in momentum of any one body during deformation
$= m_2(v - u_2)$ for m_2
$= m_1(-v + u_1)$ for m_1

Impulse of reformation:

J_R = change in momentum of any one body during reformation
$= m_2(v_2 - v)$ for m_2
$= m_1(v - v_1)$ for m_1

$$e = \frac{\text{Impulse of reformation } (J_R)}{\text{Impulse of deformation } (J_D)} = \frac{v_2 - v}{v - u_2}$$

[substituting v from Eq. (i)]

$$e = \frac{v_2 - \left(\frac{m_1v_1 + m_2v_2}{m_1 + m_2}\right)}{\left(\frac{m_1u_1 + m_2u_2}{m_1 + m_2}\right) - u_2} = \frac{v_2(m_1 + m_2) - (m_1v_1 + m_2v_2)}{(m_1u_1 + m_2u_2) - u_2(m_1 + m_2)}$$

$$\Rightarrow e = \frac{m_1(v_2 - v_1)}{m_1(u_1 - u_2)} = \frac{(v_2 - v_1)}{(u_1 - u_2)} \quad \text{or} \quad e = \frac{v_2 - v}{v - u_2} = \frac{v_2 - v_1}{u_1 - u_2}$$

$$e = \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

This is also known as Newton's experimental law.

Note: e is independent of shape and mass of the object but depends on the material.

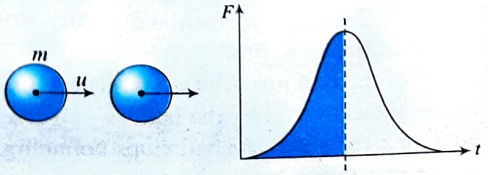
The coefficient of restitution is constant for two particular objects.

1. For $e = 1$
   - $\Rightarrow$ Impulse of reformation = Impulse of deformation
   - $\Rightarrow$ Velocity of separation = Velocity of approach
   - $\Rightarrow$ Kinetic energy is conserved
   - $\Rightarrow$ Elastic collision
2. For $e = 0$
   - $\Rightarrow$ Impulse of reformation = 0
   - $\Rightarrow$ Velocity of separation = 0
   - $\Rightarrow$ Kinetic energy is not conserved
   - $\Rightarrow$ Perfectly inelastic collision
3. For $0 < e < 1$
   - $\Rightarrow$ Impulse of reformation < Impulse of deformation
   - $\Rightarrow$ Velocity of separation < Velocity of approach
   - $\Rightarrow$ Kinetic energy is not conserved
   - $\Rightarrow$ Inelastic collision

Note: In case of contact collisions, e is always less than or equal to unity.

ILLUSTRATION 2.13

A sphere 'A' having mass m is moving with velocity u makes a head on collision with another sphere 'B'. After collision velocity of the sphere 'A' was found to be v in the direction of its original motion. The interaction force between the two spheres during their collision has been shown in the graph. The area of the shaded part of the graph is same as the area of the unshaded part. Find the velocity of the spheres at the instant they were having equal velocity.



Sol. The area under F-t graph is the impulse. Hence, the shaded area in the graph = Impulse during the period of deformation. And unshaded area = impulse during the period of restitution

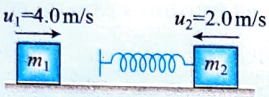
At the time of maximum deformation both the spheres move with common velocity v_0. Considering the impulse of sphere 'A'

$\therefore$ $(\text{Impulse})_{\text{deformation}} = (\text{Impulse})_{\text{restitution}}$

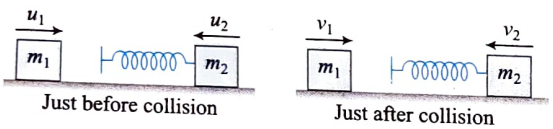
$$m(v_0 - u) = m(v - v_0) \quad \therefore \quad v_0 = \frac{u+v}{2}$$

ILLUSTRATION 2.14

A block of mass $m_1 = 2.0$ kg initially moving to right with a speed of $u = 4.0$ m/s on a frictionless, horizontal track, collides with a light spring attached to a second block of mass $m_2 = 1.0$ kg initially moving to left with a speed of $u_2 = 2.0$ m/s as shown in figure. Find the velocities of the blocks when spring again comes to original length.



Sol. If we take the 'block + spring' as a system, no external forces acting on the system. Hence linear momentum of the system will remain conserved. Let the velocities of the blocks when the spring is again relaxed be v_1 and v_2 respectively. Now using conservation of linear momentum



$$m_1u_1 - m_2u_2 = m_1v_1 + m_2v_2$$

$$2\times4 - 1\times2 = 2\times v_1 + 1\times v_2$$

or $2v_1 + v_2 = 6$ ...(i)

The collision of the spring block system can be treated as perfectly elastic collision. Hence we can use the coefficient of restitution equation

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

Here $e = 1$, $u_1 = 4.0$ m/s, $u_2 = -2.0$ m/s

Hence $1 = \dfrac{v_2 - v_1}{4-(-2)}$

or $v_2 - v_1 = 6$ ...(ii)

From Eqs. (i) and (ii), $v_1 = 0$ and $v_2 = 6$ m/s

ILLUSTRATION 2.15

A ball is released from rest from a height h_0. It collides with a horizontal inelastic floor many times and finally stops. If the coefficient of restitution is e, find

(i) the velocity of the ball just after n^{th} collision.

(ii) the total distance covered by the ball.

(iii) the total time after which the ball stops bouncing.

Sol.

(i) The velocity of the ball when it strikes the ground for the 1st time

$$u_0 = \sqrt{2gh_0} \text{ and } h_0 = \frac{1}{2}gt_0^2 \Rightarrow t_0 = \sqrt{\frac{2h_0}{g}}$$

For calculating velocity of the ball just after collision, we use coefficient of restitution equation

$$e = \left(\frac{v_2 - v_1}{u_1 - u_2}\right)_n$$

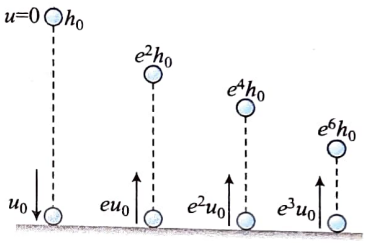
The equation is applicable along the direction of impulsive force.

Here $u_1 = u_0, u_2 = 0$; $v_1 = -u_1$ and $v_2 = 0$

Hence $e = \dfrac{0-(-u_1)}{u_0 - 0} \Rightarrow u_1 = eu_0 = e\sqrt{2gh_0}$

Similarly, velocity of ball just after second collision

$$u_2 = e\left(e\sqrt{2gh_0}\right) = e^2\sqrt{2gh_0}$$



Hence the velocity of ball just after n^{th} collision

$$u_n = e^n\sqrt{2gh_0}$$

(ii) The maximum height attain after 1st collision

$$h_1 = \frac{u_1^2}{2g} = \frac{e^2 \cdot 2gh_0}{2g} = e^2h_0$$

The distance covered between 1st and 2nd collision

$$d_1 = 2h_1 = 2e^2h_0$$

Similarly, $d_2 = 2h_2 = 2e^4h_0$

$d_3 = 2h_3 = 2e^6h_0$ and so on

The total distance covered,

$$D = h_0 + d_1 + d_2 + \ldots$$

$$= h_0 + 2e^2h_0 + 2e^4h_0 + \ldots$$

$$= h_0[1 + 2(e^2 + e^4 + \ldots)]$$

$$= h_0\left[1 + 2\frac{e^2}{1-e^2}\right] = \left(\frac{1+e^2}{1-e^2}\right)h_0$$

(iii) Time of flight of the ball after first collision,

$$t_1 = \frac{2u_1}{g} = \frac{2e\sqrt{2gh_0}}{g} = 2e\sqrt{\frac{2h_0}{g}}$$

Similarly time of flight after 2nd collision,

$$t_2 = \frac{2u_2}{g} = \frac{2e^2\sqrt{2gh_0}}{g} = 2e^2\sqrt{\frac{2h_0}{g}}$$

and $t_3 = 2e^3\sqrt{\dfrac{2h_0}{g}}$ and so on

Total time of flight,

$$T = t_0 + t_1 + t_2 + \cdots$$

$$T = \sqrt{\frac{2h_0}{g}} + 2e\sqrt{\frac{2h_0}{g}} + 2e^2\sqrt{\frac{2h_0}{g}} + \cdots$$

$$= \sqrt{\frac{2h_0}{g}}[1 + 2e(1 + e + e^2 + \cdots)]$$

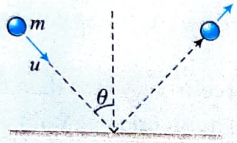
$$= \sqrt{\frac{2h_0}{g}}\left[1 + 2e\left(\frac{1}{1-e}\right)\right] = \sqrt{\frac{2h_0}{g}}\left[\frac{1+e}{1-e}\right]$$

ILLUSTRATION 2.16

A smooth ball of mass m strikes an inelastic smooth floor with a velocity u at an angle θ with the normal to the floor. If the coefficients of restitution is e, determine:

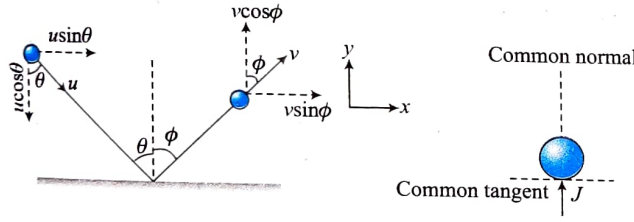
(a) the rebound velocity and angle of rebound of the particle.

(b) the impulse on the ball.



Sol.

(a) Let the ball rebound with velocity is v at angle ϕ with normal. The impulse of normal reaction during the collision acts on the ball in vertical direction.



The component of velocity along common tangent direction will remain unchanged, i.e.,

$$v_t = v\sin\phi = u\sin\theta \quad \ldots(i)$$

The component of velocity along common normal direction after collision. Applying relative speed of separation = $e \times$ (relative speed of approach) along common normal direction, we get

$$v_n = v\cos\phi = eu\cos\theta \quad \ldots(ii)$$

From Eqs. (i) and (ii): $\tan\phi = \dfrac{v_t}{v_n} = \dfrac{\tan\theta}{e}$

and rebound velocity $v = u\sqrt{\sin^2\theta + e^2\cos^2\theta}$

(b) The linear momentum will change in normal direction. Using impulse momentum equation

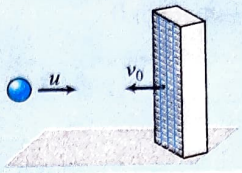
$$(\vec{p}_{\text{initial}})_y + \vec{J} = (\vec{p}_{\text{final}})_y$$

$$-mu\cos\theta + J = meu\cos\theta$$

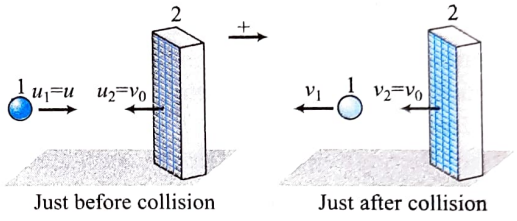
Therefore $J = m(1+e)u\cos\theta$

ILLUSTRATION 2.17

A particle is moving with velocity $u = 2$ m/s towards a heavy wall moving towards the particle with constant speed $v_0 = 1$ m/s as shown in figure. Assuming collision to be elastic, find the velocity of the particle immediately after the collision.



Sol. Let the velocity of the particle immediately after the collision be v_1.



For calculating velocity of the ball just after collision, we use coefficient of restitution equation

$$e = \left(\frac{v_2 - v_1}{u_1 - u_2}\right) = \frac{\text{relative velocity of separation}}{\text{relative velocity of approach}}$$

This equation is applicable along the direction of impulsive force.

Here $u_1 = u$, $u_2 = -v_0$, $v_1 = -v$ and $v_2 = -v_0$

Here $v_2 = -v_0$, since the wall being very heavy (infinite mass), moves with constant velocity,

Hence $e = \dfrac{-v_0 - (-v)}{u - (-v_0)} = \dfrac{v - v_0}{u + v_0}$ ...(i)

As collision is elastic. Putting the value of $e = 1$, we obtain

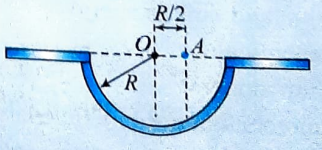
$$1 = \frac{v - v_0}{u + v_0} \quad \Rightarrow v = u + 2v_0$$

Putting u =2 m/s, v_0 = 1 m/s, we get

$$v = 2 + 2(1) = 4\,\text{m/s}$$

ILLUSTRATION 2.18

In the figure shown, a semi cylindrical frictionless track of radius R, centered at point O. A particle is released from point A ($OA = R/2$). Find the coefficient of restitution e, if



(a) after collision with the track, the particle moves along the track.

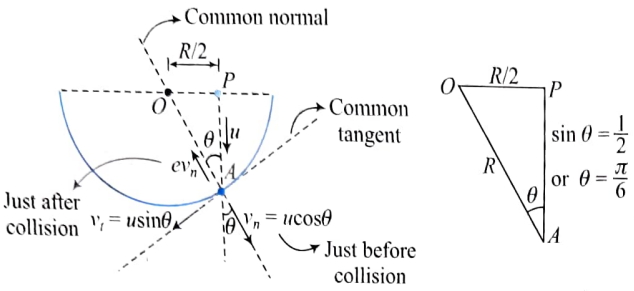
(b) the velocity of the particle becomes horizontal just after collision with the track.

Sol.

(a) After collision the particle moves along the track. This means there is no normal component of velocity. Hence $e = 0$.

(b) Just before collision, the components of velocity along the normal and along the tangent are

$v_n = u\cos\theta$ and $v_t = u\sin\theta$



The component of velocity in common tangent direction will remain unchanged as in this direction there is no impulse. Hence $v_t = u\sin\theta$ ... (i)

Velocity of the particle just after collision along common normal reaction

$$v_n' = ev_n = eu\cos\theta \quad \ldots(ii)$$

After collision the particle moves in horizontal direction, hence the net vertical component of particle velocity should be zero for which

$$ev_n\cos\theta = v_t\sin\theta \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii) we get

$$e(u\cos\theta)\cdot\cos\theta = (u\sin\theta)\cdot\sin\theta$$

or $e = \tan^2\theta = \tan^2\frac{\pi}{6} = \frac{1}{3}$

CONCEPT APPLICATION EXERCISE 2.2

1. Which one of the following is true in the case of inelastic collision?

	Linear momentum	Kinetic energy	Total energy
(a)	Remain conserved	Remain conserved	Remain conserved
(b)	Remain conserved	Not conserved	Not conserved
(c)	Not conserved	Remain conserved	Remain conserved
(d)	Remain conserved	Not conserved	Remain conserved

2. A moving bullet hits a solid target resting on a frictionless surface and gets embedded in it. What is conserved in this process?

(a) Momentum and kinetic energy
(b) Kinetic energy alone
(c) Momentum alone
(d) Neither momentum nor kinetic energy

3. Which one of the following statements is correct with reference to elastic collision between two bodies?

(a) Momentum and total energy are conserved but kinetic energy may be changed into some other form of energy.
(b) Kinetic energy and total energy are both conserved but momentum is conserved only if the two bodies have equal masses.
(c) Momentum, kinetic energy and total energy are all conserved.
(d) Neither momentum nor kinetic energy need be conserved but total energy must be conserved.

4. During collision between a closed system of particles, in the absence of external forces:

(a) The total kinetic energy of the system remains constant
(b) The momentum of each particle remains constant
(c) Momentum is exchanged between different particles
(d) Momentum and kinetic energy, both are exchanged between different particles.

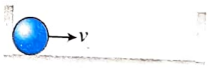
5. If two balls collide in air while moving vertically, then linear momentum of the system is conserved because

(a) gravity does not affect the linear momentum of the system
(b) force of gravity is very less compared to the impulsive force
(c) impulsive force is very less than the gravity
(d) gravity is not acting during collision

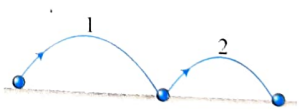
6. In an inelastic collision,

(a) linear momentum of the system is always conserved
(b) velocity of separation is less than the velocity of approach
(c) the coefficient of restitution can be zero
(d) all of the above

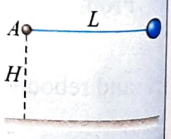
7. A ball rolls with a velocity v on a smooth horizontal floor. If e = coefficient of restitution, find the average speed of the ball for to and fro journey.



8. A ball is projected from the ground at some angle with horizontal. Coefficient of restitution between the ball and the ground is e. Let a, b and c be the ratio of times of flight, horizontal range and maximum height in two successive paths. Find a, b and c in terms of e.



9. A is a fixed point a height H above a perfectly inelastic smooth horizontal plane. A light inextensible string of length L ($> H$) has one end attached to A and other to a heavy particle. The particle is held at the level of A with string just taut and released from rest. Find the height of the particle above the plane when it is next instantaneously at rest.



10. A small ball rolls off the top landing of a staircase. It strikes the mid-point of the first step and then mid-point of the second step. The steps are smooth and identical in height and width. Find the coefficient of restitution between the ball and the first step.

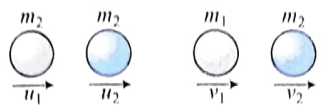
ANSWERS

1. (d) 2. (c) 3. (a) 4. (c) 5. (b)
6. (d) 7. $\frac{2ev}{e+1}$ 8. $a=b=\frac{1}{e}, c=\frac{1}{e^2}$ 9. $\frac{H^5}{L^4}$
10. $\frac{3}{4}$

GENERAL EQUATION FOR DIRECT IMPACT

If u_1, u_2 are the velocities before the impact of the masses m_1, m_2 and v_1, v_2 are the velocities after the impact, then applying conservation of momentum, we get

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \ldots(i)$$



We use coefficient of restitution equation

$$e = \left(\frac{v_2 - v_1}{u_1 - u_2}\right) = \frac{\text{relative velocity of separation}}{\text{relative velocity of approach}}$$

or $v_2 - v_1 = e(u_1 - u_2)$ $\ldots(ii)$

From Eqs. (i) and (ii) we get

$$v_1 = \frac{m_1 - em_2}{m_1 + m_2}u_1 + \frac{(1+e)m_2}{m_1 + m_2}u_2$$

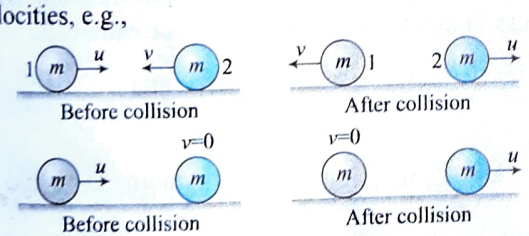
$$v_2 = \frac{(1+e)m_1}{m_2 + m_1}u_1 + \frac{m_2 - em_1}{m_2 + m_1}u_2$$

Important Points:

- For a perfectly elastic collision, we can substitute $e = 1$ and we get

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}u_1 + \frac{2m_2}{m_1 + m_2}u_2; \quad v_2 = \frac{2m_1}{m_2 + m_1}u_1 + \frac{m_2 - m_1}{m_2 + m_1}u_2$$

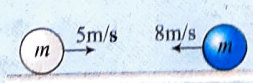
Special case: For $e = 1$ and $m_1 = m_2 = m$, we get $v_1 = u_2$ and $v_2 = u_1$ i.e., when two particles of equal mass collide elastically and the collision is head-on, they exchange their velocities, e.g.,



- For a perfectly inelastic collision, we can substitute $e = 0$ and we get $v_1 = v_2 = \frac{m_1}{m_1 + m_2}u_1 + \frac{m_2}{m_1 + m_2}u_2$

ILLUSTRATION 2.19

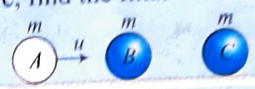
Two identical balls are approaching towards each other on a straight line with velocity 5 m/s and 10 m/s, respectively. Find the final velocities, after elastic head-on collision between them.



Sol. When two bodies of equal mass collide elastically and the collision is head-on, they exchange their velocities. So in this case, two velocities will be exchanged and the final motion is the reverse of the initial motion for both.

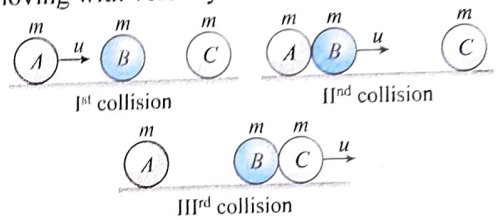
ILLUSTRATION 2.20

Three balls A, B and C of same mass 'm' are placed on a frictionless horizontal plane in a straight line as shown. Ball A is moved with velocity u towards the middle ball B. If all the collisions are elastic, find the final velocities of all the balls.



Sol. We know when two bodies of equal mass collide elastically and the collision is head-on, they exchange their velocities. In this case, A collides elastically with B and comes to rest but B starts moving with velocity u.

After a while B collides elastically with C and comes to rest but C starts moving with velocity u.

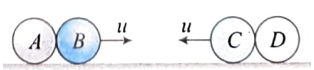


Therefore, final velocities, $v_A = 0$, $v_B = 0$ and $v_C = u$

ILLUSTRATION 2.21

Four identical balls A, B, C and D are placed in a line on a frictionless horizontal surface. A and D are moved with the same speed 'u' towards the middle as shown. Assuming elastic collisions, find the final velocities.

Sol. When two bodies of equal mass collide elastically and the collision is head-on, they exchange their velocities. In this case, A and D collide elastically with B and C, respectively, and come to rest but B and C start moving with velocity u towards each other as shown.



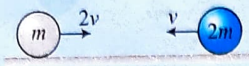
Then B and C collide elastically and exchange their velocities to move in the opposite directions

Finally, B and C collide elastically with A and D, respectively, and come to rest but A and D start moving with velocity u away from each other as shown.

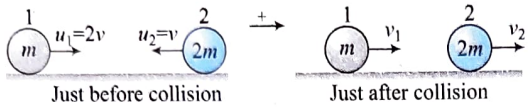
Therefore, finally ball A moves with velocity u towards left and ball D moves with velocity u towards right.

ILLUSTRATION 2.22

Two particles of masses m and $2m$ moving in opposite directions collide elastically with velocity $2v$ and v, respectively. Find their velocities after collision.



Sol. Let us assume the final velocities of the particles m and $2m$ be v_1 and v_2, respectively, as shown in figure



Just before collision Just after collision

By conservation of linear momentum, we get

$$m(2v)+2m(-v)=m(v_1)+2m(v_2)$$

or $0=mv_1+2mv_2$

or $v_1+2v_2=0$ ...(i)

and since the collision is elastic,

$$v_2-v_1=2v-(-v)$$

or $v_2-v_1=3v$ ...(ii)

Solving the above two equations, we get

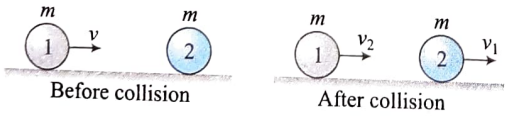
$v_2=v$ and $v_1=-2v$

That is, mass $2m$ returns with velocity v while mass m returns with velocity $2v$ in the direction shown in figure.

ILLUSTRATION 2.23

A ball of mass m moving at a speed v makes a head-on collision with an identical ball at rest. The kinetic energy of the balls after the collision is $\left(\frac{3}{4}\right)^{\text{th}}$ of the original. Find the coefficient of restitution.

Sol. As we have seen in the above discussion that under the given conditions:



Before collision After collision

$$v_1=\left(\frac{1+e}{2}\right)v \text{ and } v_2=\left(\frac{1-e}{2}\right)v$$

Given that $K_f=\frac{3}{4}K_i$

or $\frac{1}{2}mv_1^2+\frac{1}{2}mv_2^2=\frac{3}{4}\left(\frac{1}{2}mv^2\right)$

Substituting the values, we get $\left(\frac{1+e}{2}\right)^2+\left(\frac{1-e}{2}\right)^2=\frac{3}{4}$

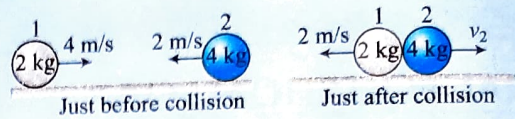
or $(1+e)^2+(1-e)^2=3$ or $2+2e^2=3$

or $e^2=\frac{1}{2}$

or $e=\frac{1}{\sqrt{2}}$

ILLUSTRATION 2.24

Two balls of masses 2 kg and 4 kg are moved towards each other with velocities 4 m/s and 2 m/s, respectively, on a frictionless surface. After colliding, the 2 kg ball returns back with velocity 2 m/s. Then find



Just before collision Just after collision

(a) velocity of the 4 kg ball after collision;
(b) coefficient of restitution e;
(c) impulse of deformation J_D;
(d) maximum potential energy of deformation;
(e) impulse of reformation J_R.

Sol.

(a) By conservation of linear momentum,

$$m_1u_1+m_2u_2=m_1v_1+m_1v_2$$

$$2(4)-4(2)=2(-2)+4(v_2) \Rightarrow v_2=1\ \text{ms}^{-1}$$

(b) Coefficient of restitution,

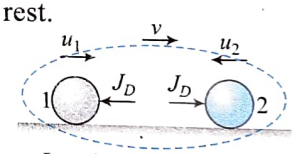
$$e=\frac{\text{velocity of separation}}{\text{velocity of approach}}=\frac{1-(-2)}{4-(-2)}=\frac{3}{6}=0.5$$

(c) Linear momentum will be conserved always. Let at the time of maximum deformation the velocity of combined body be v.

$$m_1u_1+m_2u_2=(m_1+m_2)v$$

$$2(4)-4(2)=(2+4)v \Rightarrow v=0$$

At the maximum deformed state, combined body will be momentary at rest.



Impulse of deformation

Using impulse momentum equation for particle '1'

$$(\vec{p}_{\text{initial}})_x+\vec{J}_D=(\vec{p}_{\text{final}})_x$$

$$m_1u_1-J_D=0 \Rightarrow J_D=m_1u_1=2\times4=8\ \text{Ns}$$

(d) Potential energy at the maximum deformed state, (U) = loss in kinetic energy during deformation

or $U=\left(\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2\right)-\frac{1}{2}(m_1+m_2)v^2$

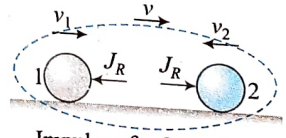
$$=\left(\frac{1}{2}2(4)^2+\frac{1}{2}4(2)^2\right)-\frac{1}{2}(2+4)(0)^2=24\ \text{J}$$

(e) Using impulse momentum equation for particle '2'

$$(\vec{p}_{\text{initial}})_x+\vec{J}_R=(\vec{p}_{\text{final}})_x$$

$$m_2v_2-J_R=-m_2v$$

$$J_R=m_2(v_2+v)=4(1+0)=4\ \text{Ns}$$

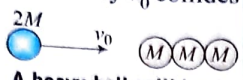


Impulse of reformation

Also, $J_R=eJ_D=0.5\times8=4\ \text{Ns}$

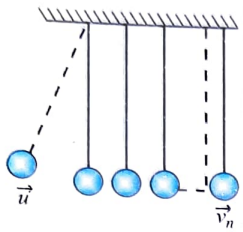
CONCEPT APPLICATION EXERCISE 2.3

1. A heavy ball of mass $2M$ moving with a velocity v_0 collides elastically head-on with a cradle of three identical balls each of mass M as shown in figure. Determine the velocity of each ball after collision.

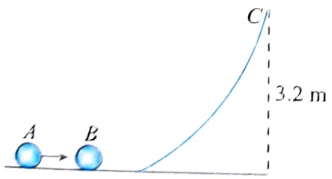


A heavy ball collides a cradle of three identical balls.

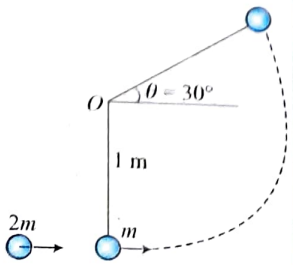
2. In figure, there are n identical spheres of mass m suspended with wires of equal length. The spheres are almost in contact with each other. Sphere 1 is pulled aside and released. If sphere 1 strikes sphere 2 with velocity u, find an expression for velocity v_n of the n^{th} sphere immediately after being struck by the one adjacent to it. The coefficient of restitution for all the impacts is e.



3. A mass m_1 moves with a great velocity. It strikes another mass m_2 at rest in head-on collision. It comes back along its path with low speed after collision. Then find out whether $m_1 < m_2$ or $m_1 > m_2$.
4. A ball of mass 4 kg moving with a velocity of 12 m/s impinges directly on another ball of mass 8 kg moving with velocity of 4 m/s in the same direction. Find their velocities after impact and calculate the loss of KE due to impact if $e = 0.5$.
5. Three balls of masses m_1, m_2 and m_3 are lying in a straight line. The first ball is moved with a certain velocity so that it strikes the second ball directly and itself comes to rest. The second ball collides with the third and is itself reduced to rest. If e is the coefficient of restitution for each ball, write down the relation of m_3 in terms of m_1 and m_2.
6. Two identical balls A and B lie on a smooth horizontal surface, which gradually merges into a curve to a height 3.2 m. Ball A is given a velocity of 10 m/s, to collide head-on with ball B, which then takes up the curved path. What is the minimum coefficient of restitution, e, for the collision between A and B in order that ball B reaches the highest point C of the curve.



7. A smooth ball of mass m is suspended from a light string of length 1 m. Another ball of mass $2m$ strikes the ball of mass m horizontally with a speed of $u = \sqrt{35}$ m/s. The coefficient of restitution for the collision is e. The string becomes loose, when it makes an angle of 30° with the horizontal; find the value of e.



8. Two balls of masses m and $2m$ and momenta $4p$ and $2p$ collides as shown in figure. During collision, the value of linear impulse between them is J. In terms of J and p find coefficient of restitution 'e'. Under what condition collision is elastic. Also find the condition of perfectly inelastic collision.

ANSWERS

2. $\left(\frac{1+e}{2}\right)^{n-1} u$
3. $m_1 < m_2$
4. 4 m/s, 8 m/s; 64 J
5. $m_2^2 = m_1 m_3$
6. $\frac{3}{5}$
7. 0.5

COLLISION IN TWO DIMENSIONS (OBLIQUE IMPACT)

If the velocities of colliding masses are not linear, then it is known as oblique collision. In such situation:

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remains constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.
2. No component of impulse acts along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remains unchanged along this direction.
3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remains conserved before and after collision in any direction.
4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction.

Relative speed of separation = $e \times$ (relative speed of approach)

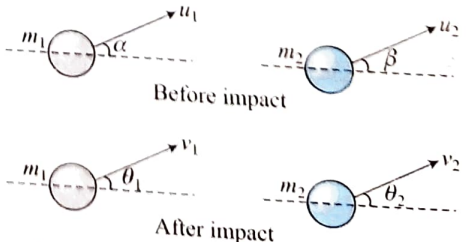
From law of conservation of momentum, we have the following.

Along x-axis:

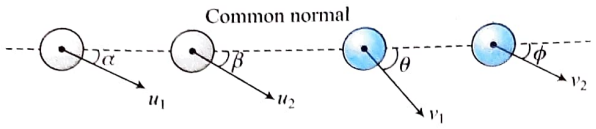
$$m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2$$

Along y-axis:

$$m_1 u_1 \sin\alpha + m_2 u_2 \sin\beta = m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2$$



First approach of analyzing oblique collision: When two bodies collide obliquely, their relative velocity, resolved along their common normal after the impact, is in a constant ratio to their relative velocity before impact (resolved along common normal) and is in the opposite direction.



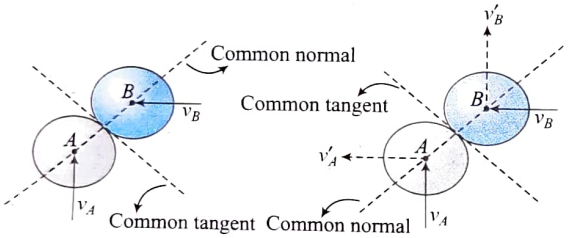
$$\frac{v_1\cos\theta - v_2\cos\phi}{u_1\cos\alpha - u_2\cos\beta} = -e$$

$$\Rightarrow\ v_1\cos\theta - v_2\cos\phi = -e(u_1\cos\alpha - u_2\cos\beta)$$

To write down the expression for coefficient of restitution, e, we first draw the line of impact. Then we resolve the components of velocities of points of contact of both the bodies along line of impact just before and just after the collision. Then,

$$e = \frac{v_{2x} - v_{1x}}{u_{1x} - u_{2x}}$$

Second approach of analyzing oblique collision: Let us consider an oblique collision between two particles. We assign different axis for oblique collision, normal axis (n-axis) along the line of impact and tangential axis (t-axis) along tangent to surfaces in contact. We assume the particles to be smooth and frictionless, so that the impulsive force exerted by particles on each other is internal force for the system. These impulsive force act along line of impact (n-axis).



We can form the following equations:

1. Since no force acts along t-axis on each particle, considered separately, the component of momentum along t-axis is conserved; hence t component of the velocity of each particle remains unchanged. Therefore, we obtain

$$(v_A)_t = (v_A')_t;\ (v_B)_t = (v_B')_t \qquad \ldots(i)$$

2. Total momentum of the two particles is conserved along n-axis.

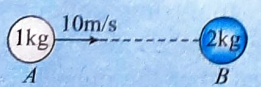
$$m_A(v_A)_n + m_B(v_B)_n = m_B(v_A')_n + m_B(v_B')_n \qquad \ldots(ii)$$

3. From definition of coefficient of restitution,

$$(v_B')_n - (v_A')_n = e[(v_A)_n - (v_B)_n] \qquad \ldots(iii)$$

ILLUSTRATION 2.25

Two point particles A and B are placed in line on a frictionless horizontal plane. If particle A (mass 1 kg) is moved with velocity 10 m/s towards stationary particle B (mass 2 kg) and after collision the two move at an angle of 45° with the initial direction of motion, then find:



(a) velocities of A and B just after collision.
(b) coefficient of restitution.

Sol. The very first step to solve such problems is to find the line of impact which is along the direction of force applied by A on B, resulting the stationary particle B to move. Thus, by watching the direction of motion of B, line of impact can be determined. In this case, the line of impact is along the direction of motion of B, i.e., 45° with the initial direction of motion of A.

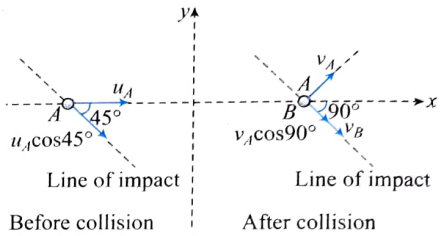
(a) Let us apply the principle of conservation of momentum.

Along x-direction:

$$m_Au_A = m_Av_A\cos 45° + m_Bv_B\cos 45°$$

or $\quad 1(10) = 1(v_A\cos 45°) + 2(v_B\cos 45°)$

or $\quad v_A + 2v_B = 10\sqrt{2} \qquad \ldots(i)$



Along y-direction:

$$0 = m_Av_A\sin 45° - m_Bv_B\sin 45°$$

or $\quad 0 = 1(v_A\sin 45°) - 2(v_B\sin 45°)$

or $\quad v_A = 2v_B \qquad \ldots(ii)$

Solving the two equations,

$$v_A = \frac{10}{\sqrt{2}}\ \text{m/s} \quad \text{and} \quad v_B = \frac{5}{\sqrt{2}}\ \text{m/s}$$

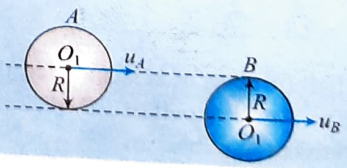
(b) Coefficient of restitution

$$e = \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

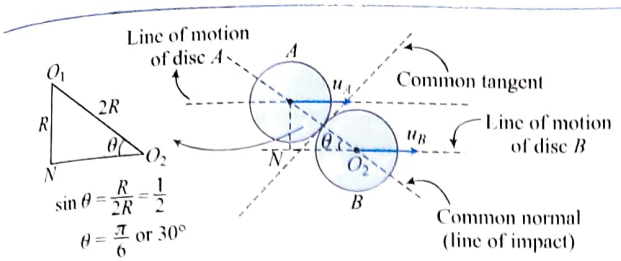
$$e = \frac{v_B - v_A\cos 90°}{u_A\cos 45°} = \frac{\frac{5}{\sqrt{2}} - 0}{\frac{10}{\sqrt{2}}} = \frac{1}{2}$$

ILLUSTRATION 2.26

Two identical discs 'A' and 'B' are moving on smooth horizontal surface as shown in the figure. The velocities of discs are $u_A = 8$ m/s and $u_B = 4$ m/s respectively. If both the disc collide elastically ($e = 1$), then find the velocities of discs just after collision.



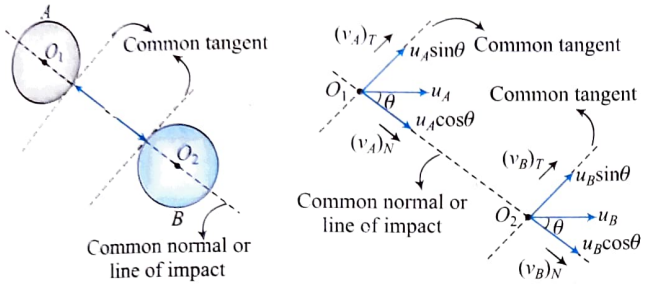
Sol. If we take both the disc as system, the impulse between the discs acts along common normal. This impulse is internal to the system. Hence we can conserve linear momentum of the system in any direction. In common tangent direction there is no component of impulse on any individual disc. It means linear momentum of the disc 'A' and 'B' will not change along common tangent.



Hence the velocities of the discs along common tangent will not change after collision.

Hence $(v_A)_{\text{tangent}} = u_A \sin\theta = 8 \times \sin 30° = 4$ m/s

and $(v_B)_{\text{tangent}} = u_B \sin\theta = 4 \times \sin 30° = 2$ m/s



Now we can apply conservation of linear momentum along common normal, just before collision and just after collision.

$$m_A u_A \cos\theta + m_B u_B \cos\theta = m_A (v_A)_{\text{normal}} + m_B (v_B)_{\text{normal}}$$

as $m_A = m_B$ and $\cos\theta = \frac{\sqrt{3}}{2}$

$$\Rightarrow (v_A)_n + (v_B)_n = 8 \times \frac{\sqrt{3}}{2} + 4 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$$

or $(v_A)_n + (v_B)_n = 6\sqrt{3}$ ...(i)

Now we can apply restitution equation which is applicable only along common normal

$$e = \frac{v_2 - v_1}{u_1 - u_2} \Rightarrow 1 = \frac{(v_B)_N - (v_A)_N}{u_A \cos\theta - u_B \cos\theta}$$

$$\Rightarrow (v_B)_n - (v_A)_n = 8 \times \frac{\sqrt{3}}{2} - 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Hence $(v_B)_n - (v_A)_n = 2\sqrt{3}$ ...(ii)

On solving Eqs. (i) and (ii) we get

$(v_A)_n = 2\sqrt{3}$ m/s and $(v_B)_n = 4\sqrt{3}$ m/s

Hence final speed of the disc are

$$v_A = \sqrt{(v_A)_n^2 + (v_A)_t^2}$$

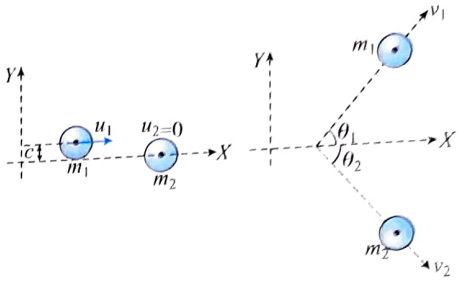
$$= \sqrt{(2\sqrt{3})^2 + 4^2} = 2\sqrt{7} \text{ m/s}$$

and $v_B = \sqrt{(v_B)_n^2 + (v_B)_t^2}$

$$= \sqrt{(4\sqrt{3})^2 + 2^2} = 2\sqrt{13} \text{ m/s}$$

IMPACT PARAMETER

Now consider the situation shown in figure, in which the second mass m_2 is at rest. The object m_1 with an initial velocity u_1 strikes m_2. These objects could be billiard balls, subatomic particles, whatever you like, and the two need not have the same mass. After the collision, we observe that the object 1, which was initially in motion along the x-direction, diverge at an angle θ_1 and the one which was at rest, will start from the reference line at an angle θ_2. Because a particle at rest, starts moving in that direction where the force acts on it. If initially m_2 was moving parallel to m_1, the direction of motion m_2 after collision will be at an angle less than θ_2.



Here the distance between the two lines of motion of the particles before collision (if parallel) is called as "impact parameter". In this case impact parameter is 'c'. For collision to occur, it must be less than the sum of the radii of the two balls.

In this situation, we can again conserve momentum in x and y directions as

$$m_1 u_1 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \quad \text{...(i)}$$

$$0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2 \quad \text{...(ii)}$$

If collision is elastic we can use energy conservation here as

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad \text{...(iii)}$$

If $m_1 = m_2$ and m_2 is at rest then Eqs. (i), (ii) and (iii) become

$$u_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2 \quad \text{...(iv)}$$

$$0 = v_1 \sin\theta_1 - v_2 \sin\theta_2 \quad \text{...(v)}$$

$$u_1^2 = v_1^2 + v_2^2 \quad \text{...(vi)}$$

Squaring and adding Eqs. (iv) and (v)

$$u_1^2 = v_1^2(\sin^2\theta_1 + \cos^2\theta_2) + v_2^2(\sin^2\theta_2 + \cos^2\theta_2) + 2v_1 v_2 \cos(\theta_1 + \theta_2)$$

or $u_1^2 = v_1^2 + v_2^2 + 2v_1 v_2 \cos(\theta_1 + \theta_2)$ ...(vii)

If $\theta_1 + \theta_2 = 90°$ then $\cos(\theta_1 + \theta_2) = 0$

and Eq. (vii) becomes, $u_1^2 = v_1^2 + v_2^2$ which is same as Eq. (vi), which is true in case of perfectly elastic collision.

Hence the particles will move perpendicular to each other after collision.

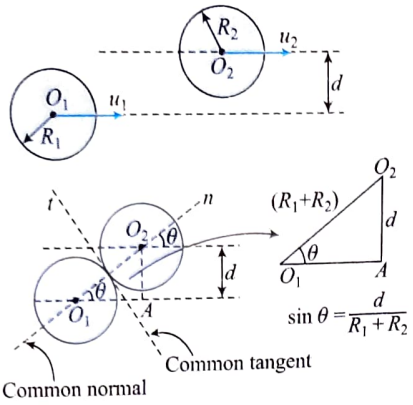
It is important to note that if two bodies of masses m_1 and m_2 have an elastic collision and $m_1 = m_2$ with m_2 at rest then $\theta_1 + \theta_2 = 90°$.

Note :

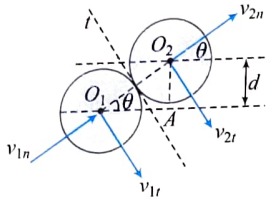
- When two identical bodies have an oblique elastic collision, with one particle at rest before collision, then the two particles will go in perpendicular directions.
- The colliding balls of the same mass in an elastic direct impact interchange their velocities.

OBLIQUE COLLISION OF TWO DISCS

Two discs of masses m_1 and m_2 and radius R_1 and R_2 are moving with velocities u_1 and u_2 respectively. Both discs collides, let us analyse velocities of discs just after collision.



Let the velocity of disc 1 and disc 2 just after collision in the direction of normal and tangent are $(v_{1,n}, v_{1,t})$ and $(v_{2,n}, v_{2,t})$ respectively. Then the velocity of discs will be unchanged in the direction of tangent



$$v_{1,t} = u_1 \sin\theta \text{ and } v_{2,t} = u_2 \sin\theta \quad \ldots(i)$$

Conservation of linear momentum is applicable in any direction as there is no external impulse on the system.

We have already calculated the velocity in tangent direction. We need to calculate velocity in normal direction.

Hence conserving linear momentum in normal direction.

$$m_1u_1\cos\theta + m_2u_2\cos\theta = m_1v_{1,n} + m_2v_{2,n} \quad \ldots(ii)$$

Now using Newton's law of restitution, we get

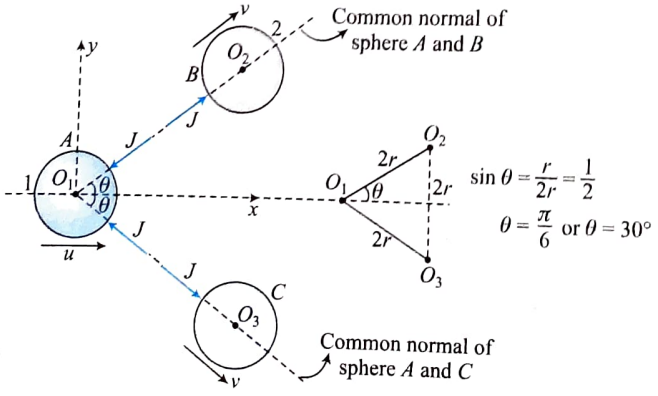
$$(v_{2,n} - v_{1,n}) = e(u_1\cos\theta - u_2\cos\theta) \quad \ldots(iii)$$

From Eqs. (ii) and (iii), we can calculate the value of $v_{1,n}$ and $v_{2,n}$

ILLUSTRATION 2.27

Two equal spheres of mass m are in contact on a smooth horizontal table. A third identical sphere impinges symmetrically on them and is reduced to rest. Prove that $e = 2/3$ and find the loss in KE.

Sol. If we take all three spheres A, B and C as system, the impulse between the spheres will be internal to the system. Hence no external impulse acting on the system, it means we can conserve the linear momentum of the system in any direction.



From impulse diagram it is clear that the spheres B and C will move along common normal direction and motion of sphere A will be confined along x-axis only.

The sphere A may move forward, move backward or may come to rest. According to question it comes to rest.

Let u be the velocity of sphere A before impact. As the spheres are identical, the triangle $O_1O_2O_3$ formed by joining their centres is equilateral. The spheres B and C will move in directions O_1O_2 and O_1O_3 after impact making an angle of 30° with the original line of motion of sphere A. Let v be the speed of the spheres B and C after impact.

From impulse diagram of sphere A:

$$(\vec{p}_{\text{initial}}) + \vec{J} = (\vec{p}_{\text{final}})$$

$$mu - 2J\cos\theta = m \times 0$$

$$\Rightarrow J = \frac{mu}{2\cos\theta} = \frac{mu}{\sqrt{3}} \quad \ldots(i)$$

Now impulse momentum equation in sphere B gives

$$J = mv \Rightarrow \frac{mu}{\sqrt{3}} = mv$$

or $$v = \frac{u}{\sqrt{3}} \quad \ldots(ii)$$

Here we can also apply conservation of linear momentum in x-direction.

$$(\vec{p}_{\text{initial}})_x = (\vec{p}_{\text{final}})_x$$

$$mu + 0 + 0 = 0 + mv\cos\theta + mv\cos\theta$$

or $$v = \frac{u}{2\cos\theta} = \frac{u}{\sqrt{3}}$$

Now applying restitution equation which is applicable along common normal

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

Here $v_2 = v = \frac{u}{\sqrt{3}}$; $v_1 = 0$; $u_2 = 0$ and $u_1 = u\cos 30°$

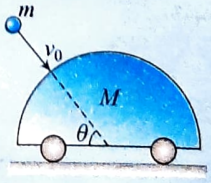
Hence $$e = \frac{v - 0}{u\cos 30° - 0} = \frac{u/\sqrt{3}}{u\sqrt{3}/2} = \frac{2}{3}$$

Loss in KE, $$\Delta K = \frac{1}{2}mu^2 - \left(2 \times \frac{1}{2}mv^2\right)$$

$$= \frac{1}{2}mu^2\left(1 - \frac{2}{3}\right) = \frac{1}{6}mu^2$$

ILLUSTRATION 2.28

A particle (a mud pallet, say) of mass m strikes a smooth stationary wedge of mass M with a velocity v_0, at an angle θ with horizontal. If the collision is perfectly inelastic, find the:



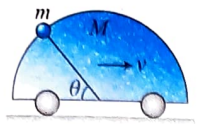
(a) velocity of the wedge just after the collision.

(b) change in KE of the system $(M + m)$ during collision.

Sol.

(a) Let the system $(M + m)$ moves as a single mass with a velocity v. Conserving the momentum of the system in horizontal, we have

$$mv_0 \cos\theta = (M+m)v$$



This gives $v = \dfrac{mv_0 \cos\theta}{M+m}$

(b) The change in KE of the system is

$$\Delta K = \frac{1}{2}(M+m)v^2 - \frac{1}{2}mv_0^2$$

where $v = \dfrac{mv_0 \cos\theta}{M+m}$

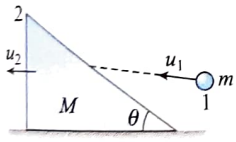
This gives,

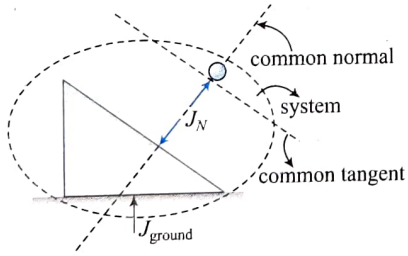
$$\Delta K = \frac{(M+m)}{2}\left(\frac{mv_0\cos\theta}{M+m}\right)^2 - \frac{1}{2}mv_0^2$$

$$= -\frac{mv_0^2}{2}\left(1 - \frac{m}{M+m}\cos^2\theta\right) = -\frac{(M+m\sin^2\theta)\,mv_0^2}{2(M+m)}$$

COLLISION OF WEDGE AND PARTICLE

If we take particle and the wedge as a system, the normal impulse between the particle and wedge will be internal to the system. In this system there will be impulse due to normal reaction of the ground on the wedge in vertical direction. It means we can conserve the linear momentum of the system in horizontal direction.





$$m_1u_{1,x} + m_2u_2 = m_1v_{1,x} + m_2v_2$$

$u_{1,x}$ = component of initial velocity of the particle in horizontal direction.

$v_{1,x}$ = component of velocity of the particle just after collision in horizontal direction.

Along common tangent the velocity of the particle will remain unchanged.

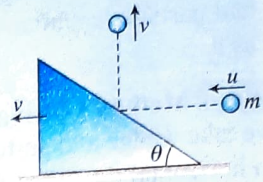
We can apply coefficient of restitution equation along common normal direction.

$$e = \frac{v_{2,n} - v_{1,n}}{u_{1,n} - u_{2,n}}$$

Let us learn the application of this concept through some illustrations.

ILLUSTRATION 2.29

A ball of mass 'm' moving horizontally which velocity 'u' hits a wedge of mass 'M'. The wedge is situated on a smooth horizontal source. If after striking with wedge the ball starts moving in vertical direction and the wedge starts moving in horizontal plane. Calculate:



(a) the velocity of wedge V.

(b) the velocity (v) with which the ball moves in vertical direction.

(c) the impulse imparted by the ball to the wedge.

(d) the coefficient of restitution e.

Sol.

(a) As no external force is acting on the system in horizontal direction the linear momentum should be constant and conserved in horizontal direction.

Let velocity of wedge after collision be V.

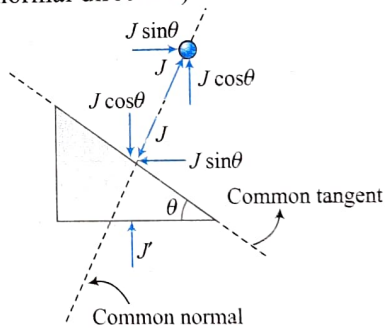
Then $mu = MV \Rightarrow V = \dfrac{mu}{M}$ ...(i)

(b) As there is no impulse on the ball in the direction parallel to sloping side hence the velocity of ball along the slope (tangent direction) should remain unchanged,

$u\cos\theta = v\sin\theta$ ...(ii)

which gives $v = u\cot\theta$.

(c) We can find the values of v and V by impulse approach, when ball hits the wedge the impulse is generated between ball and wedge in the direction perpendicular to sloping surface. (normal direction)



For ball: $mu - J\sin\theta = 0$

$J\sin\theta = mu$ ...(iii)

$J\cos\theta = mv$ ...(iv)

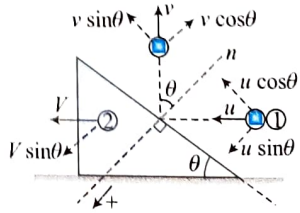
From Eqs. (i) and (ii), $v = u\cot\theta$

From Eq. (iii), $J = mu\ \text{cosec}\,\theta$

$$J\sin\theta = MV \Rightarrow V = \frac{J\sin\theta}{M}$$

which gives $V = \dfrac{mu}{M}$

(d) The coefficient of restitution is given as

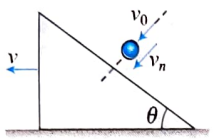


$$e = \frac{V\sin\theta - (-v\cos\theta)}{u\sin\theta - 0} = \frac{m\sin^2\theta + M\cos^2\theta}{M\sin^2\theta} = \frac{m}{M} + \cot^2\theta$$

ILLUSTRATION 2.30

A wedge of mass M is kept at rest on smooth surface, a particle of mass m hits the wedge normally. Find the velocity of wedge and particle just after collision. Take coefficient of restitution as e.

Sol. Method 1 (using conservation of linear momentum): If we take wedge and particle as a system, no external impulse acts in horizontal direction, the linear momentum should be constant conserved in horizontal direction. As the particle is moving in normal direction, it has no component of velocity in sloping surface direction (tangent direction).



Hence after collision particle will have no velocity component in common tangent direction. The velocity of the particle will confine along common normal direction only. Let velocity of the particle and wedge after collision be v_n and V respectively.

Conserving linear momentum in horizontal direction

$$mv_0 \sin\theta = MV + mv_n \sin\theta \quad \ldots(i)$$

Now, using Newton's law of collision.

$$(v_2 - v_1)_n = e(u_1 - u_2)_n$$

$$(V\sin\theta - v_n) = e(v_0 - 0) \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

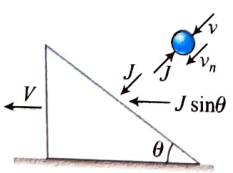
$$V = \frac{(1+e)mv_0 \sin\theta}{M + m\sin^2\theta}$$

and $$v_n = V\sin\theta - ev_0 = \frac{(m\sin^2\theta - Me)v_0}{M + m\sin^2\theta}$$

Method 2 (using impulse method): The impulse acting on the ball and wedge can be represented as

$$J\sin\theta = MV \quad \ldots(i)$$

$$mv_0 - J = mv_n \quad \ldots(ii)$$



and Newton's law of restitution

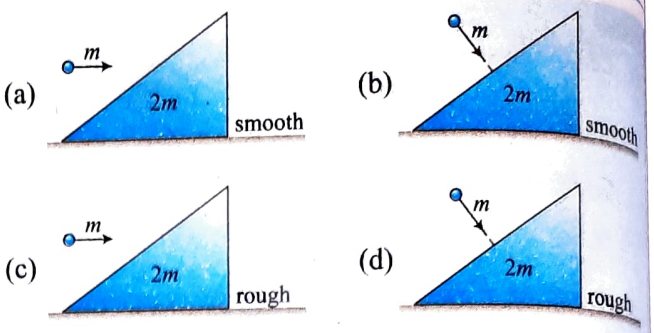
$$(V\sin\theta - v_n) = e(v_0 - 0) \quad \ldots(iii)$$

Solving Eqs. (i), (ii) and (iii), we get

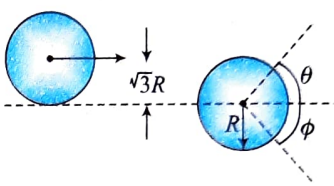
$$V = \frac{(1+e)mv_0 \sin\theta}{M + m\sin^2\theta}$$

CONCEPT APPLICATION EXERCISE 2.4

1. In which of the following system(s) linear momentum cannot be conserved during collision along horizontal line (parallel to base)?

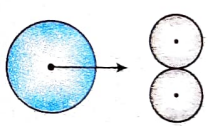


2. A disc moving on a frictionless horizontal table collide elastically with another identical disc as shown. The directions of motion of the two discs make angle θ and ϕ with the initial line of motion as shown. Then:

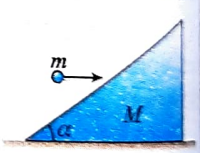


(a) $\theta = 30°$ (b) $\theta = 60°$
(c) $\phi = 30°$ (d) $\phi = 60°$

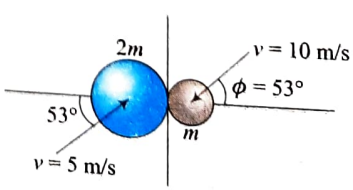
3. Two equal discs are in contact on a table. A third disc of same mass but of double radius strikes them symmetrically and remains at rest after impact. Find the coefficient of restitution.



4. A wedge (free to move) of mass 'M' has one face making an angle α with horizontal and is resting on a smooth rigid floor. A particle of mass 'm' hits the inclined face of the wedge with a horizontal velocity v_0. It is observed that the particle rebounds in vertical direction after impact. Neglect friction between particle and the wedge and take $M = 2m$, v_0 = 10 m/s, $\tan\alpha = 2$, g = 10 m/s^2. Determine the coefficient of restitution for the impact.

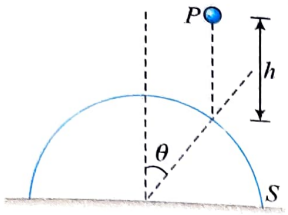


5. Two smooth spheres made of identical material having masses 'm' and '$2m$' undergoes an oblique impact as shown in figure. The initial velocities of the masses are also shown. The impact force is along the line joining their centres. The coefficient of restitution is 5/9. Calculate the velocities of the masses after the impact.

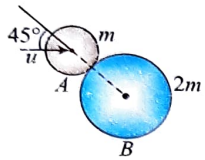


6. A hemisphere 'S' and a particle 'P' are of same mass 'm'. P is dropped from a height 'h'. A hemisphere 'S' is kept on a smooth horizontal surface. The particle 'P' collides elastically with hemisphere at the point shown in the figure. If $\theta = 45°$ and after collision the velocity

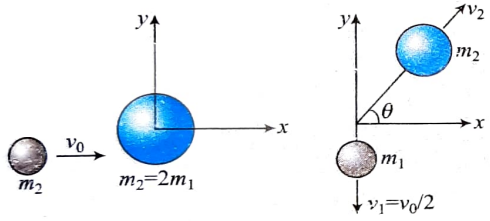
of the particle becomes horizontal. Find the velocities of hemisphere 'S' and particle 'P' after collision.



7. Disc A of mass m collides with stationary disc B of mass $2m$ as shown in figure. Find the value of coefficient of restitution for which the two discs move in perpendicular direction after collision.



8. A mass m_1 with initial speed v_0 in the positive x-direction collides with a mass $m_2 = 2m_1$ which is initially at rest at the origin, as shown in figure. After the collision m_1 moves off with speed $v_1 = \frac{v_0}{2}$ in the negative y-direction, and m_2 moves off with speed v_2 at angle θ.

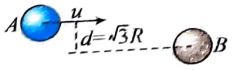


Before collision After collision

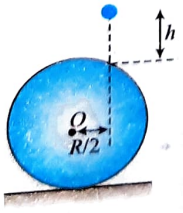
(a) Find the velocity (magnitude and direction) of the centre of mass before the collision, as well as its velocity after the collision.

(b) Find v_2 in terms of v_0.

9. A smooth ball 'A' travels towards another identical ball 'B' with a velocity u. Ball B is at rest and the impact parameter d is equal to $\sqrt{3}R$ where R is radius of each ball. Due to impact the direction of motion of ball A changes by 30°. Find the velocities of the balls 'A' and 'B' after collision. It is given that collision is elastic.



10. A small particle of mass m is released from a height h on a large smooth sphere kept on a perfectly smooth surface, as shown in the figure. Collision between particle and sphere is perfectly inelastic. Determine the velocities of particle and sphere after collision.



ANSWERS

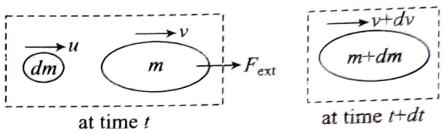
1. (c, d) **2.** (a, d) **3.** $\frac{9}{16}$ **4.** $\frac{3}{4}$ **6.** $\sqrt{\frac{gh}{2}}, \sqrt{\frac{gh}{2}}$

7. $\frac{1}{2}$ **8.** (a) $\frac{v_0}{3}, \frac{v_0}{3}$ (b) $\frac{\sqrt{5}}{4}v_0$ **9.** $\frac{u\sqrt{3}}{2}, \frac{u}{2}$

10. $\sqrt{\frac{6m^2gh}{4(4M+m)^2} + \frac{gh}{2}}; \frac{m\sqrt{6gh}}{(4M+m)}$

VARIABLE MASS SYSTEM

Consider at any time t, a system of mass m travelling with velocity v as shown. After a small time dt, let its mass becomes $m + dm$ and velocity $v + dv$. Here a small mass dm travelling with velocity u adds into this system of mass m. F_{ext} is the force on m due to an external agent. For the system of $(m + dm)$:



at time t at time $t+dt$

Impulse = change in momentum

$$\Rightarrow \quad F_{ext}\,dt = (m+dm)(v+dv) - mv - dm\,u$$
$$= mdv + vdm + dm.dv - dmu$$

but $dmdv$ is very small, so it is negligible.

$$\Rightarrow \quad F_{ext}\,dt = mdv - (u-v)dm$$

$$\Rightarrow \quad F_{ext}\,dt + v_{rel}\,dm = mdv$$

where $v_{rel} = u - v$ is the relative velocity of the elementary mass dm w.r.t. the initial mass of system.

$$\Rightarrow \quad F_{ext} + v_{rel}\frac{dm}{dt} = m\frac{dv}{dt}$$

The above equation is also valid in vector form

$$\vec{F}_{ext} + \vec{v}_{rel}\frac{dm}{dt} = m\frac{d\vec{v}}{dt}$$

$$\vec{F}_{ext} + \vec{F}_{thrust} = \vec{F}_{net}$$

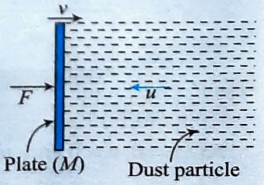
$\vec{F}_{ext}$ = Net external force acting on the mass 'm'

$\vec{v}_{rel}\frac{dm}{dt}$ = Thrust force acting on the system. The direction of this force is in the direction of $\vec{v}_{rel}$, if the mass is increasing.

It acts opposite direction of $\vec{v}_{rel}$ if mass is decreasing. The value of v_{rel} will be zero if a mass just dropped from a moving body, as dropped mass has same velocity as the moving body at the time of dropping. In this case no thrust force will act.

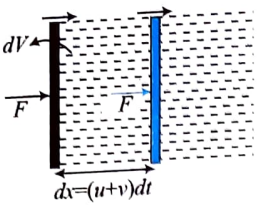
ILLUSTRATION 2.31

A square plate of mass M and area A, is moving with constant velocity v against dust particles moving with velocity u in opposite direction as shown in figure. The density of the dust particles in the medium is ρ (in kg/m^3). Find the force F required to keep the plate moving with constant velocity.



Sol. Velocity of plate relative to dust particles.

$\vec{v}_{p,d} = \vec{v}_p - \vec{v}_d = v - (-u) = (v+u)$ (In rightward direction)



In time 'dt' plate comes into contact with dust particles stored in volume,

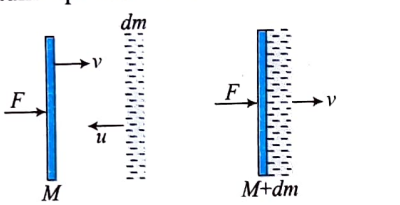
$dV = Adx = A(v+u)\,dt$

The mass of dust striking to plate in time dt is

$dm = \rho dV = \rho A(v+u)\,dt$

$$\Rightarrow \quad \frac{dm}{dt} = \rho A(v+u) \qquad \ldots(i)$$

Let us consider "plate + mass of dust" as system, and apply impulse momentum equation over time dt.



Just before striking the mass dm with the plate

Just after striking the mass dm with the plate

Impulse $= Fdt = (M+dm)v - [Mv - dmu]$

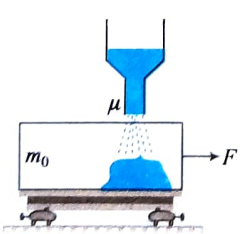
$$Fdt = Mv + dmv - Mv + dmu$$

$$F = \frac{dmu}{dt}(u+v)$$

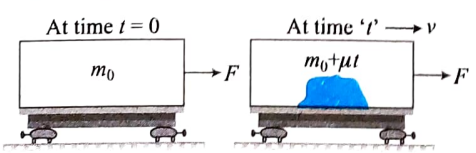
From Eq. (i), we have $F = \rho A(v+u)^2$

ILLUSTRATION 2.32

A flat car of mass m_0 starts moving to the right due to a constant horizontal force F. Sand spills on the flat car from a stationary hopper. The rate of loading is constant and equal to μ kg/s. Find the time dependence of the velocity and the acceleration of the flat car in the process of loading. The friction is negligibly small.



Sol. Initial velocity of the flat car is zero. Let v be its velocity at time t and m be its mass at that instant. Then



At $t = 0, v = 0$ and $m = m_0$

At $t = t, v = v$ and $m = m_0 + \mu t$

Here, $v_r = u - v = 0 - v = -v$

Here u is the velocity of mass being added in the horizontal direction which is zero.

Rate of adding the mass, $\frac{dm}{dt} = \mu$

Apply $F + v_{rel}\frac{dm}{dt} = \frac{mdv}{dt}$, we get

$$F + (-v)\mu = (m_0 + \mu t)\frac{dv}{dt} \qquad \ldots(i)$$

or $$\int_0^v \frac{dv}{F - \mu v} = \int_0^t \frac{dt}{m_0 + \mu t}$$

$$\Rightarrow \quad -\frac{1}{\mu}[\ln(F - \mu v)]_0^v = \frac{1}{\mu}[\ln(m_0 + \mu t)]_0^t$$

$$\Rightarrow \quad \ln\left(\frac{F}{F - \mu v}\right) = \ln\left(\frac{m_0 + \mu t}{m_0}\right)$$

$$\Rightarrow \quad \frac{F}{F - \mu v} = \frac{m_0 + \mu t}{m_0}$$

or $$v = \frac{Ft}{m_0 + \mu t} \qquad \ldots(ii)$$

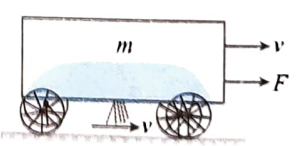
From Eq. (i), $a = \frac{F - \mu v}{m} = \left(F - \frac{F\mu t}{m_0 + \mu t}\right)\frac{}{m_0 + \mu t}$

or $$a = \frac{Fm_0}{(m_0 + \mu t)^2}$$

ILLUSTRATION 2.33

A cart loaded with sand moves along a horizontal floor due to a constant force F coinciding in direction with the cart's velocity vector. In the process, the sand spills through a hole in the bottom with a constant rate μ kg/s. Find the acceleration and velocity of the cart at the moment t, if at the initial moment $t = 0$ the cart with loaded sand had mass m_0 and its velocity was equal to zero. Friction is to be neglected.

Sol. In this problem, the sand spills through a hole at the bottom of the cart. Hence, the relative velocity of the sand v_r will be zero, because it will acquire the same velocity as that of the cart at the moment of separation.



$v_{rel} = 0$

Apply $F + v_{rel}\frac{dm}{dt} = m\frac{dv}{dt}$

$$\Rightarrow \quad F + 0 \times \left(\frac{dm}{dt}\right) = (m_0 - \mu t)\frac{dv}{dt} \qquad \ldots(i)$$

$$\Rightarrow \quad (m_0 - \mu t)\frac{dv}{dt} = F$$

or $$\int_0^v dv = \int_0^t \frac{F\,dt}{m_0 - \mu t}$$

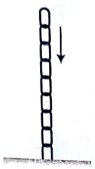
$$\Rightarrow v = \frac{F}{-\mu}\left[\ln(m_0 - \mu t)\right]_0^t \text{ or } v = \frac{F}{\mu}\ln\left(\frac{m_0}{m_0 - \mu t}\right)$$

From Eq. (i), acceleration of the cart,

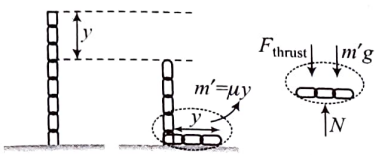
$$a = \frac{dv}{dt} = \frac{F}{m} \text{ or } a = \frac{F}{m_0 - \mu t}$$

ILLUSTRATION 2.34

The upper end of the uniform chain of linear mass density μ is lowered with a speed v as shown in the figure. Find the reaction force offered by the ground after the chain falls through a distance y.



Sol. The each falling link collides with the fallen links with a relative velocity, $\vec{v}_{rel} = -v\hat{j}$



Initially Finally FBD of fallen part

where v = velocity of the falling chain, we have the formula for the variable mass system "m" given as: $\vec{F}_{ext} + \vec{F}_{thrust} = \vec{F}_{net}$, where,

$$\vec{F}_{thrust} = \vec{v}_{rel}\frac{dm'}{dt}$$

where $\vec{v}_{rel} = -v\hat{j}$ and $\frac{dm'}{dt} = \mu v$ (positive, as mass m' is increasing)

This gives $\vec{F}_{thrust} = -\mu v^2\hat{j}$

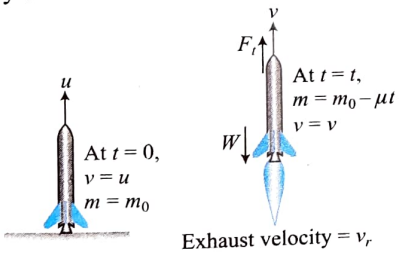
Force equation on the heap: $N - m'g + F_{thrust} = 0$

where $m' = \mu y$ and $F_{thrust} = \mu v^2$

or $N = \mu(v^2 + gy)$

ROCKET PROPULSION

Let m_0 be the mass of the rocket at time $t = 0$, m its mass at any time t and v its velocity at that moment. Initially, let us suppose that the velocity of the rocket is u.



Exhaust velocity = v_r

Further, let $-dm/dt$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases with respect to rocket. Usually, $-dm/dt$ and v_{rel} are kept constant throughout the journey of the rocket. Now, let us write a few equations which can be used in the problems of rocket propulsion.

At time $t = t$,

Thrust force on the rocket, $F_{thrust} = v_{rel}\left(-\frac{dm}{dt}\right)$ (upwards),

Weight of the rocket, $W = mg$ (downwards)

Net force on the rocket,

$$F_{net} = F_{thrust} - W \text{ (upwards)} = v_{rel}\left(-\frac{dm}{dt}\right) - mg$$

Net acceleration of the rocket, $a = \frac{F}{m}$

or $$\frac{dv}{dt} = \frac{v_{rel}}{m}\left(-\frac{dm}{dt}\right) - g$$

or $$dv = \frac{v_{rel}}{m}(-dm) - g\,dt$$

or $$\int_u^v dv = v_{rel}\int_{m_0}^{m} -\frac{dm}{m} - g\int_0^t dt$$

Thus, $$v = u - gt + v_{rel}\ln\left(\frac{m_0}{m}\right)$$

Note: If gravity is ignored and initial velocity of the rocket $u = 0$ then $v = v_{rel}\ln\left(\frac{m_0}{m}\right)$.

ILLUSTRATION 2.35

A rocket, with an initial mass of 1000 kg, is launched vertically upwards from rest under gravity. The rocket burns fuel at the rate of 10 kg per second. The burnt matter is ejected vertically downwards with a speed of 2000 m/s relative to the rocket. Find the velocity of the rocket after 1 min of start.

Sol. The velocity equation is given by

$$v = u - gt + v_r\ln\left(\frac{m_0}{m}\right)$$

Here $u = 0$, $t = 60$ s, $g = 10$ m/s^2, $v_r = 2000$ m/s,

$m_0 = 1000$ kg and $m = 1000 - 10\times 60 = 400$ kg

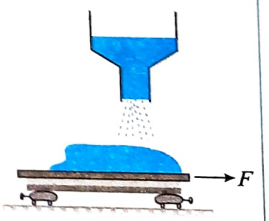
Putting these values, we get

$$v = 0 - 600 + 2000\ln\left(\frac{1000}{400}\right)$$

$$= 2000\ln 2.5 - 600 = 1232.6 \text{ m/s}$$

CONCEPT APPLICATION EXERCISE 2.5

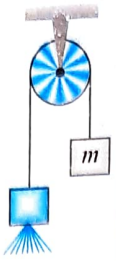
1. Sand is falling on a flat car being pulled with constant speed. The rate of mass falling on the cart is constant. Then the horizontal component of force exerted by the falling sand on the cart



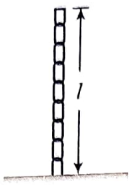
   (a) increases
   (b) decreases
   (c) remains constant
   (d) increases and then decreases

2. A gun fires 40 shots in one second with a muzzle velocity of 300 m/s. If mass of each shot is 40 g, find the average impact force on the gun.

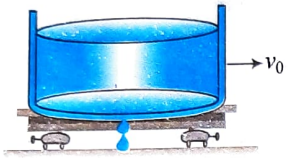
3. A mass m_1 is connected by a weightless cable passing over a frictionless pulley to a container of water, whose mass is m_0 at $t = 0$. If the container ejects water in downward direction at a constant rate b kg/s. With a velocity v_0 relative to the container, determine the acceleration of m_1 as a function of time.



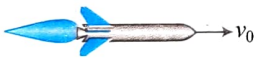
4. A uniform chain of mass m and length l hangs on a thread and touches the surface of a table by its lower end. Find the force exerted by the table and the chain when half of its length has fallen on the table. The fallen part does not form heap.



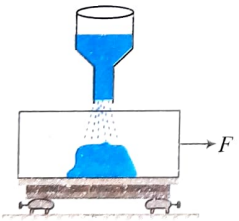
5. A smooth container filled with water moves with a velocity v_0. If water leaks at the base with a constant rate μ kg/s, find the velocity and acceleration of the container as the function of time. Assume M = initial mass of the system (container + water).



6. A rocket has initial velocity v_0 and mass m_0. If it travels in free space, find the velocity of the rocket as the function of its existing mass M. Assume that that rocket expels the fuel with a velocity u_{rel} with respect to itself.

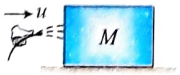


7. A cart is pulled with a constant force F with a constant velocity v on a smooth horizontal ground. A Hooper loads the cart with sands at a time of μ kg/s. Find the:



(i) External force F
(ii) The power delivered by the external force.
(iii) The rate of change of kinetic energy.

8. A block of mass M is kept on a smooth horizontal surface. A jet of water emerging from a nozzle of cross-section, it strikes the block horizontally with a speed u. Assuming ρ as the density of water.



(i) Find the maximum power applied on the block.
(ii) Calculate the thrust imparted by water jet on the block at this time.
(ii) Obtain an expression for the speed of the block as the function of time.

ANSWERS

1. (c) 2. 48 N 3. $\frac{(m_1 - m_0 + bt)g + bv_0}{m_0 + m_1 - bt}$ 4. $\frac{3}{2}mg$

5. $v = v_0$ and $a = 0$ 6. $v_0 - v_{rel} \ln \frac{M}{m_0}$

7. (i) μv (ii) μv^2 (iii) $\frac{\mu v^2}{2}$

8. (i) $\frac{4}{27}A\rho u^3$ (ii) $\frac{4}{9}A\rho u^2$ (iii) $\frac{u}{1+\frac{M}{A\rho ut}}$

Solved Examples

EXAMPLE 2.1

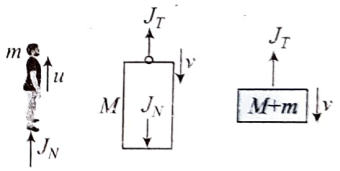
A man of mass m stands in a stationary lift of mass M, counterpoised by a mass $(M + m)$. The man suddenly jumps up with a velocity $u = \sqrt{2gh}$ relative to the ground. Find:

(i) the velocity of the man relative to the lift immediately after jumping.

(ii) the height to which the man rises relative to the floor.

Sol.

(i) While jumping, the man suddenly exerts a large impulsive force for a brief interval of time; therefore we can ignore gravitational force on our system (man + lift + counterpoise).



Applying impulse momentum equation for system elements

$$\vec{p}_{initial} + \vec{J} = \vec{p}_{final}$$

For man: $0 + J_N = mu$

or $J_N = mu$ ...(i)

For lift: $0 + (J_N - J_T) = Mv$

or $J_N - J_T = Mv$ ...(ii)

For counterpoise: $0 + J_T = (M + m)v$

or $J_T = (M + m)v$ ...(iii)

From Eqs. (i), (ii) and (iii), $v = \frac{mu}{(m + 2M)}$

But $u = \sqrt{2gh}$, hence $v = \frac{m\sqrt{2gh}}{(m + 2M)}$

where v is the common velocity attained by lift and counterpoise.

Velocity of the man with respect to lift,

$$\vec{u}_{m,l} = \vec{v}_m - \vec{v}_l$$

$$u_{m,l} = u - v = \sqrt{2gh} - \left(-\frac{m\sqrt{2gh}}{(m+2M)}\right)$$

$$\vec{u}_{m,l} = \frac{2(M+m)}{(2M+m)}\sqrt{2gh} \quad \text{(upward)}$$

(ii) After the man jumps, the acceleration of the system is

$$a = \frac{(M+m)-M}{(M+m)+M}g = \frac{mg}{(2M+m)}$$

Hence acceleration of the lift $a_l = \frac{mg}{(2M+m)}$ (upward)

The acceleration of the man w.r.t. lift;

$$\vec{a}_{m,l} = \vec{a}_m - \vec{a}_l$$

$$= (-g) - \left(\frac{mg}{2M+m}\right) = -\frac{2(M+m)}{(2M+m)}g$$

or $\vec{a}_{m,l} = \frac{2(M+m)}{(2M+m)}g$ (downward)

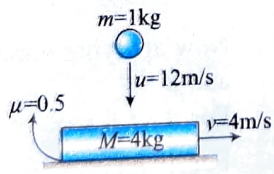
Let the height to which the man rises relative to the floor is h_{rel}. Using, $v_{m,l}^2 = u_{m,l}^2 + 2a_{m,l}h_{rel}$

$$0 = \left[\frac{2(M+m)}{(2M+m)}\sqrt{2gh}\right]^2 - 2\left[\frac{2(M+m)}{(2M+m)}g\right]h_{rel}$$

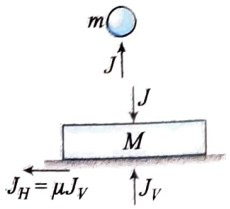
$$\Rightarrow h_{rel} = \frac{2(M+m)}{(2M+m)}h$$

EXAMPLE 2.2

A plank of mass $M = 4$ kg is, moving on a horizontal rough surface with coefficient of friction $\mu = 0.5$. A clay ball of mass $m = 1$ kg is dropped on the plank, hitting it with a vertical velocity of $u = 12$ m/s. At the instant of hit, the plank was having a horizontal velocity of $v = 4$ m/s. After an interval of Δt, another similar clay ball hits the plank and the system comes to rest immediately after the hit. Assume that the clay balls stick to the plank and collision is momentary. Find Δt. Take $g = 10$ m/s^2.



Sol. When clay ball hits the plank, the normal reaction between the ball and plank is impulsive in nature. As the normal reaction applied by ground on plank is impulsive, it makes the friction between plank and ground also impulsive.



During collision

During time interval Δt

Vertical impulse due to normal reaction

$$J_V = mu = 1\times12 = 12\,\text{kg m/s}$$

Horizontal impulse on the plank due to friction

$$J_H = \mu J_V = 0.5\times12 = 6\,\text{kg m/s}$$

Let the velocity of the 'plank + ball' system just after collision of 1st clay ball be v_1.

Using impulse-momentum equation,

$$(\vec{p}_{\text{initial}})_x + \vec{J} = (\vec{p}_{\text{final}})_x$$

$$Mv - J_H = (M+m)v_1$$

or $4\times4 - 6 = (4+1)v_1 \Rightarrow v_1 = 2$ m/s

After 1st collision the plank moves on rough surface for the time Δt, during this period its speed reduces to v_2. Now 2nd collision occurs, and just after collision the speed of the system becomes zero.

Again using impulse momentum equation

$$(M+m)v_2 - J_H = 0$$

$$\Rightarrow v_2 = \frac{J_H}{(M+m)} = \frac{6}{(4+1)} = \frac{6}{5}\,\text{m/s}$$

Note that the values of J_H and J_V remain same for second collision.

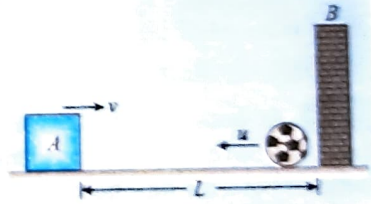
During time interval between two collisions (Δt), the friction causes retardation of plank. Now using

$$v = u + at \Rightarrow v_2 = v_1 - \mu g\Delta t$$

or $\Delta t = \frac{v_1 - v_2}{\mu g} = \frac{2-(6/5)}{0.5\times10} = \frac{4}{25}$ s

EXAMPLE 2.3

A heavy block A is made to move uniformly along a smooth floor with velocity $V = 2$ cm/s towards left. A ball of mass $m = 100$ g is projected towards the block with a velocity of $u = 200$ m/s. The ball keeps bouncing back and forth between the block A and a fixed wall B. After the ball has made 1000 collisions with the block and wall each, the distance between the block and the wall was found to be $L = 2.4$ m. Calculate the average force being experienced by the block due to collision at this instant. Consider each of the collisions are elastic and instantaneous.



Sol. Let us consider the collision of the ball with block.

(a) Just before collision of ball and block (b) Just after collision of ball and block (c) Collision of ball with wall

Using $e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{v - V}{V - (-u)}$

or $v = u + 2V$

After each collision of the ball with the block, speed of the ball will increases by $2V$. The collision of the ball with wall will not change the speed of the ball but changes its direction as all collisions are elastic.

After $n = 1000$ collisions, speed of the balls is

$$v' = u + 2nV = 200 + 2\times1000\times0.02 = 240\ \text{m/s}$$

The speed of the ball changes marginally during each collision but the change becomes significant after a large number of collisions. Change in the linear momentum of the ball during n^{th} collision with the body is

$$\Delta p = m[u + 2nV] - [-m(u + 2(n-1)V)]$$

$$= m[2u + 4nV - 2V]$$

As n is large (1000), we can neglect $2V$

$$\therefore \quad \Delta p = m[2u + 4nV]$$

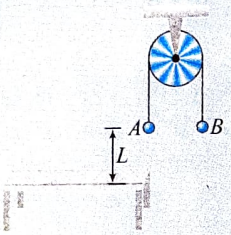
$$= \frac{100}{1000}\times[2\times200 + 4\times1000\times0.02] = 48\ \text{kg m/s}$$

At this instant frequency of collision of the ball with body A is $f = \dfrac{240\ \text{m/s}}{2.4\ \text{m}} = 100\ \text{s}^{-1}$

$$\therefore \quad \text{Force} = \Delta p \times f = 48\times100 = 4800\ \text{N}$$

EXAMPLE 2.4

Two particles A and B, of mass $3m$ and m respectively, are attached to the ends of a light inextensible string which passes over a smooth fixed pulley of negligible mass. After the system is released and the particle A falls through a distance L=1.6 m, it hits a horizontal inelastic table so that its speed is immediately reduced to zero. Assuming that the particle B never hits the table or the pulley. Find:



(a) the time for which the particle A is resting on the table after its collision with table and before it is jerked off,

(b) the speed of the particles just after the string become taut again.

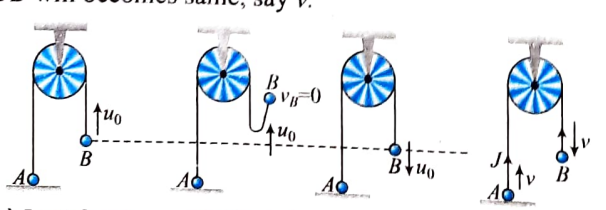
Sol. As both the particles are connected with a single string, they will have common acceleration. The acceleration of the particles before A hits the table is

$$a = \left(\frac{3m-m}{3m+m}\right)g = \frac{g}{2} = 5\ \text{m/s}^2$$

The speed of the particle A just before it hits the table is

$$u_0 = \sqrt{2aL} = \sqrt{2\times5\times1.6} = 4\ \text{m/s}$$

This is also the speed of particle B at this moment. Now particle A comes to rest and string becomes slack. Now particle B moves up with retardation g. Particle B stops and then starts falling down. When particle B acquires speed u_0 and comes to position shown in Fig. (b), the string is about to regain tension. As soon as the string becomes taut, the speed of both the particles A and B will becomes same, say v.



(a) Just after A hits the table (b) The string gets stack and the particle A moves under gravity (c) Just after string is taut

(a) Time for which particle rests on table = time of flight of particle B projected vertically up with speed u_0

$$= \frac{2u_0}{g} = \frac{2\times4}{10} = 0.8\ \text{s}$$

(b) The impulse, J applied by the string causes sudden change in velocities of particle A and particle B.

Using impulse-momentum equation

$$(\vec{p}_{\text{initial}})_y + \vec{J}_y = (\vec{p}_{\text{final}})_y$$

For particle A: $0 + J = 3mv$ ...(i)

For particle B: $-mu_0 + J = -mv$ ...(ii)

From Eqs. (i) and (ii); $v = \dfrac{u_0}{4} = \dfrac{4}{4} = 1\ \text{m/s}$

EXAMPLE 2.5

Two small identical balls A and B of mass 'm' are interconnected by an inextensible string of length L. The balls are resting on a frictionless horizontal surface, separated by a distance $l = 3L/5$. In this position, ball A is suddenly given a horizontal velocity v_0 perpendicular to the line joining the two balls.

(a) Find the speed of ball B just after the string becomes taut.

(b) Find the impulse of the tension in string when the string becomes taut.

(c) Find the steady tension in string much after the string has become taut.

Sol.

(a) The velocity of the balls will not change in the direction perpendicular to impulse. It means the velocity of ball B perpendicular to impulse will be zero.

The velocity of the ball A perpendicular to impulse

$$v_\perp = v_0 \sin\theta = \frac{3}{5}v_0$$

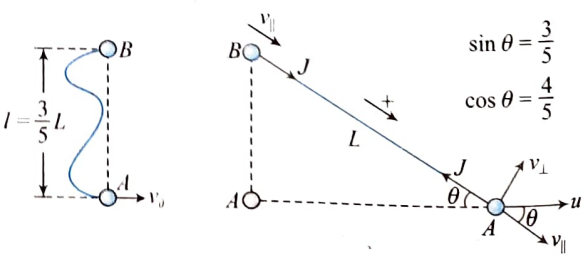
Now applying impulse momentum equation to the balls A and B

For B: $J = mv_\parallel$ ...(i)

For A: $mv_0\cos\theta - J = mv_\parallel$ ...(ii)

From (i) and (ii), $v_\parallel = \dfrac{v_0\cos\theta}{2} = \dfrac{2}{5}v_0$

Hence speed of ball B just after string become taut is $\dfrac{4}{5}v_0$

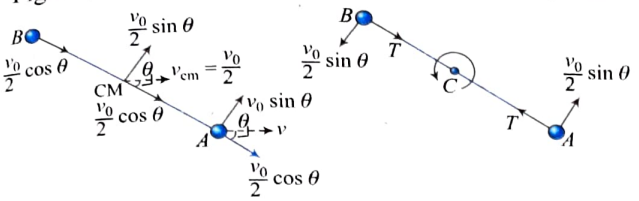


(a) (b)

(b) The impulse of the tension $J = mv_\parallel = \dfrac{mv_0\cos\theta}{2} = \dfrac{2}{5}mv_0$

(c) When string is jerked, the motion of two ball system is superposition of translation and rotation of two balls about

CM. In CM frame two balls system will rotate about CM. Figure (c) and (d) show ground frame and CM frame.

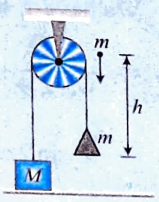


(c) From ground frame **(d) From CM frame**

$$\text{Tension in string } T=\frac{m\left(\frac{v_0\sin\theta}{2}\right)^2}{L/2}=\frac{mv_0^2\sin^2\theta}{2L}=\frac{9mv_0^2}{50L}$$

EXAMPLE 2.6

A block of mass M and a pan of mass m are fastened to the two ends of a string passing over a light smooth pulley. A small particle of mass m, is dropped into the pan from a height h. Assuming, the coefficient of restitution for the collision to be e, determine:

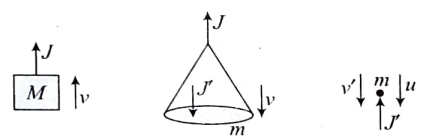


(a) the initial velocity, with which the mass M rises,

(b) the velocity of the particle after collision.

Sol.

(a) The particle apply impulsive reaction on the pan. It will make tension in string which is also impulsive. Let the block rise with the velocity 'v' and the velocity of the particle after collision be 'v''.



Impulse on the block | Impulse on the pan | Impulse on the particle

Writing impulse momentum equation

$$\vec{p}_{\text{initial}}+\vec{J}=\vec{p}_{\text{final}}$$

For block: $J=Mv$ ...(i)

For pan: $J'-J=mv$ ...(ii)

For particle: $mu-J'=mv'$ ...(iii)

From Eqs. (i), (ii) and (iii): $mu=(M+m)v+mv'$ ...(iv)

Writing restitution equation for the particle and pan

$$e=\frac{v-v'}{u-0}\quad\text{or}\quad v-v'=eu \qquad ...(v)$$

From Eqs. (iv) and (v), we get $v=\dfrac{mu(1+e)}{(M+2m)}$

Since $u=\sqrt{2gh}$, hence $v=\dfrac{m\sqrt{2gh}\,(1+e)}{(M+2m)}$ ...(vi)

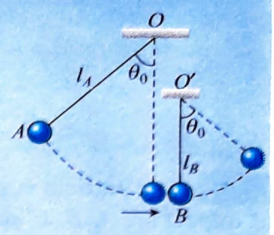
(b) In Eq. (v), substituting the value of v

$$v'=v-eu=\frac{m\sqrt{2gh}(1+e)}{(M+2m)}-e\sqrt{2gh}$$

$$v'=\sqrt{2gh}\left[\frac{m-e(M+m)}{M+2m}\right]$$

EXAMPLE 2.7

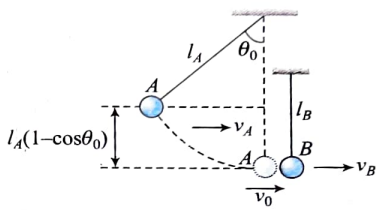
There are two pendulums with bobs having identical size and mass. The pendulum A is released from rest in the position as shown in the figure. If the maximum angle formed by cord BO' with vertical in the subsequent motion of bob B is equal to the angle θ_0 and the coefficient of restitution between sphere A and sphere B is e, find:



(a) the velocities of sphere A and sphere B just after collisions

(b) the ratio of lengths of pendulums l_B/l_A.

Sol. Let the velocity of bob A at lowest position be v_0. Using conservation of mechanical energy between position '1' and '2'.



$$\Delta K+\Delta U=0$$

$$\left(\frac{1}{2}mv_0^2-0\right)+\left[0-mgl_A(1-\cos\theta_0)\right]=0$$

which gives $v_0=\sqrt{2gl_A(1-\cos\theta_0)}$ ...(i)

(a) Collision of A and B:

Now using conservation of linear momentum of the system along horizontal direction.

$$m\cdot v_0+0=mv_A+mv_B$$

$$v_A+v_B=v_0 \qquad ...(ii)$$

Using Newton's restitution law

$$v_B-v_A=e(v_0-0)$$

$$v_B-v_A=ev_0 \qquad ...(iii)$$

Solving Eqs. (i) and (ii),

$$v_A=\frac{v_0}{2}(1-e)\quad\text{and}\quad v_B=\frac{v_0}{2}(1+e)$$

(b) As 'B' swings angle θ_0, hence

$$v_B=\sqrt{2gl_B(1-\cos\theta_0)} \qquad ...(iv)$$

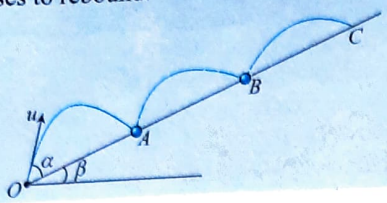
and $\dfrac{v_0}{2}(1+e)=\sqrt{2gl_B(1-\cos\theta_0)}$

$$\frac{\sqrt{2gl_A(1-\cos\theta_0)}}{2}(1+e)=\sqrt{2gl_B(1-\cos\theta_0)}$$

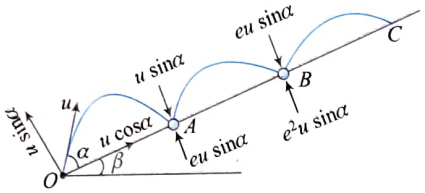
$$\Rightarrow\ \frac{l_B}{l_A}=\left(\frac{1+e}{2}\right)^2$$

EXAMPLE 2.8

A particle is projected from the base of a inclined plane with initial velocity u as shown in figure. Find the total time till the particle ceases to rebound.



Sol. The particle is projected from O. The successive points of impact are A, B, C etc. Initially at O the resolved parts of the velocity of projection 'u' along and perpendicular to the inclined plane are $u\cos\alpha$ and $u\sin\alpha$ respectively.



Due to impacts at A, B, C... the velocity components perpendicular to the inclined plane will only be altered while the parallel to inclined component will remain unaffected.

At A, B, C... the velocity components perpendicular to the inclined plane after impact will be

$eu\sin\alpha$, $e^2u\sin\alpha$, $e^3u\sin\alpha$.....

And times of flight of the successive parabolic path from O to A, A to B... will be

$$\frac{2u\sin\alpha}{g\cos\beta}, \frac{2eu\sin\alpha}{g\cos\beta}, \text{... etc.}$$

Rebounding ceases after an infinite number of jumps i.e., if T be the required time till the particle ceases to rebound then:

$$T = \frac{2u\sin\alpha}{g\cos\beta} + \frac{2eu\sin\alpha}{g\cos\beta} + \cdots\infty$$

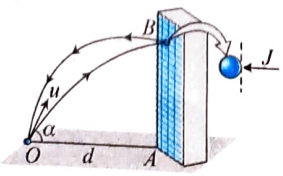
$$= \frac{2u\sin\alpha}{g\cos\beta}(1+e+e^2\cdots\infty)$$

$$= \frac{2u\sin\alpha}{g\cos\beta}\left(\frac{1}{1-e}\right)$$

EXAMPLE 2.9

A ball is projected with a velocity u at an elevation α from a point at distance d from a smooth vertical wall in a plane perpendicular to it. After rebounding from the wall, it returns to the point of projection. The coefficient of restitution between the ball and wall is e. Find the maximum distance d for which the ball can return to the point of projection.

Sol. The only force on the ball is the gravitational force, throughout its motion. During collision it experiences a horizontal impulse from the wall which changes the velocity of the ball in horizontal direction. We can use



$$u_y t - \frac{1}{2}gt^2 = v_y$$

Let total time of flight of the ball is T.

$$\therefore\quad 0 = u\sin\alpha T - \frac{1}{2}gT^2 \Rightarrow T = \frac{2u\sin\alpha}{g}$$

Due to collision with the wall at B, the normal component (i.e., horizontal component) of velocity is reversed and becomes e times.

Horizontal velocity before impact = $u\cos\alpha$ and horizontal velocity after impact = $eu\cos\alpha$

Time taken to reach the wall, $t_1 = \dfrac{d}{(u\cos\alpha)}$

and time taken to come back to O from B, $t_2 = \dfrac{d}{(eu\cos\alpha)}$

We have $t_1 + t_2 = T$

$$\Rightarrow \frac{d}{u\cos\alpha} + \frac{d}{eu\cos\alpha} = \frac{2u\sin\alpha}{g}$$

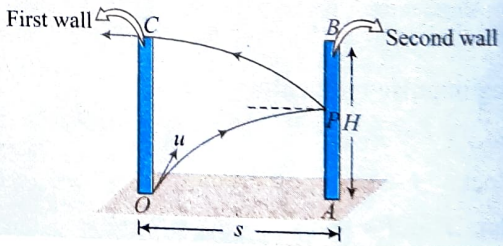
$$\Rightarrow u^2\sin 2\alpha = gd\left[1+\frac{1}{e}\right]$$

$$\text{or}\quad \sin 2\alpha = \frac{gd}{u^2}\left[1+\frac{1}{e}\right]$$

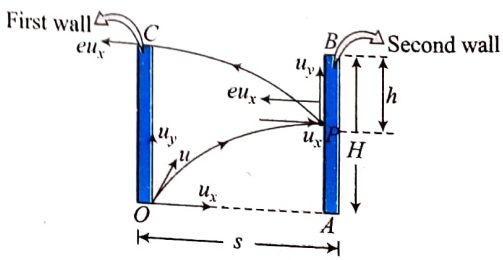
$$\text{As } \sin 2\alpha \le 1,\ \frac{gd}{u^2}\left[1+\frac{1}{e}\right] \le 1 \Rightarrow d \le \frac{eu^2}{g(1+e)}$$

EXAMPLE 2.10

Two parallel vertical walls of height 'H' stand on a horizontal plane. A ball is projected from one foot of one wall towards the other and after impact it just clears the top of the first wall. Prove that the point of impact with the second wall is at a depth $\dfrac{H}{(1+e)^2}$ below its top, where e is coefficient of restitution.



Sol. Walls OC and AB are distance 's' apart. Let the ball is projected from O with u_x and u_y as horizontal and vertical components of initial velocity. In the subsequent motion of the ball, let it strike the 2nd wall at P and afterwards just reach the top C of the first wall.



Let $PB = h$. Due to the impact with wall, only the horizontal component of velocity will be affected and therefore the horizontal velocity of the ball (in moving from P to C) $= eu_x$

Time taken by the ball (in moving from O to P), $t_1 = \frac{s}{u_x}$

and time taken by the ball (in moving from P to C), $t_2 = \frac{s}{eu_x}$

$\therefore$ Total time taken by the ball in moving from O to C

$$t = \frac{s}{u_x} + \frac{s}{eu_x} = \frac{s}{u_x}\left(1 + \frac{1}{e}\right) = \frac{s(1+e)}{eu_x} \quad \ldots(i)$$

Also as the ball just reaches C, its vertical velocity = 0.
Hence, considering the vertical motion of the ball from O to C:

$$0 = u_y - gt$$

where 't' is time taken by the ball to move from O to C

or $t = \frac{u_y}{g}$ which is same as given by Eq. (i)

$$\frac{u_y}{g} = \frac{s(1+e)}{eu_x} \Rightarrow \frac{s}{u_x} = \frac{eu_y}{g(e+1)} \quad \ldots(ii)$$

or $t_1 = \frac{eu_y}{g(e+1)}$ and $t_2 = \frac{u_y}{g(e+1)}$ $\ldots(iii)$

Also $0 = u_y^2 - 2gH$ or $u_y^2 = 2gH$ $\ldots(iv)$

and for vertical motion from O to P we have

$$(H-h) = u_y t_1 - \frac{1}{2} g t_1^2$$

where t_1 = time from O to P

or $$(H-h) = u_y\left(\frac{eu_y}{g(e+1)}\right) - \frac{g}{2}\left(\frac{eu_y}{g(e+1)}\right)^2 \quad \text{(Using Eq.(iii))}$$

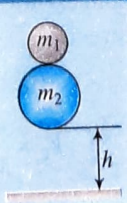
$$= \frac{eu_y^2}{g(e+1)}\left[1 - \frac{e}{2(e+1)}\right]$$

$$= \frac{e2gH}{g(e+1)}\left[\frac{e+2}{2(e+1)}\right] = \frac{e(e+2)H}{(e+1)^2} \quad \text{(Using Eq.(iv))}$$

or $$h = H - \frac{e(e+2)H}{(e+1)^2} = \frac{H}{(e+1)^2}$$

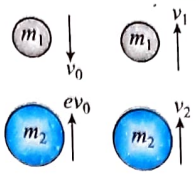
EXAMPLE 2.11

Two spherical bodies of mass m_1 and m_2 fall freely through a distance h, before the body m_2 collides with the ground. If the coefficient of restitution of all collisions is e, find the velocity of m_1 just after it collides with m_2. [Use $m_1 = m_2$, $e = 1/2$]



Sol. As the bodies m_1 and m_2 fall freely through a distance h, m_2 collides with ground with a velocity $v_0 = \sqrt{2gh}$.

Hence m_2 bounces up with a velocity ev_0 just after the collision with the ground. Then it collides with the freely falling body m_1. Just after the collision, let us assume that v_1 and v_2 are the velocities of m_1 and m_2 respectively.



Ignoring the impulses due to the weights of the bodies during the period of collision, let us conserve the momentum of the system $(m_1 + m_2)$.

Conservation of linear momentum

$$-m_1 v_0 + m_2 ev_0 = m_1 v_1 + m_2 v_2 \quad \ldots(i)$$

Newton's impact formula, $-e[u_1 - u_2] = v_1 - v_2$

where $u_1 = -v_0$ and $u_2 = ev_0$

Then $$-e[(-v_0) - (ev_0)] = v_1 - v_2 \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we have

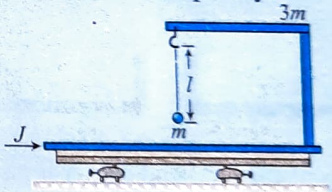
$$v_1 = \frac{-(m_1 - em_2)}{m_1 + m_2} v_0 + \frac{(1+e)m_2}{m_1 + m_2} ev_0$$

Putting $m_1 = m_2$, $e = 1/2$ and $v_0 = \sqrt{2gh}$, we have

$$v_1 = \sqrt{\frac{gh}{32}} \uparrow$$

EXAMPLE 2.12

A small bob of mass m is suspended by a light inextensible string of length l, inside a toy car of mass $3m$ which can move on a smooth horizontal surface. Initially the car and bob is at rest. A sharp horizontal impulse $6m\sqrt{\frac{gl}{11}}$ is applied on the car and it starts moving. Find the maximum angle θ_0 that the string will make with the vertical subsequently.

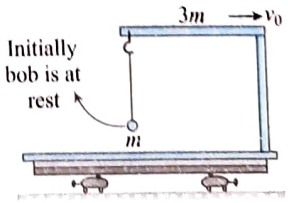


Sol. The impulse will change the linear momentum of the car only. There will not be change in state of the bob due to the impulse.

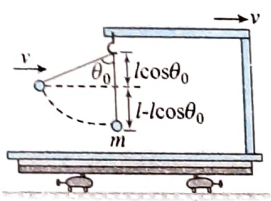
$$J = 3mv_0 \Rightarrow v_0 = \frac{J}{3m}$$

or $$v_0 = \frac{6m\sqrt{\frac{gl}{11}}}{3m} = 2\sqrt{\frac{gl}{11}} \quad \ldots(i)$$

After the motion of the car, the bob will starts moving in circular path as seen from car.



(i) Just after impulse



(ii) At the time of maximum deflection of the bob

The string will make maximum angle with the vertical when there is no relative motion between the particle and the car and both move horizontally with common velocity v.

Conserving linear momentum of the system (car + bob) just after impulse and at the time of maximum reflection of the bob.

$$mv_0 + 0 = mv + 3mv \Rightarrow v = \frac{v_0}{4} \quad ...(ii)$$

Let the maximum angle made by string with vertical is θ_0. Now using C.O.M.E for the system (car + bob)

$$\Delta K + \Delta U = 0$$

$$\left[\left(\frac{1}{2}(3m)v^2 + \frac{1}{2}mv^2\right) - \frac{1}{2}(3m)v_0^2\right] + mgl(1-\cos\theta) = 0 \quad ...(iii)$$

From Eqs. (ii) and (iii),

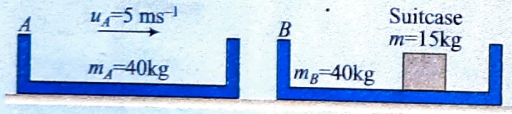
$$\frac{11}{8}v_0^2 = gl(1-\cos\theta_0)$$

$$\Rightarrow 1-\cos\theta_0 = \frac{11v_0^2}{8gl}$$

$$\Rightarrow \cos\theta_0 = 1 - \frac{11}{8gl}\cdot\frac{4}{11gl} = \frac{1}{2} \text{ or } \theta_0 = 60°$$

EXAMPLE 2.13

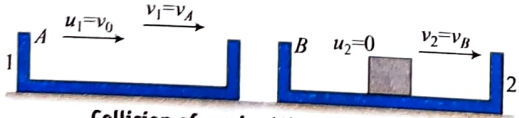
An empty luggage carrier A of mass $M = 40$ kg slide with a velocity $v_0 = 5$ m s^{-1} on horizontal floor (without friction) hits an identical carrier B containing $m = 15$ kg suitcase. The impact causes the suitcase to slide on the floor of carrier B and collide with the left wall of carrier B. Knowing that the coefficient of restitution between the two carriers is 4/5 and that the coefficient of restitution between the suitcase and the wall of carrier B is 2/9. Assume no friction anywhere. Find:



(a) the velocities of carrier A and carrier B just after collision,

(b) the velocity of carrier B after the suitcase hits the wall for the first time.

Sol. If we consider carrier A, carrier B and suitcase as system, there is no external force acting on system in horizontal direction. It means the linear momentum of the system will be conserved in horizontal direction.



Collision of carrier 'A' with carrier 'B'

Let us consider the collision of carrier A and B:

Let velocities of carrier A and B just after collision be v_A and v_B respectively.

$$m_A u_1 + m_B u_2 = m_A v_1 + m_B v_2$$

$$40 \times 5 + 40 \times 0 = 40v_A + 40v_B$$

$$v_A + v_B = 5 \quad ...(i)$$

Using Newton's restitution equation

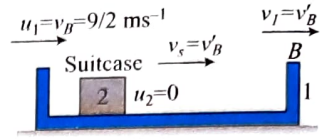
$$v_B - v_A = (4/5)(5-0)$$

$$v_B - v_A = 4 \quad ...(ii)$$

Solving Eqs. (i) and (ii), we get

$$v_A = \frac{1}{2}\text{m s}^{-1},\ v_B = \frac{9}{2}\text{m s}^{-1}$$

As there is no friction, the suitcase will not slide with respect to ground, hence velocity of suitcase with respect to the ground will be zero. Now left wall of carrier 'B' be collide with suitcase



Collision of suitcase with left wall of carrier 'B'

Now consider collision of suitcase with carrier. Using conservation of linear momentum of the system along horizontal direction.

$$40 \times \frac{9}{2} + 0 = 40v'_B + 15v_s$$

Here v'_B and v_s are the velocities of carrier B and suitcase just after collision

$$180 = 40v'_B + 15v_s$$

$$36 = 8v'_B + 3v_s \quad ...(i)$$

Using Newton's restitution equation

$$v_s - v'_B = \frac{2}{9}(v_B - 0) \Rightarrow v_s - v'_B = \frac{2}{9}\left(\frac{9}{2} - 0\right)$$

$$v_s - v'_B = 1 \quad ...(ii)$$

After solving $v'_B = 3\text{m s}^{-1}$ and $v_s = 4$ m s^{-1}

EXAMPLE 2.14

A ball of mass m is pushed with a horizontal velocity v_0 from one end of a sledge of mass M and length l. If the ball stops after its first collision with the sledge, find the speeds of the ball and sledge after the second collision of the ball with the sledge. Assume no friction anywhere.

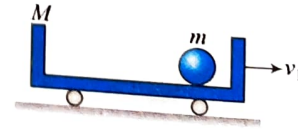
Sol. If we take the ball and the sledge as system. There is no external force acting on the system in horizontal direction. Hence linear momentum of the system can be conserved in horizontal direction.

$$(\vec{p}_{\text{initial}})_x = (\vec{p}_{\text{final}})_x$$

Considering the 1st collision of the ball and right wall of sledge. Let velocity of the sledge just after 1st collision be V_1. Then using conservation of linear momentum of the system along horizontal direction.

$$mv_0 + M \times 0 = m \times 0 + MV_1$$

or $$V_1 = \frac{m}{M}v_0 \quad ...(i)$$

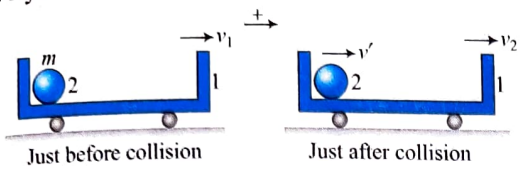


Using coefficient of restitution equation

$$e = \frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}}$$

$$e = \frac{V_1}{v_0} = \frac{m}{M} \text{ (from Eq. (i))} \qquad \ldots(ii)$$

After 1st collision the ball comes to rest, as there is no friction, the ball will not slide with respect to sledge. Hence velocity of ball with respect to the ground will be zero just before 2nd collision. But the sledge moves with velocity V_1. Now left wall of the sledge collide with the ball. Let us consider the velocities of the ball and sledge just after second collision be v' and V_2 respectively.



Just before collision Just after collision

Now applying conservation of linear momentum to the system in horizontal direction for 2nd collision.

$$m \times 0 + MV_1 = mv' + MV_2$$

or $mv' + MV_2 = mv_0 \qquad \ldots(iii)$

Now applying restitution equation again

$$e = \frac{m}{M} = \frac{v' - V_2}{V_1 - 0} \quad \text{(from Eq. (ii))}$$

$$\Rightarrow v' - V_2 = \frac{m}{M} V_1 \qquad \ldots(iv)$$

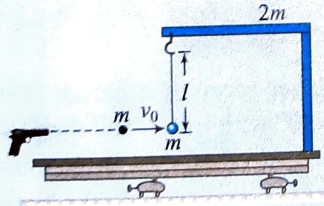
From Eqs. (i) and (iv), $v' - V_2 = \left(\frac{m}{M}\right)^2 v_0 \qquad \ldots(v)$

Solving Eqs. (iii) and (v) we get

$$v' = \frac{m}{M} v_0 \text{ and } V_2 = \frac{mv_0[M-m]}{M^2}$$

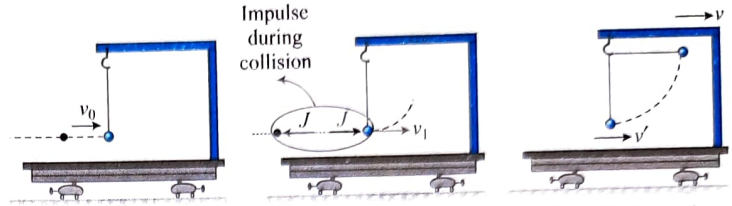
EXAMPLE 2.15

A small bob of mass m is suspended by a slight inextensible sting of length l, which is attached with a roof of a trolley of mass $2m$, as shown in figure. Initially trolley is resting on a smooth horizontal surface. A bullet of mass m moving with velocity v_0 collides with the bob and sticks to the bob. Find the minimum velocity of the bullet so that the combined body (bob + bullet) can touch the roof of the trolley.



Sol. There is no external force acting on the system in horizontal direction and impulse during collision of bullet and bob is horizontal, also internal to the system. Hence we can conserve the linear momentum of the system in horizontal direction.

The combined body (bob + bullet) just manages to touch the roof. It means that velocity of combined body (bob + bullet) relative to the car is zero when the string gets horizontal. Let this common velocity be v.



Just before collision During collision Final position of combined body

Conservation of linear momentum in horizontal direction gives

$$(m + m + 2m)v = mv_0 \Rightarrow v = \frac{v_0}{4} \qquad \ldots(i)$$

During collision of the bullet and the bob, there is loss in mechanical energy due to impulse. Let velocity of (bullet + bob) be v_1 just after collision.

Again using conservation of linear momentum in horizontal direction.

$$(m + m)v_1 = mv_0 \Rightarrow v_1 = \frac{v_0}{2}$$

KE just after collision $K = \frac{1}{2}(2m)\left(\frac{v_0}{2}\right)^2 = \frac{mv_0^2}{4}$

The mechanical energy of the system will be conserved after collision. Hence using conservation of M.E. theorem

$$\Delta K + \Delta U = 0$$

$$\left(\frac{1}{2}(4m)v^2 - \frac{1}{4}mv_0^2\right) + 2mgl = 0$$

Using Eq. (i) $\left(2m\left(\frac{v_0}{4}\right)^2 - \frac{mv_0^2}{4}\right) + 2mgl = 0$

$$\Rightarrow v_0 = 4\sqrt{gL}$$

Exercises

Single Correct Answer Type

1. A ball impinges directly on another ball at rest. The first ball is brought to rest by the impact. If half of the kinetic energy is lost by the impact, the value of coefficient of restitution is

(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{3}}$

(3) $\frac{1}{\sqrt{2}}$ (4) $\frac{\sqrt{3}}{2}$

2. A particle of mass 2 kg moving with a velocity of 3 m/s is acted upon by a force which changes its direction of motion by an angle of 90° without changing its speed. What is the magnitude of impulse experienced by the particle?

(1) 6 N s (2) 2 N s

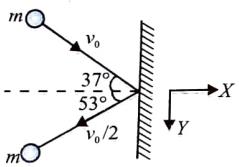
(3) $3\sqrt{2}$ N s (4) $6\sqrt{2}$ N s

3. A body of mass 3 kg moving with a velocity of 4 m/s towards left collides head on with a body of mass 4 kg moving in opposite direction with a velocity of 3 m/s. After collision the two bodies stick together and move with a common velocity which is

(1) zero (2) 12 m/s towards left

(3) 12 m/s towards right (4) $\frac{12}{7}$ m/s towards left

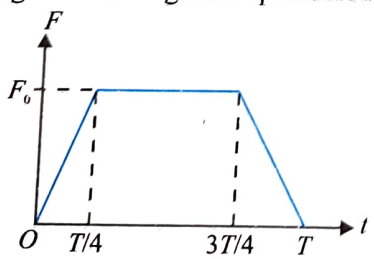
4. A ball of mass m moving with velocity v_0 collides with a wall as shown in figure. After impact it rebounds with a velocity $v_0/2$. The component of impulse acting on the ball along the wall is



(1) $\frac{mv_0}{2}\hat{j}$ (2) $-\frac{mv_0}{2}\hat{j}$

(3) $-\frac{mv_0}{5}\hat{j}$ (4) none of these

5. A particle of mass m moving with a velocity v makes an elastic one dimensional collision with a stationary particle of mass m establishing a contact with it for extremely small time T. Their force of contact increases from zero to F_0 linearly in time $T/4$, remains constant for a further time $T/2$ and decreases linearly from F_0 to zero in further time $T/4$ as shown in figure. The magnitude possessed by F_0 is



(1) $\frac{mu}{T}$ (2) $\frac{2mu}{T}$

(3) $\frac{4mu}{3T}$ (4) $\frac{3mu}{4T}$

6. A ball of mass m moving with speed u undergoes a head-on elastic collision with a ball of mass nm initially at rest. The fraction of the incident energy transferred to the second ball is

(1) $\frac{n}{1+n}$ (2) $\frac{n}{(1+n)^2}$

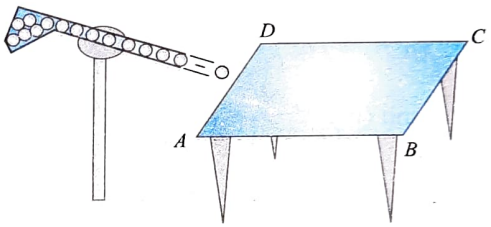
(3) $\frac{2n}{(1+n)^2}$ (4) $\frac{4n}{(1+n)^2}$

7. In the above question, the ratio of the kinetic energies of the first ball to the second ball after collision is

(1) $\frac{(1-n)^2}{2n}$ (2) $\frac{(1-n)^2}{4n}$

(3) $\frac{(1+n)^2}{2n}$ (4) $\frac{(1+n)^2}{4n}$

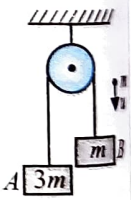
8. A gun which fires small balls of mass 20 g is firing 20 balls per second on the smooth horizontal table surface $ABCD$. If the collision is perfectly elastic and balls are striking at the centre of table with a speed of 5 m/s at an angle of 60° with the vertical just before collision, then force exerted by one of the legs on ground is (assume total weight of the table is 0.2 kg)



(1) 0.5 N (2) 1 N

(3) 0.25 N (4) 0.75 N

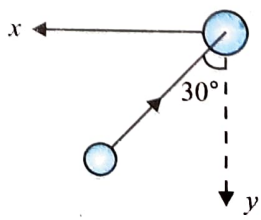
9. A system of two blocks A and B are connected by an inextensible massless string as shown in figure. The pulley is massless and frictionless. Initially, the system is at rest. A bullet of mass 'm' moving with a velocity 'u' as shown hits block 'B' and gets embedded into it. The impulse imparted by tension force to the block of mass $3m$ is



(1) $\frac{5mu}{4}$ (2) $\frac{4mu}{5}$

(3) $\frac{2mu}{5}$ (4) $\frac{3mu}{5}$

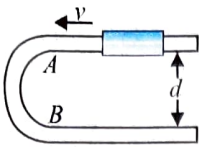
10. A hockey player receives a corner shot at a speed of 15 m/s at angle 30° with y-axis and then shoots the ball along x-axis with the speed 30 m/s. If the mass of the ball is 150 g and it remains in contact with the hockey stick for 0.01 s, the force exerted on the ball along x-axis is



(1) 281 N (2) 187.5 N

(3) 562.5 N (4) 375N

11. A U-shaped wire has a semicircular bending between A and B as shown in figure. A bead of mass m moving with uniform speed v through a wire enters the semicircular bend at A and leaves at B with velocity $v/2$ after time T. The average force exerted by the bead on the part AB of the wire is



(1) 0 (2) $\frac{3mv}{2T}$

(3) $\frac{3mv}{T}$ (4) none of these

12. A force exerts an impulse I on a particle changing its speed from initial velocity u to final velocity $2u$. The applied force and the initial velocity are oppositely oriented along the same line. The work done by the force is

(1) $\frac{3}{2}Iu$ (2) $\frac{1}{2}Iu$

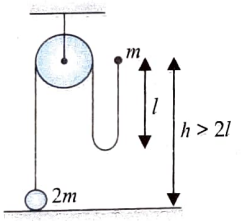
(3) Iu (4) $2Iu$

13. A particle of mass m is made to move with uniform speed v_0 along the perimeter of a regular hexagon, inscribed in a circle of radius R. The magnitude of impulse applied at each corner of the hexagon is

(1) $2mv_0 \sin\frac{\pi}{6}$ (2) $mv_0 \sin\frac{\pi}{6}$

(3) $mv_0 \sin\frac{\pi}{3}$ (4) $2mv_0 \sin\frac{\pi}{3}$

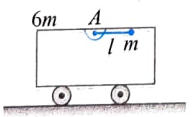
14. In the figure shown, the heavy ball of mass $2m$ rests on the horizontal surface and the lighter ball of mass m is dropped from a height $h > 2l$. At the instant the string gets taut, the upward the velocity of the heavy ball will be



(1) $\frac{2}{3}\sqrt{gl}$ (2) $\frac{4}{3}\sqrt{gl}$

(3) $\frac{1}{3}\sqrt{gl}$ (4) $\frac{1}{2}\sqrt{gl}$

15. In the figure shown, the cart of mass $6m$ is initially at rest. A particle of mass m is attached to the end of the light rod which can rotate freely about A. If the rod is released from rest in a horizontal position shown, determine the velocity v_{rel} of the particle with respect to the cart when the rod is vertical.



(1) $\sqrt{\frac{14}{3}gl}$ (2) $\sqrt{\frac{7}{6}gl}$

(3) $\sqrt{\frac{7}{3}gl}$ (4) $\sqrt{\frac{8}{3}gl}$

16. Two identical particles collide in air inelastically. One moves horizontally and the other moves vertically with equal speed just before collision. The fractional loss in kinetic energy of the system of particles is equal to

(1) $1/\sqrt{2}$ (2) 1/2

(3) $\sqrt{3}/2$ (4) none of these

17. A particle at rest is constrained to move on a smooth horizontal surface. Another identical particle hits the stationary particle with a velocity v at an angle $\theta = 60°$ with horizontal. If the particles move together, the velocity of the combination just after the impact is equal to

(1) v (2) $v/2$

(3) $\sqrt{3}v/4$ (4) $v/4$

18. A particle loses 25% of its energy during collision with another identical particle at rest. the coefficient of restitution will be

(1) 0.25 (2) $\sqrt{2}$

(3) $\frac{1}{\sqrt{2}}$ (4) 0.5

19. A body of mass 3 kg collides elastically with another body at rest and then continues to move in the original direction with one half of its original speed. What is the mass of the target body?

(1) 1 kg (2) 1.5 kg

(3) 2 kg (4) 5 kg

20. Two particles of equal masses moving with same speed collide perfectly inelastically. After the collision the combined mass moves with half of the speed of the individual masses. The angle between the initial momenta of individual particle is

(1) 60° (2) 90°

(3) 120° (4) 45°

21. A ping-pong ball of mass m is floating in air by a jet of water emerging out of a nozzle. If the water strikes the ping-pong ball with a speed v and just after collision water falls dead, the rate of flow of water in the nozzle is equal to

(1) $\frac{2mg}{V}$ (2) $\frac{mV}{g}$

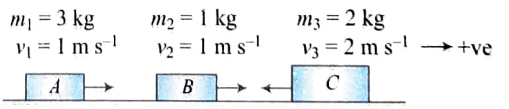
(3) $\frac{mg}{V}$ (4) none of these

22. A ball collides with a fixed inclined plane of inclination θ after falling through a distance h. If it moves horizontally just after the impact, the coefficient of restitution is

(1) $\tan\theta$ (2) $\tan^2\theta$

(3) $\cot\theta$ (4) $\cot^2\theta$

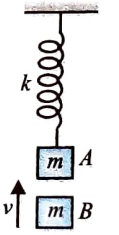
23. Three carts move on a frictionless track with masses and velocities as shown. The carts collide and stick together after successive collisions. Find the total magnitude of the impulse experienced by A.



(1) 1 Ns (2) 2 Ns

(3) 3 Ns (4) 4 Ns

24. Block A is hanging from a vertical spring and is at rest. Block B strikes the block A with velocity v and sticks to it. Then the value of v for which the spring just attains natural length is



(1) $\sqrt{\frac{60\,mg^2}{k}}$ (2) $\sqrt{\frac{6\,mg^2}{k}}$

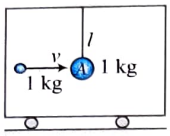
(3) $\sqrt{\frac{10\,mg^2}{k}}$ (4) none of these

25. A pendulum consists of a wooden bob of mass m and length l. A bullet of mass m_1 is fired towards the pendulum with a speed v_1. The bullet emerges out of the bob with a speed of $v_1/3$ and the bob just completes motion along a vertical circle, then v_1 is

(1) $\frac{3m}{2m_1}\sqrt{5gl}$ (2) $\frac{m_1}{m}\sqrt{5gl}$

(3) $\frac{2m}{3m_1}\sqrt{5gl}$ (4) $\frac{m_1}{m}\sqrt{gl}$

26. In the arrangements shown in the figure masses of each ball is 1 kg and mass of trolley is 4 kg. In the figure, shell of mass 1 kg moving horizontally with velocity $v = 6\ ms^{-1}$ collides with the ball and get stuck to it then its maximum deflection of the thread (length 1.5 m) with vertical is



(1) 53° (2) 37°

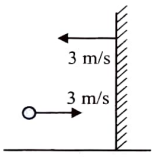
(3) 30° (4) 60°

27. A 3000 kg space probe is moving in a gravity free space at a constant velocity of 300 m/s. To change the direction of space probe, rockets have been fired in a direction perpendicular to the direction of initial motion of the space probe, the rocket firing exerts a thrust of 4000 N for 225 s. The space probe will turn by an angle of (neglect the mass of the rockets fired)

(1) 30° (2) 60°

(3) 45° (4) 37°

28. A highly elastic ball moving at a speed of 3 m/s approaches a wall moving towards it with a speed of 3 m/s. After the collision, the speed of the ball will be



(1) 3 m/s (2) 6 m/s

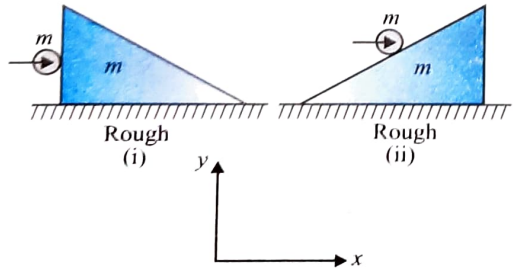
(3) 9 m/s (4) zero

29. Two identical billiard balls undergo an oblique elastic collision. Initially, one of the balls is stationary. If the initially stationary ball after collision moves in a direction which makes an angle of 37° with direction of initial motion of the moving ball, then the angle through which initially moving ball will be deflected is

(1) 37° (2) 60°

(3) 53° (4) > 53°

30. A ball of mass m collides horizontally with a stationary wedge on a rough horizontal surface, in the two orientations as shown. Neglect friction between the ball and the wedge. The students comment on the system of ball and wedge in these situations



Saurav: Momentum of the system in x-direction will change by significant amount in both the cases.

Rahul: There are no impulsive external forces in x-direction in both cases, hence the total momentum of the system in x-direction can be treated as conserved in both cases.

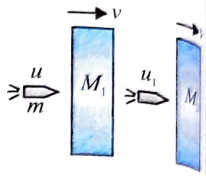
(1) Saurav is incorrect and Rahul is correct

(2) Saurav is correct and Rahul is incorrect

(3) Both are correct

(4) Both are incorrect

31. A 20 g bullet pierces through a plate of mass $M_1 = 1$ kg and then comes to rest inside a second plate of mass $M_2 = 2.98$ kg as shown in figure. It is found that the two plates, initially at rest, now move with equal velocities. Find the percentage loss in the initial velocity of the bullet when it is between M_1 and M_2. Neglect any loss of material of the plates due to the action of bullet.



(1) 50% (2) 25%

(3) 100% (4) 75%

32. A body X with a momentum p collides with another identical stationary body Y one dimensionally. During the collision, Y gives an impulse J to body X. Then coefficient of restitution is

(1) $\frac{2J}{p} - 1$ (2) $\frac{J}{p} + 1$

(3) $\frac{J}{p} - 1$ (4) $\frac{J}{2p} - 1$

33. A smooth sphere is moving on a horizontal surface with velocity vector $2\hat{i} + 2\hat{j}$ immediately before it hits a vertical wall. The wall is parallel to $\hat{j}$ vector and the coefficient of restitution between the sphere and the wall is $e = 1/2$. The velocity vector of the sphere after it hits the wall is

(1) $\hat{i} - \hat{j}$ (2) $-\hat{i} + 2\hat{j}$

(3) $-\hat{i} - \hat{j}$ (4) $2\hat{i} - \hat{j}$

34. A ball is dropped from a height of 45 m from the ground. The coefficient of restitution between the ball and the ground is 2/3. What is the distance travelled by the ball in 4th second of its motion? Assume negligible time is spent in rebounding. Let $g = 10\ m\ s^2$.

(1) 5 m (2) 20 m

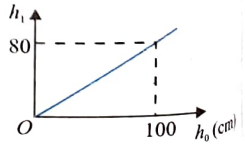
(3) 15 m (4) 10 m

35. A ball falls vertically onto a floor with momentum p, and then bounces repeatedly. If the coefficient of restitution is e, then the total momentum imparted by the ball on the floor till the ball comes to rest is

(1) $p(1 + e)$ (2) $\frac{p}{1 - e}$

(3) $p\left(1 + \frac{1}{e}\right)$ (4) $p\left(\frac{1 + e}{1 - e}\right)$

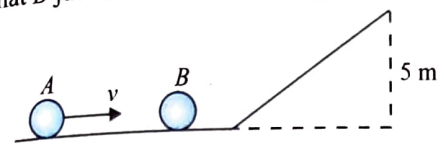
36. A ball released from a height h_0 above a horizontal surface rebounds to a height h_1 after one bounce. The graph that relates h_0 to h_1 is shown in figure. If the ball (of the mass m) was dropped from an initial height h and made three bounces, the kinetic energy of the ball immediately after the third impact with the surface was



(1) $(0.8)^3\ mgh$ (2) $(0.8)^2\ mgh$

(3) $0.8\ mg(h/3)$ (4) $[1-(0.8)^3]mgh$

37. Two identical balls, of equal masses A and B, are lying on a smooth surface as shown in figure. Ball A hits ball B (which is at rest) with a velocity $v = 16$ m/s. What should be the minimum value of coefficient of restitution between A and B so that B just reaches the highest point of inclined plane?



(1) $\frac{2}{3}$ (2) $\frac{1}{4}$

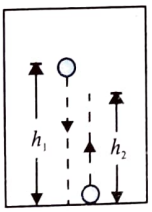
(3) $\frac{1}{2}$ (4) $\frac{1}{3}$

38. A particle at rest is constrained to move on a smooth horizontal surface. Another identical particle hits the fractional particle with a velocity v at an angle $\theta = 60°$ with horizontal. If the particles move together, the velocity of the combination just after impact is equal to

(1) v (2) $\frac{v}{2}$

(3) $\frac{\sqrt{3}v}{4}$ (4) $\frac{v}{4}$

39. A ball of mass m is released from rest relative to elevator at a height h_1 above the floor of the elevator. After making collision with the floor of the elevator it rebounces to height h_2. The coefficient of restitution for collision is e. For this situation, mark the correct statement(s).



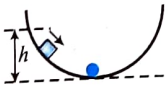
(1) If elevator is moving down with constant velocity v_0, then $h_2 = e^2h_1$.

(2) If elevator is moving down with constant velocity v_0, then $h_2 = e^2h_1 - \frac{v_0^2}{2g}$.

(3) If elevator is moving down with constant velocity v_0, then impulse imparted by floor of the elevator to the ball is $m(\sqrt{2gh_2} + \sqrt{2gh_1} + 2v_0)$ in the upward direction.

(4) If elevator is moving with constant acceleration of $g/4$ in upward direction, then it is not possible to determine a relation between h_1 and h_2 from the given information.

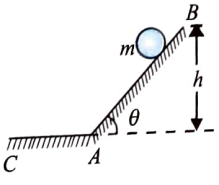
40. A block of mass m starts from rest and slides down a frictionless semi-circular track from a height h as shown. When it reaches the lowest point of the track, it collides with a stationary piece of putty also having mass m. If the block and the putty stick together and continue to slide, the maximum height that the block–putty system could reach is



(1) $h/4$ (2) $h/2$

(3) h (4) independent of h

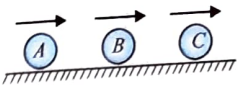
41. A particle of mass m comes down on a smooth inclined plane from point B at a height of h from rest. The magnitude of change in momentum of the particle between position A (just before arriving on horizontal surface) and C (assuming the angle of inclination of the plane as θ with respect to the horizontal) is



(1) 0 (2) $2m\sqrt{(2gh)}\sin\theta$

(3) $2m\sqrt{(2gh)}\sin\left(\frac{\theta}{2}\right)$ (4) $2m\sqrt{(2gh)}$

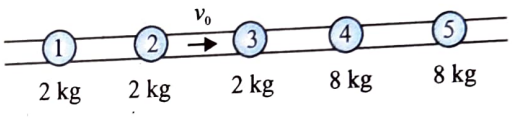
42. Three balls A, B and C of masses 2 kg, 4 kg and 8 kg, respectively, move along the same straight line and in the same direction, with velocities 4 m/s, 1 m/s and 3/4 m/s. If A collides with B and subsequently B collides with C, find the velocity of ball A and ball B after collision, taking the coefficient of restitution as unity.



(1) $v_A = 3, v_B = 9/4$ (2) $v_A = 0, v_B = 3$

(3) $v_A = 3, v_B = 0$ (4) $v_A = 0, v_B = 0$

43. Five balls are placed one after the other along a straight line as shown in the figure. Initially, all the balls are at rest. Then the second ball has been projected with speed v_0 towards the third ball. Mark the correct statements. (Assume all collisions to be head-on and elastic.)



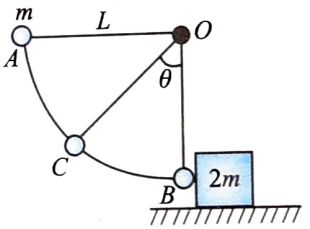
(1) Total number of collisions in the process is 5

(2) Velocity of separation between the first and fifth ball after the last possible collision is v_0

(3) Finally, three balls remain stationary

(4) All of the above

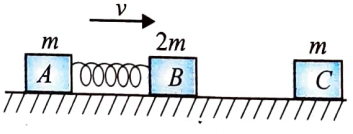
44. A ball of mass m is attached to a cord of length L, pivoted at point O, as shown in figure. The ball is released from rest at point A, swings down and makes an inelastic collision with a block of mass $2m$ kept on a rough horizontal floor. The coefficient of restitution of collision is $e = 2/3$ and coefficient of friction between block and surface is μ. After collision, the ball comes momentarily to rest at C when cord makes an angle of θ with the vertical and block moves a distance of $3L/2$ on rough horizontal floor before stopping. The values of μ and θ are, respectively,



(1) $\frac{50}{243}, \cos^{-1}\left(\frac{80}{81}\right)$ (2) $\frac{50}{81}, \cos^{-1}\left(\frac{80}{81}\right)$

(3) $\frac{2}{81}, \cos^{-1}\left(\frac{80}{243}\right)$ (4) $\frac{2}{243}, \cos^{-1}\left(\frac{80}{243}\right)$

45. Two blocks A and B of masses m and $2m$, respectively, are connected with the help of a spring having spring constant, k as shown in figure. Initially, both the blocks are moving with same velocity v on a smooth horizontal plane with the spring in its natural length. During their course of motion, block B makes an inelastic collision with block C of mass m which is initially at rest. The coefficient of restitution for the collision is 1/2. The maximum compression in the spring is



(1) $\sqrt{\frac{2m}{k}}\,v$ (2) will never be attained

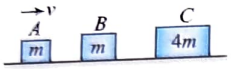
(3) $\sqrt{\frac{m}{12k}}\,v$ (4) $\sqrt{\frac{m}{6k}}\,v$

46. After a totally inelastic collision, two objects of the same mass and same initial speeds are found to move together at half of their initial speeds. The angle between the initial velocities of the objects is

(1) 120° (2) 60°

(3) 150° (4) 45°

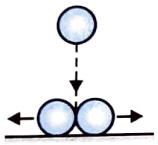
47. Three blocks are initially placed as shown in the figure. Block A has mass m and initial velocity v to the right. Block B with mass m and block C with mass $4m$ are both initially at rest. Neglect friction. All collisions are elastic. The final velocity of blocks A is



(1) $0.6v$ to the left (2) $1.4v$ to the left

(3) v to the left (4) $0.4v$ to the right

48. In the figure shown, the two identical balls of mass M and radius R each, are placed in contact with each other on the frictionless horizontal surface. The third ball of mass M and radius $R/2$, is coming down vertically and has a velocity = v_0 when it simultaneously hits the two balls and itself comes to rest. Then, each of the two bigger balls will move after collision with a speed equal to



(1) $4v_0/\sqrt{5}$ (2) $2v_0/\sqrt{5}$

(3) $v_0/\sqrt{5}$ (4) none

49. In the above question, suppose that the smaller ball does not stop after collision, but continues to move downwards with a speed = $v_0/2$, after the collision. Then, the speed of each bigger ball after collision is

(1) $4v_0/\sqrt{5}$ (2) $2v_0/\sqrt{5}$

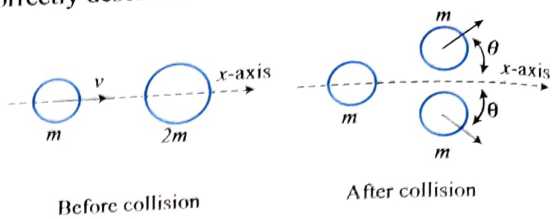
(3) $v_0/\sqrt{5}$ (4) none

50. Three blocks are placed on smooth horizontal surface and lie on same horizontal straight line. Block 1 and block 3 have mass m each and block 2 has mass M ($M > m$). Block 2 and block 3 are initially stationary, while block 1 is initially moving towards block 2 with speed v as shown. Assume that all collisions are head on and perfectly elastic. What value of M/m ensures that block 1 and block 3 have the same final speed?

(1) $5+\sqrt{2}$ (2) $5-\sqrt{2}$

(3) $2+\sqrt{5}$ (4) $3+\sqrt{5}$

51. A particle of mass m is moving along the x-axis with speed v when it collides with a particle of mass $2m$ initially at rest. After the collision, the first particle has come to rest and the second particle has split into two equal-mass pieces that are shown in the figure. Which of the following statements correctly describes the speeds of the two places? ($\theta > 0$)



Before collision After collision

(1) Each piece moves with speed v

(2) Each piece moves with speed $v/2$

(3) One of the pieces moves with speed $v/2$, the other moves with speed greater than $v/2$

(4) Each piece moves with speed greater than $v/2$

52. A ball is projected in a direction inclined to the vertical and bounces on a smooth horizontal plane. The range of one rebound is R. If the coefficient of restitution is e, then range of the next rebound is

(1) $R' = eR$ (2) $R' = e^2R$

(3) $R' = \frac{R}{e}$ (4) $R' = R$

53. A glass ball collides with a smooth horizontal surface (xz plane) with a velocity $V = a\hat{i} - b\hat{j}$. If the coefficient of restitution of collision be e, the velocity of the ball just after the collision will be

(1) $\sqrt{e^2a^2+b^2}$ at angle $\tan^{-1}\left(\frac{a}{eb}\right)$ to the vertical

(2) $\sqrt{a^2+e^2b^2}$ at angle $\tan^{-1}\left(\frac{a}{eb}\right)$ to the vertical

(3) $\sqrt{a^2+\frac{b^2}{e^2}}$ at angle $\tan^{-1}\left(\frac{ea}{b}\right)$ to the vertical

(4) $\sqrt{\frac{a^2}{e^2}+b^2}$ at angle $\tan^{-1}\left(\frac{a}{eb}\right)$ to the vertical

54. A mass m_1 moves with a great velocity. It strikes another mass m_2 at rest in head-on collision. It comes back along its path with low speed after collision. Then

(1) $m_1 > m_2$

(2) $m_1 < m_2$

(3) $m_1 = m_2$

(4) there is no relation between m_1 and m_2

55. A body is hanging from a rigid support by an inextensible string of length 'l'. It is struck inelastically by an identical body of mass m with horizontal velocity $v = \sqrt{2gl}$, the tension in the string increases just after the striking by

(1) mg (2) $3\,mg$

(3) $2\,mg$ (4) none of these

56. A steel ball of mass 0.5 kg is fastened to a cord 20 cm long and fixed at the far end and is released when the cord is horizontal. At the bottom of its path the ball strikes a 2.5 kg steel block initially at rest on a frictionless surface. The collision is elastic. The speed of the block just after the collision will be

(1) $\frac{10}{3}$ ms^{-1} (2) $\frac{20}{3}$ ms^{-1}

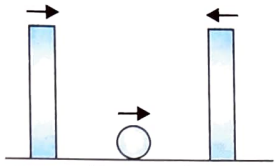
(3) 5 ms^{-1} (4) $\frac{5}{3}$ ms^{-1}

57. A bullet of mass 0.01 kg and travelling at a speed of 500 ms^{-1} strikes a block of mass 2 kg which is suspended by a string of length 5 m. The centre of gravity of the block is found to raise a vertical distance of 0.2 m. What is the speed of the bullet after it emerges from the block?

(1) 15 ms^{-1} (2) 20 ms^{-1}

(3) 100 ms^{-1} (4) 50 ms^{-1}

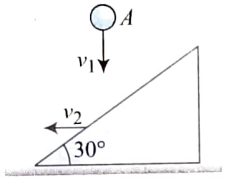
58. A small ball is projected horizontally between two large blocks. The ball is given a velocity v ms^{-1} and each of the large blocks move uniformly with a velocity of $2v$ ms^{-1}. The ball collides elastically with the blocks. If the velocity of the blocks do not change due to the collision, then find out the velocity of the ball after the 2nd collision.


Assume friction to be absent

(1) $5v$ (2) $7v$

(3) $9v$ (4) none of these

59. A ball A is falling vertically downwards with velocity v_1. It strikes elastically with a wedge moving horizontally with velocity v_2 as shown in figure. What must be the ratio v_1/v_2 so that the ball bounces back in vertically upwards direction relative to wedge?



(1) $\sqrt{3}$ (2) $\frac{1}{\sqrt{3}}$

(3) 2 (4) $\frac{1}{2}$

Multiple Correct Answers Type

1. In an elastic collision between two particles

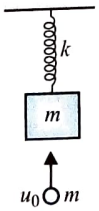
(1) the total kinetic energy of the system is always conserved

(2) the kinetic energy of the system before collision is equal to the kinetic energy of the system after collision

(3) the linear momentum of the system is conserved

(4) the mechanical energy of the system before collision is equal to the mechanical energy of the system after collision

2. A block of mass 'm' is hanging from a massless spring of spring constant k. It is in equilibrium under the influence of gravitational force. Another particle of same mass 'm' moving upwards with velocity u_0 hits the block and sticks to it. For the subsequent motion, choose the *incorrect* statements:



(1) Velocity of the combined mass must be maximum at natural length of the spring.

(2) Velocity of the combined mass must be maximum at the new equilibrium position.

(3) Velocity of the combined mass must be maximum at the instant particle hits the block.

(4) Velocity of the combined mass must be maximum at a point lying between old equilibrium position and natural length.

3. A body moving towards a body of finite mass at rest collides with it. It is possible that

(1) both bodies come to rest

(2) both bodies move after collision

(3) the moving body stops and the body at rest starts moving

(4) the stationary body remains stationary and the moving body rebounds

4. A ball strikes a smooth horizontal floor obliquely and rebounds inelastically.

(1) The kinetic energy of the ball just after hitting the floor is equal to the potential energy of the ball at its maximum height after rebound.

(2) Total energy of the ball is not conserved.

(3) The angle of rebound with the vertical is greater than the angle of incidence.

(4) None of the above.

5. A particle A suffers an oblique elastic collision with a particle B that is at rest initially. If their masses are the same, then after the collision

(1) their KE may be equal

(2) A continues to move in the original direction while B remains at rest

(3) they will move in mutually perpendicular directions

(4) A comes to rest and B starts moving in the direction of the original motion of A

6. A ball strikes a wall with a velocity $\vec{u}$ at an angle θ with the normal to the wall surface and rebounds from it at an angle β with the surface. Then

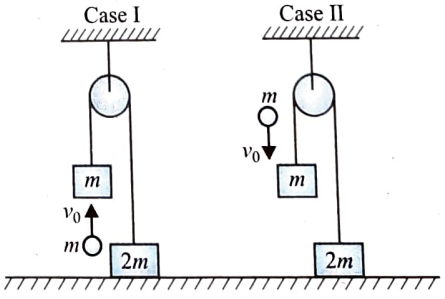
(1) $(\theta + \beta) < 90°$, if the wall is smooth

(2) if the wall is rough, coefficient of restitution = $\tan\beta/\cos\theta$

(3) if the wall is rough, coefficient of restitution < $\tan\beta/\cot\theta$

(4) none of the above

7. Two masses $2m$ and m are connected by an inextensible light string. The string is passing over a light frictionless pulley. The mass $2m$ is resting on a surface and mass m is hanging in air as shown in figure. A particle of mass m strikes the mass m from below in case (I) with a velocity v_0 and in case (II) strikes mass m with a velocity v_0 from top and sticks to it.



(1) The conservation of linear momentum can be applied in both the cases just before and just after collision.

(2) The conservation of linear momentum can be applied in case I but cannot be applied in case II just before and just after collision.

(3) The ratio of velocities of mass m just after collision in first and second cases is 1/2.

(4) The ratio of velocities of mass m just after collision in first and second case is 2.

8. A ball moving with a velocity v hits a massive wall moving towards the ball with a velocity u. An elastic impact lasts for time Δt.

(1) The average elastic force acting on the ball is $[m(u+v)]/\Delta t$.

(2) The average elastic force acting on the ball is $[2m(u+v)]/\Delta t$.

(3) The kinetic energy of the ball increases by $2mu\,(u+v)$.

(4) The kinetic energy of the ball remains the same after the collision.

9. A particle strikes a horizontal smooth floor with a velocity u making an angle θ with the floor and rebounds with velocity v making an angle ϕ with the floor. If the coefficient of restitution between the particle and the floor is e, then

(1) the impulse delivered by the floor to the body is $mu(1+e)\sin\theta$

(2) $\tan\phi = e\tan\theta$

(3) $v = u\sqrt{1-(1-e)^2\sin^2\theta}$

(4) the ratio of final kinetic energy to the initial kinetic energy is $(\cos^2\theta + e^2\sin^2\theta)$

10. A body of mass 2 kg moving with a velocity 3 m/s collides with a body of mass 1 kg moving with a velocity of 4 m/s in opposite direction. If the collision is head on and completely inelastic, then

(1) both particles move together with velocity (2/3) m/s

(2) the momentum of system is 2 kg m/s throughout

(3) the momentum of system is 10 kg m/s

(4) the loss of KE of system is (49/3) J

11. Velocity of a particle of mass 2 kg changes from $\vec{v}_1 = (-2\hat{i} - 2\hat{j})$ m/s to $\vec{v}_2 = (\hat{i} - \hat{j})$ m/s after colliding with a plane surface.

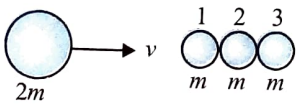
(1) The angle made by the plane surface with the positive x-axis is $90^\circ + \tan^{-1}\left(\frac{1}{3}\right)$.

(2) The angle made by the plane surface with the positive x-axis is $\tan^{-1}\left(\frac{1}{3}\right)$.

(3) The direction of change in momentum makes an angle $\tan^{-1}\left(\frac{1}{3}\right)$ with the positive x-axis.

(4) The direction of change in momentum makes an angle $90^\circ + \tan^{-1}\left(\frac{1}{3}\right)$ with the plane surface.

12. A steel ball of mass $2m$ suffers one-dimensional elastic collision with a row of three steel balls, each of mass m. If mass $2m$ has collided with velocity v and the three balls numbered 1, 2, 3 were initially at rest, then after the collision



(1) balls 1, 2 and 3 would start moving to the right, each with velocity $v/3$

(2) balls 2 and 3 would start moving to the right, each with velocity $v/2$

(3) balls 2 and 3 would start moving to the right, each with velocity v

(4) ball 1 and ball of mass $2m$ would remain at rest

13. Two particles of masses m_1 and m_2 and velocities u_1 and αu_1 $(\alpha \neq 0)$ make an elastic head on collision. If the initial kinetic energies of the two particles are equal and m_1 comes to rest after collision, then

(1) $\frac{u_1}{u_2} = \sqrt{2} + 1$ (2) $\frac{u_1}{u_2} = \sqrt{2} - 1$

(3) $\frac{m_2}{m_1} = 3 + 2\sqrt{2}$ (4) $\frac{m_2}{m_1} = 3 - 2\sqrt{2}$

14. A ball of mass 1 kg is dropped from a height of 3.2 m on smooth inclined plane. The coefficient of restitution for the collision is $e = 1/2$. The ball's velocity become horizontal after the collision.

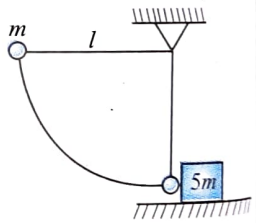
(1) The angle $\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

(2) The speed of the ball after the collision = $4\sqrt{2}$ ms^{-1}

(3) The total loss in kinetic energy during the collision is 8 J.

(4) The ball hits the inclined plane again while travelling vertically downward.

15. A pendulum bob of mass m connected to the end of an ideal string of length l is released from rest from horizontal position as shown in figure.



At the lowest point, the bob makes an elastic collision with a stationary block of mass $5m$, which is kept on a frictionless surface. Mark out the correct statement(s) for the instant just after the impact.

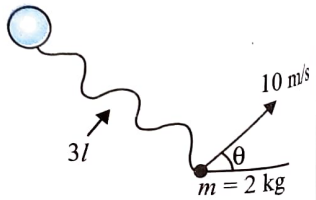
(1) Tension in the string is (17/9) mg.

(2) Tension in the string is 3 mg.

(3) The velocity of the block is $\sqrt{2gl}/3$.

(4) The maximum height attained by the pendulum bob after impact is (measured from the lowest position) $4l/9$.

16. A string of length $3l$ is connected to a fixed cylinder whose top view is shown in figure. The string is initially slack. The other end of the string (connected to a marble) is moving at a constant velocity of 10 m/s as shown. The string will get stretched at some instant and impulsive tension occurs in the string. If hinge is exerting a force of 40000 N for 0.25 ms on the cylinder to bear up the impact of impulsive tension, then mark the correct statements. (Take string to be light, breaking tension of the string is 2×10^5 N.)



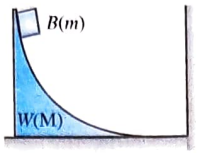
(1) The angle made by the velocity of marble with the length of string when it is just stretched is 60°.

(2) The marble will move in a circular path of varying radius with constant speed of $5\sqrt{3}$ m/s, after the string is taut.

(3) To answer above two options, the volume of θ must be given.

(4) The string will break if impulse duration is less than 0.05 ms.

17. In the figure, the block B of mass m starts from rest at the top of a wedge W of mass M. All surfaces are without friction. W can slide on the ground. B slides down onto the ground, moves along it with a speed v, has an elastic collision with the wall, and climbs back on to W.



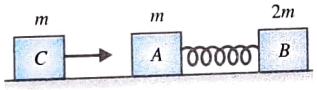
(1) From the beginning, till the collision with the wall, the centre of mass of 'B plus W' does not move horizontally.

(2) After the collision, the centre of mass of 'B plus W' moves with the velocity $\frac{2mv}{m+M}$.

(3) When B reaches its highest position of W, the speed of W is $\frac{2mv}{m+M}$.

(4) When B reaches its highest position of W, the speed of W is $\frac{mv}{m+M}$.

18. Two blocks A and B of masses m and $2m$ respectively placed on a smooth floor are connected by a spring. A third body C of mass m moves with velocity v_0 along the line joining A and B and collides elastically with A. At a certain instant of time after collision it is found that the instantaneous velocities of A and B are same then



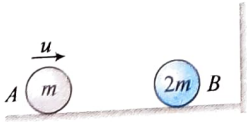
(1) the common velocity of A and B at time t_0 is $v_0/3$

(2) the spring constant is $k = \frac{3mv_0^2}{2x_0^2}$

(3) the spring constant is $k = \frac{2mv_0^2}{3x_0^2}$

(4) none of these

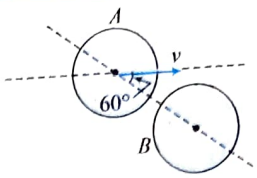
19. The wall is at extreme right is fixed as shown in figure. Coefficient of restitution for collision between two balls is 1/2 and between ball and wall is 1. Then speed of A and B after all possible collisions are



(1) $v_A = \frac{u}{2}$ (2) $v_A = \frac{u}{4}$

(3) $v_B = \frac{u}{8}$ (4) $v_B = \frac{u}{4}$

20. A smooth sphere A of mass m collides elastically with an identical sphere B at rest. The velocity of A before collision is 10 m/s in a direction making 60° with the line of centres at the time of impact.



(1) The sphere A comes to rest after collision.

(2) The sphere B will move with a speed of 10 m/s after collision.

(3) The directions of motion A and B after collision are at right angles.

(4) The speed of B after collision is 5 m/s.

21. A particle of mass m makes an elastic head-on collision with a stationary particle of mass $2m$. If initial kinetic energy is moving particle is $6J$, then during the impact,

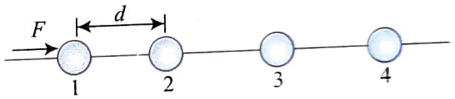
(1) the minimum kinetic energy of the system is $2J$

(2) the maximum elastic potential energy of the system is $4J$

(3) momentum and total energy are conserved at every instant

(4) the ratio of kinetic energy to potential energy of the system first decreases and then increases

22. Four identical beads of mass m are arranged in a smooth string. The beads are equally separated by a distance d. The beads are free to slide without friction on a thin wire. A constant force F acts on the first bead initially at rest till it makes collision with the second bead. The second bead then collides with the third and so on. Suppose that all collisions are elastic,



(1) speed of the first bead immediately before and immediately after its collision with the second bead is $\sqrt{\frac{2Fd}{m}}$ and zero respectively

(2) speed of the first bead immediately before and immediately after its collision with the second bead is $\sqrt{\frac{2Fd}{m}}$ and $\frac{1}{2}\sqrt{\frac{2Fd}{m}}$ respectively

(3) speed of the second bead immediately after its collision with third bead is zero

(4) the average speed of the first bead is $\frac{1}{2}\sqrt{\frac{2Fd}{m}}$

23. In a one dimensional collision between two identical small sphere A and B, B is stationary and A has linear momentum p before impact. During impact, B gives impulse J to A.

(1) The total linear momentum of the 'A plus B' system is p before and after the impact, and $(p-1)$ during the impact

(2) During the impact A gives impulse J to B

(3) The coefficient of restitution is $\frac{2J}{p} - 1$

(4) The coefficient of restitution is $\frac{J}{p} + 1$

24. Two balls A & B of mass m_1 and m_2 are kept on a horizontal smooth surface. A is given a velocity towards B so that they perform head on collision.

(1) If $m_1 = m_2$ and collision is elastic A stops and B moves with the velocity of A after collision

(2) Impulses on A and B during collision are same in magnitude whether it is elastic or inelastic collision

(3) Impulses on A and B during collision are equal in magnitude only if $m_1 = m_2$

(4) If $m_1 >> m_2$, though impulses are same in magnitude velocity of A is still in the same direction

25. Two small spheres 'A' and 'B' mass 1 kg and 2 kg are moving with speeds 21 m/s and 4 m/s respectively in opposite direction collide head on. After collision A moves with speed 1 m/s in its initial direction. Which is/are correct?

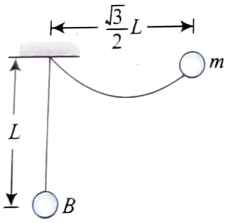
(1) Velocity of B after collision is 6 m/s opposite to its direction before collision

(2) The coefficient of restitution $e = 0.2$

(3) Loss of kinetic energy due to collision is 200 J

(4) Impulse of force between two balls is 400 Ns

26. A small bob of mass $m = 200$ g is suspended from a point A by an inextensible string of length L. The bob is drawn to a side and held at same level as A but at a distance $\frac{\sqrt{3}}{2}L$ from A as shown. Now the bob is released. Then: (assume string applies only that much jerk which is required so that velocity along string becomes zero).



(1) speed of ball just after experiencing jerk is $\frac{\sqrt{3gL}}{2}$

(2) impulse applied by string $\frac{\sqrt{gL}}{10}$

(3) ball will experience jerk after reaching to point B

(4) speed of ball just before experiencing jerk is $\sqrt{gL}$

27. A body of mass 2 kg moving with a velocity 3 m/s collides with a body of mass 1 kg moving with a velocity of 4 m/s in opposite direction, if the collision is head on and completely inelastic, then

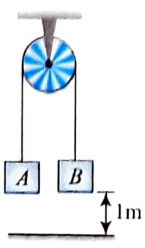
(1) the momentum of system is 2 kg m/s throughout

(2) the momentum of system is 10 kg m/s throughout

(3) the loss of KE of system is (49/3) J in collision

(4) both particles move together with velocity (2/3) m/s after collision

28. Two masses A and B of 3 kg and 2 kg are connected by a long inextensible string which passes over a massless and frictionless pulley. Initially the height of both the masses from the ground is same and equal to 1 metre. When the masses are released, mass A hits the ground and gets stuck to the ground. Consider the length of the string as large, so that the pulley does not obstruct the motion of masses A and B. [$g = 10$ m/s^2]



(1) The impulse A exerts on ground is 2 Ns

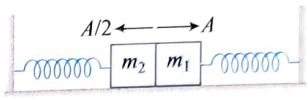
(2) B reaches a maximum height of 2.2 m from the ground

(3) B reaches maximum height from the ground 1.2 s after being released

(4) The impulse A exerts on ground is 6 Ns

29. Two identical blocks P and Q have mass m each. They are attached to two identical springs (of spring constant k) initially unstretched. Both the blocks are initially in contact as shown.

Now the left spring (attached with block P) is compressed by $A/2$ and the right spring (attached with block Q) is compressed by A. Both the blocks are then released simultaneously. Then choose the correct option(s).



Smooth horizontal floor

(1) The speed of block P just before P and Q are about to collide for the first time is $A\sqrt{\frac{k}{m}}$.

(2) The speed of block Q just before P and Q are about to collide for the first time is $A\sqrt{\frac{k}{m}}$.

(3) The blocks shall collide for the first time, $\frac{\pi}{2}\sqrt{\frac{m}{k}}$ time after release.

(4) If the collision is perfectly inelastic then maximum elongation in right spring is $\frac{A}{4}$.

30. A particle of mass m_1 elastically collides with another stationary particle of mass m_2. Then if

(1) $\frac{m_1}{m_2} = \frac{1}{2}$ and the collision is head on, then the particles fly apart in the opposite direction with unequal speed.

(2) $\frac{m_1}{m_2} = \frac{1}{3}$ and the collision is head on, then the particles fly apart in the opposite direction with equal speed.

(3) $\frac{m_1}{m_2} = \frac{2}{1}$ and the collision is head on, then the particles fly apart in the opposite direction with unequal speed.

(4) $\frac{m_1}{m_2} = \frac{3}{1}$ and the collision is head on, then the particles fly apart in the opposite direction with equal speed.

Linked Comprehension Type

For Problems 1–3

Two identical balls, each of mass m, are tied with a string and kept on a frictionless surface. Initially, the string is slack. They are given velocities $2u$ and u in the same direction. Collision between the balls is perfectly elastic.

1. After the first collision, what is the total loss in kinetic energy of the balls?
(1) $2mu^2$ (2) mu^2
(3) $3mu^2$ (4) zero

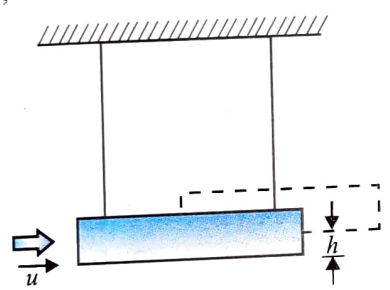
2. What is the final loss in kinetic energy of the balls?
(1) $mu^2/4$ (2) $mu^2/2$
(3) $3mu^2/4$ (4) None of these

3. What is the impulse generated in the string during the second collision?
(1) $mu/2$ (2) $mu/4$
(3) $2mu/3$ (4) None of these

For Problems 4–7

A ballistic pendulum is a device that was used to measure the speeds of bullets before the development of electronic timing devices. The device consists of a large block of wood of mass M, hanging from two long cords. A bullet of mass m is fired into the block, the bullet comes quickly into rest and the block + bullet rises to a vertical distance h before the pendulum comes momentarily to rest as the ends of the arc.

In the process, the linear momentum is conserved. In such a collision, some kinetic energy is dissipated as heat; so mechanical energy is not conserved. When there is a loss in mechanical energy, the collision is said to be inelastic. Further when two bodies coalesce, the collision is said to be perfectly inelastic.



4. After collision what is the combined velocity of the bullet + block system?
(1) $\frac{m}{M}u$ (2) $\frac{m}{M+m}u$
(3) $\frac{M}{M+m}u$ (4) $\frac{M+m}{M}u$

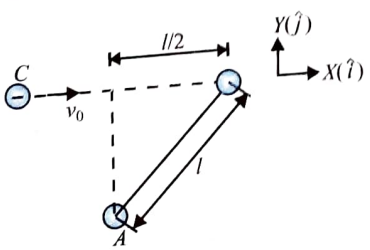
5. What is the initial speed of the bullet in terms of height h?
(1) $\frac{m}{M}\sqrt{2gh}$ (2) $\frac{m}{M+m}\sqrt{2gh}$
(3) $\frac{M+m}{m}\sqrt{2gh}$ (4) $\sqrt{\frac{2m}{M+m}gh}$

6. The collision of block–bullet system is
(1) perfectly elastic (2) partially inelastic
(3) partially elastic (4) perfectly inelastic

7. The energy dissipated as heat in the collision is
(1) $\frac{1}{2}mu^2 - mgh$ (2) $\frac{1}{2}mu^2 - (M+m)gh$
(3) $\frac{1}{2}(M+m)u^2 - (M+m)gh$ (4) cannot be estimated

For Problems 8–12

Three spheres, each of mass m, can slide freely on a frictionless, horizontal surface. Spheres A and B are attached to an inextensible, inelastic cord of length 'l' and are at rest in the position shown where sphere B is struck by sphere C which is moving to the right with a velocity v_0. Knowing that the cord is taut where sphere B is struck by sphere C and assuming 'head on' inelastic impact between B and C, we cannot conserve kinetic energy of the entire system.



8. The velocity of B immediately after collision is along unit vector
(1) $\hat{i}$ (2) $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$
(3) $\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$ (4) none of these

9. Velocity of A immediately after collision is along unit vector
(1) $\hat{i}$ (2) $\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}$
(3) $\hat{j}$ (4) none of these

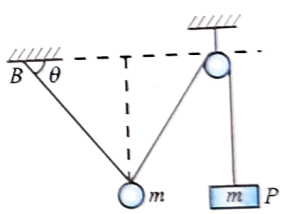
10. If velocity of C immediately after collision becomes $v_0/2$ in the initial direction of motion, the impulse due to string on sphere A is
(1) $\frac{mv_0}{8}$ (2) $\frac{mv_0}{2}$
(3) $\frac{\sqrt{mv_0}}{4}$ (4) none of these

11. The magnitude of velocity of A immediately after collision is
(1) $\frac{\sqrt{3v_0}}{4}$ (2) $\frac{v_0}{8}$
(3) $\frac{v_0}{4}$ (4) $\frac{\sqrt{3v_0}}{8}$

12. The loss in kinetic energy of the system during collision is
(1) $\frac{17mv_0^2}{64}$ (2) $\frac{15mv_0^2}{64}$
(3) $\frac{12mv_0^2}{25}$ (4) none of these

For Problems 13–15

In figure, a pulley is shown which is frictionless and a ring of mass m can slide on the string without any friction. One end of the string is attached to point B and to the other end, a block 'P' of mass m is attached. The whole system lies in vertical plane.



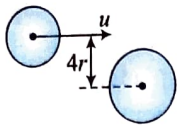
13. If the system is released from rest, it is found that the system remains at rest. What is the value of θ?
(1) 30° (2) 45°
(3) 60° (4) 75°

14. Now another block 'C' of same mass m is attached to the block 'P' and system is released from rest. If a_1 and a_2 are the magnitudes of initial accelerations of ring and blocks, respectively, then
(1) $a_1 + a_2 = g$ (2) $a_1 + 2a_2 = g$
(3) $a_1 = 2a_2$ (4) $2a_1 = a_2$

15. If block 'C' mentioned above was released from some height and collided with 'P' with some velocity u, then find the velocity of the ring just after the collision. The collision is perfectly inelastic.
(1) u (2) $u/2$
(3) $u/4$ (4) $u/3$

For Problems 16–18

A circular disc of mass '$2m$' and radius '$3r$' is resting on a flat frictionless surface. Another circular disc of mass 'm' and radius '$2r$', moving with a velocity 'u' hits the first disc as shown in the figure. The collision is elastic.



16. What is the tangential component of final velocity of the smaller disc?

(1) u (2) $\frac{u}{2}$

(3) $\frac{3u}{5}$ (4) $\frac{4u}{5}$

17. What is the magnitude of normal component of final velocity of the smaller disc?

(1) $\frac{u}{5}$ (2) $\frac{2u}{5}$

(3) $\frac{3u}{5}$ (4) $\frac{4u}{5}$

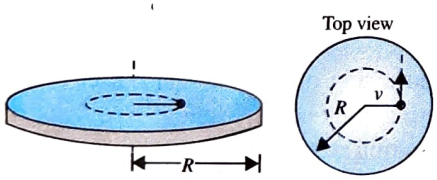
18. What is the final velocity of the heavier disc?

(1) $\frac{u}{5}$ (2) $\frac{2u}{5}$

(3) $\frac{3u}{5}$ (4) $\frac{4u}{5}$

For Problems 19–21

A small ball of mass 1 kg is kept in circular path of radius 1 m inside a concentric smooth horizontal fixed casing of radius R. Angular speed of the ball in the circular motion is 1 rad s^{-1}. At a certain moment the string, which kept the ball in the circular path breaks and the ball goes off tangentially to the wall of rigid casing and bounces off elastically and again hits the casing and bounces off. This way, the ball traces a regular hexagon. Consider all the collisions to be elastic.



19. Total impulse imparted to the casing by the ball in first six collisions will be

(1) $6\sqrt{3}$ Ns (2) $3\sqrt{3}$ Ns

(3) zero (4) 12 N s

20. Following quantities of the ball will remain a constant before and after any collision

(1) linear momentum

(2) kinetic energy, angular momentum about the centre of the circle

(3) velocity, angular momentum about the centre of the circle, kinetic energy

(4) none of these

21. Total time between the first collision and the seventh collision will

(1) $\frac{\sqrt{4}}{3}$ s (2) $4\sqrt{3}$ s

(3) $3\sqrt{3}$ s (4) none

For Problems 22–24

Two beads A and B of masses m_1 and m_2, respectively, are threaded on a smooth circular wire of radius a fixed in a vertical plane. B is stationary at the lowest point when A is gently dislodged from rest at the highest point. A collided with B at the lowest point. The impulse given to B due to collision is just great enough to carry it to the level of the centre of the circle while A is immediately brought to rest by the impact.

22. Find the ratio $m_1:m_2$.

(1) 1 (2) $\sqrt{2}$

(3) $1/\sqrt{2}$ (4) 2

23. What is the coefficient of restitution between the beads?

(1) 1 (2) 1/2

(3) 1/3 (4) $1/\sqrt{2}$

24. If m_2 again comes down and collides with m_1, then after the collision

(1) m_1 will rise the same height as risen by m_2

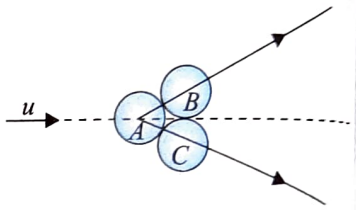
(2) m_1 will rise the less height as risen by m_2

(3) m_1 will rise the more height as risen by m_2

(4) m_1 and m_2 will move in opposite directions

For Problems 25–27

Two equal spheres B and C, each of mass m, are in contact on a smooth horizontal table. A third sphere A of same size as that of B or C but mass $m/2$ impinges symmetrically on them with a velocity u and is itself brought to rest.



25. The coefficient of restitution between the two spheres A and B (or between A and C) is

(1) 1/3 (2) 1/4

(3) 2/3 (4) 3/4

26. Find the velocity acquired by each of the spheres B and C after collision.

(1) $\frac{u}{\sqrt{3}}$ (2) $\frac{2u}{\sqrt{3}}$

(3) $\frac{u}{2\sqrt{3}}$ (4) $\frac{u}{2}$

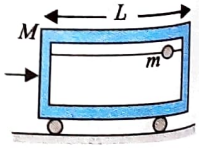
27. The loss of KE during collision is

(1) $\frac{mu^2}{8}$ (2) $\frac{mu^2}{6}$

(3) $\frac{mu^2}{3}$ (4) $\frac{mu^2}{2}$

For Problems 28–30

A horizontal frictionless rod is threaded through a bead of mass m. The length of the cart is L and the radius of the bead, r, is very small in comparison with $L(r << L)$. Initially at $(t = 0)$ the bead is at the right edge of the cart. The cart is struck and as a result, it moves with velocity v_0 towards right. When the bead collides with the cart's walls, the collisions are always completely elastic.



28. What is the velocity of the cart just after the first collision?

(1) $\frac{-mv_0}{m+M}$ (2) $\frac{Mv_0}{m+M}$

(3) $\frac{M-m}{M+m}v_0$ (4) $\frac{2M}{m+M}v_0$

29. Velocity of bead just after the first collision is

(1) $\frac{-mv_0}{m+M}$ (2) $\frac{Mv_0}{m+M}$

(3) $\frac{M-m}{M+m}v_0$ (4) $\frac{2M}{m+M}v_0$

30. The first collision takes place at time t_1 and the second collision takes place at time t_2. Find $t_2 - t_1$.

(1) $\frac{2L}{v_0}$ (2) $\frac{L}{v_0}$

(3) $\frac{L}{2v_0}$ (4) $\frac{L}{3v_0}$

For Problems 31–33

A ball A of mass m is suspended by a thread of length r = 1.2 m. Another ball B of mass $2m$ is projected from the ground with velocity u = 9 m/s such that at the highest point of its trajectory it collides head-on elastically with ball A. It is observed that during subsequent motion, tension in the thread at the highest point is equal to mg.

31. At highest point of B, the velocity of ball A is

(1) $6\sqrt{2}$ m/s (2) $2\sqrt{6}$ m/s

(3) $3\sqrt{2}$ m/s (4) $3\sqrt{6}$ m/s

32. The angle of projection (θ) of ball B is

(1) 30° (2) 60°

(3) 45° (4) 75°

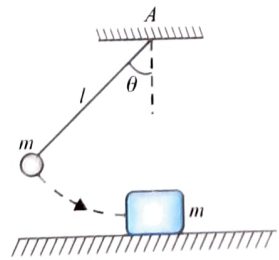
33. The height of the point of suspension of ball A from the ground is

(1) $\frac{81}{40}$ m (2) $\frac{129}{40}$ m

(3) $\frac{81}{20}$ m (4) $\frac{101}{40}$ m

For Problems 34–35

In the arrangement shown in figure, the ball and the block have the same mass m = 1 kg each, θ = 60° and length l = 2.50 m. Coefficient of friction between the block and the floor is 0.5. When the ball is released from the position shown in figure, it collides with the block and the block stops after moving a distance 2.50 m.



34. The velocity of block just after collision is

(1) 10 m/s (2) 5 m/s

(3) 2.5 m/s (4) 3 m/s

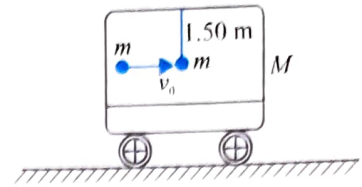
35. The coefficient of restitution for collision between the ball and the block is

(1) 0.50 (2) 0.75

(3) 1.0 (4) 0.30

For Problems 36–38

A ball of mass m = 1 kg is hung vertically by a thread of length l = 1.50 m. Upper end of the thread is attached to the ceiling of a trolley of mass M = 4 kg. Initially, the trolley is stationary and it is free to move along horizontal rails without friction. A shell of mass m = 1 kg, moving horizontally with velocity v_0 = 6 m/s, collides with the ball and gets stuck with it. As a result, the thread starts to deflect towards right.



36. The velocity of the combined body just after collision is

(1) 2 m/s (2) 3 m/s

(3) 1 m/s (4) 4 m/s

37. At the time of maximum deflection of the thread with vertical, the trolley will move with velocity

(1) 2 m/s (2) 3 m/s

(3) 1 m/s (4) 4 m/s

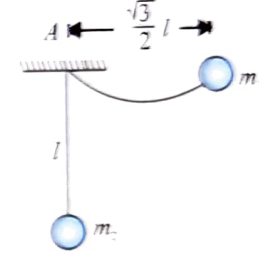
38. The maximum deflection of the thread with the vertical is

(1) $\cos^{-1}\left(\frac{4}{5}\right)$ (2) $\cos^{-1}\left(\frac{3}{5}\right)$

(3) $\cos^{-1}\left(\frac{2}{3}\right)$ (4) $\cos^{-1}\left(\frac{3}{4}\right)$

For Problems 39–40

Two balls of masses m_1 = 100 g and m_2 = 300 g are suspended from point A by two equal inextensible threads, each of length l = 32/35 m. Ball of mass m_1 is drawn aside and held at the same level as A but at a distance $(\sqrt{3}/2)l$ from A, as shown in figure. When ball m_1 is released, it collides elastically with the stationary ball of mass m_2.



39. Velocity u_1 with which the ball of mass m_1 collides with the other ball is

(1) 1 m/s (2) 2 m/s

(3) 3 m/s (4) 4 m/s

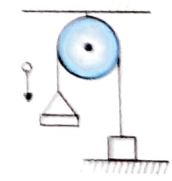
40. The maximum rise of centre of mass of the ball of mass m_2 is

(1) 0.20 m (2) 0.50 m

(3) 0.75 m (4) 1 m

For Problems 41–44

A pan of mass m = 1.5 kg and a block of mass M = 3 kg are connected to each other by a light inextensible string, passing over a light pulley as shown in figure. Initially, the block is resting on a horizontal floor. A ball of mass m_0 = 0.5 kg collides with the pan at a speed v_0 = 20 m/s. Consider this instant of collision as t = 0. Assume collision to be perfectly inelastic. Now, answer the following questions based on the above information.



41. Mark the correct statement(s) for this situation.

(1) After the collision, the pan + ball system moves downwards with decreasing speed.

(2) After the collision, the block is moving upwards with the same speed with which the ball + pan system is moving downwards.

(3) The block will jerk for a number of times during its motion.

(4) All of these.

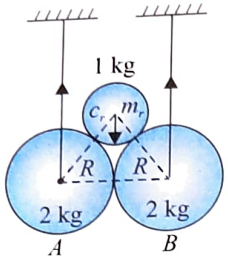
42. Find the time t at which the block strikes the floor for the first time.
(1) 1 s (2) 2 s
(3) 4 s (4) 5 s

43. Find the velocity of pan + ball system at $t = 2.6$ s. Assume that the block comes to rest instantaneously after striking the floor.
(1) 4 m/s downward (2) 4 m/s upward
(3) 0.6 m/s upward (4) 0.4 m/s downward

44. Find the maximum height reached by the block after the second jerk.
(1) 0.2 m (2) 0.64 m
(3) 0.16 m (4) No jerk for the second time

For Problems 45–47

Two identical balls A and B, each of mass 2 kg and radius R, are suspended vertically from inextensible strings as shown in figure. The third ball C of mass 1 kg and radius $r = (\sqrt{2} - 1)R$ falls and hits A and B symmetrically with 10 m/s. Speed of both A and B just after the collision is 3 m/s.



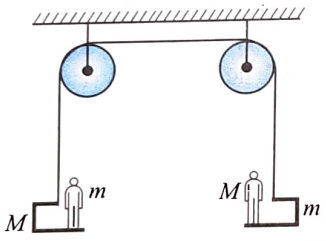
45. Speed of C just after collision is
(1) 2 m/s (2) $2\sqrt{2}$ m/s
(3) 5 m/s (4) $(\sqrt{2} - 1)$ m/s

46. Impulse provided by each string during collision is
(1) $6\sqrt{2}$ N s (2) 12 N s
(3) $3\sqrt{2}$ N s (4) 6 N s

47. The value of coefficient of restitution is
(1) $\frac{1}{4}$ (2) $\frac{1}{\sqrt{2}}$
(3) $\sqrt{2} - 1$ (4) $\frac{1}{2}$

For Problems 48–50

A system of men and trolley is shown in figure. To the left end of the string, a trolley of mass M is connected on which a man of mass m is standing. To the right end of the string another trolley of mass m is connected on which a man of mass M is standing. Initially, the system is at rest. All of a sudden both the men leap upwards simultaneously with the same velocity u w.r.t. ground.



48. Find the relative velocity of left man with respect to his trolley just after he leaps upwards.
(1) $\frac{mu}{m+M}$ (2) $\frac{Mu}{m+M}$
(3) $\frac{2mu}{m+M}$ (4) $\frac{2Mu}{m+M}$

49. Find the impulse generated in the string connecting the trolleys during this process.
(1) $\frac{Mmu}{m+M}$ (2) $\frac{(M^2+m^2)u}{m+M}$
(3) $\frac{m^2u}{m+M}$ (4) $\frac{M^2u}{m+M}$

50. Which is correct?
(1) When the men are at the highest point of their motion, the trolleys will also be instantaneously at rest.
(2) When the men are at the highest point of their motion, then the left trolley will be moving downwards.
(3) Impulse acting on both the men will be same in the given process.
(4) None of these.

Matrix Match Type

1. Column I contains physical quantity/process while column II contains formula/principle. Match columns I and II such that the formula/principle is correct corresponding to the quantity in column I.

Column I	Column II
i. Momentum	a. $m(v_2 - v_1)$
ii. Impulse	b. only momentum is conserved
iii. Elastic collision	c. momentum and kinetic energy both are conserved
iv. Inelastic collision	d. mv

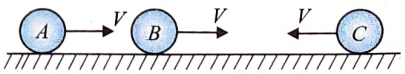
2. Match the following:

Column I	Column II
i. Inelastic collisions	a. kinetic energy of the system may decrease
ii. Elastic collisions	b. kinetic energy of the system may increase
iii. Total work done is zero	c. kinetic energy of the system may remain constant
iv. Non-conservative forces are not present	d. just before and after collision, momentum remains constant

3. In column I, nature of collision between two bodies is given while in column II some physical quantity that may remain conserved during the collision is given. Match the entries of column I with the entries of column II.

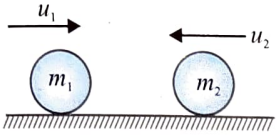
Column I	Column II
i. Elastic collision	a. kinetic energy is conserved
ii. Inelastic collision	b. kinetic energy of the system may increase
iii. Perfectly inelastic collision	c. kinetic energy is not conserved
iv. Collision between two cars moving at high speeds	d. total energy is conserved

4. Two spheres A and B move on a smooth horizontal surface with the same velocity V and have some separation between them. A third sphere C is moving in opposite direction on the same surface with same speed. All the spheres are of equal mass. The collisions are elastic. V_{CM} represents the centre of mass velocity of all the three spheres. Column II represents the values after all the possible impacts have occurred.



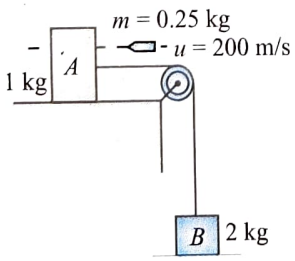
Column I	Column II
i. If A and B are not connected to each other	a. V_{CM} before collision is $V/3$
ii. If A and B are connected to each other by a massless rigid rod	b. V_{CM} after all the collision is $\frac{V}{3}$
iii. If A and B are connected by an ideal string	c. Momentum of system $(B + C)$ is not conserved
iv. If A and B are connected by an ideal spring which is initially unstretched	d. Momentum of system $(A + B)$ is conserved

5. Two balls of masses m_1 and m_2 are moving towards each other with speeds u_1 and u_2, respectively. They collide head-on and their speeds are v_1 and v_2 after collision ($m_1 = 8$ kg, $m_2 = 2$ kg, $u_2 = 3$ m/s).



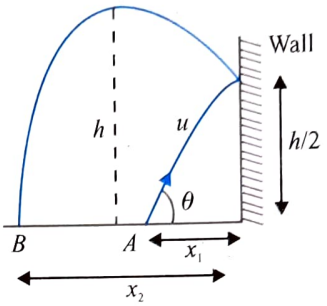
Column I	Column II
i. The speed u_1 (in m/s) so that both balls move in same direction if coefficient of restitution is $e = 0.5$	a. $\frac{1}{14}$
ii. The speed u_1 (in m/s) so that the maximum fraction of energy is transformed to m_2 (assume elastic collision)	b. $\frac{1}{8}$
iii. Coefficient of restitution if m_2 stops after collision and $u_1 = 0.5$ m/s	c. 2
iv. If collision is inelastic and $u_1 = 3$ m/s, the loss of kinetic energy (in J) after collision may be	d. 4

6. A block A of mass $M_A = 1$ kg is kept on a smooth horizontal surface and attached by a light thread to another block B of mass $M_B = 2$ kg. Block B is resting on ground, and thread and pulley are massless and frictionless. A bullet of mass $m = 0.25$ kg moving horizontally with velocity of $u = 200$ m/s penetrates through block A and comes out with a velocity of 100 m/s.



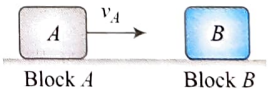
Column I	Column II (Values are in their respective SI units)
i. Velocity of the 2 kg block just after the bullet comes out	a. 50/3
ii. Maximum displacements of 1 kg block in left direction	b. 25
iii. Impulse by the string on block B	c. 25/3
iv. Impulse by the particle on block A	d. 5.2

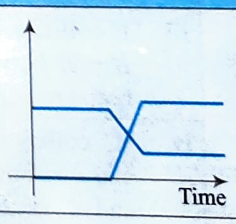
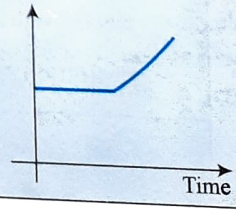
7. A particle is projected with a speed u at an angle θ with horizontal from point A. It strikes elastically with a vertical wall at height $h/2$. It rebounds and reaches maximum height h and falls back on the ground at point B as shown in figure. Distances from A to wall and from wall to B are x_1 and x_2, respectively, and time to cover x_1 and x_2 are t_1 and t_2, respectively. Match the values in column I with the expressions in column II.

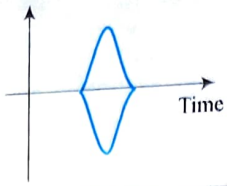
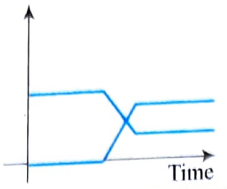
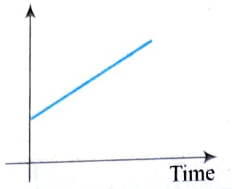


Column I	Column II
i. $\sqrt{2}$	a. $\frac{x_2 - x_1}{x_2 + x_1}$ or $\frac{x_2 + x_1}{x_2 - x_1}$
ii. $\frac{1}{\sqrt{2}}$	b. $\frac{t_2 - t_1}{t_2 + t_1}$ or $\frac{t_2 + t_1}{t_2 - t_1}$
iii. 1	c. $\frac{u \sin\theta}{g(t_1 + t_2)}$
iv. $\frac{1}{2}$	d. $\frac{u \cos\theta (t_1 + t_2)}{(x_1 + x_2)}$

8. Block A moving on frictionless horizontal plane collides head-on with block B initially at rest. The collision is NOT $(0 < e < 1)$ perfectly elastic.

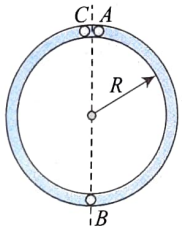


Column I	Column II
i. Forces exerted by the blocks	a. 
ii. The position of mass centres of the two block system	b. 

iii. Velocity of blocks	c. 
iv. Position of block B	d. 
	e. 

(1) (i –b); (ii – c); (iii – d); (iv – e)
(2) (i –c); (ii – e); (iii – d); (iv – b)
(3) (i –c); (ii – b); (iii – d); (iv – e)
(4) (i –c); (ii – d); (iii – e); (iv – b)

9. Three beads A, B and C of masses m, m and $2m$ respectively are held at rest inside a fixed vertical smooth circular tube of radius R. Beads A and C are released simultaneously at the position shown in the figure. The three beads undergo head-on collision simultaneously at the bottom of the tube. v_1, v_2 and v_3 be the speeds of beads A, B and C respectively just after the collision. For various types of collisions between the beads given in column I match the column I with column II. The diameter of beads to be very small compared to the radius of circular tube.

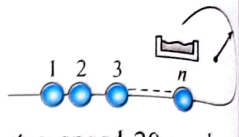


Column I	Column II
i. If collision between A and B and between B and C are perfectly elastic, then	a. $v_1 > v_2 > v_3$
ii. If the collision between A and B is perfectly elastic and that between B and C is perfectly inelastic	b. $v_1 = v_2 = v_3$
iii. If collision between A and B and between B and C both are perfectly inelastic	c. $v_1 > v_2 = v_3$
iv. If collision between A and B and between B and C both are partially elastic with coefficient of restitution, $e = \frac{1}{2}$.	d. Normal reaction at the top most point on bead A is $4mg$, if it reaches there after collision
	e. Linear momentum of $(A + B + C)$ system remain conserved during collision

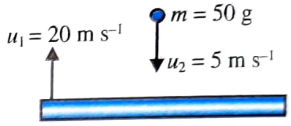
(1) (i –c,e); (ii – c,e); (iii – a,e); (iv – b,e)
(2) (i –c,e); (ii – b,e); (iii – a,d,e); (iv – b,e)
(3) (i –c,d,e); (ii – b,e); (iii – b,d); (iv –a, e)
(4) (i –c,d); (ii – d,e); (iii – b,e); (iv – b,e)

Numerical Value Type

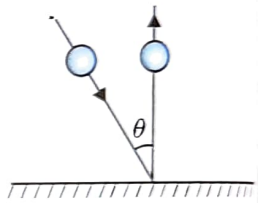
1. N beads identical beads are resting on a smooth horizontal wire which is circular at the end with radius $r = 0.5$ m as shown in the figure. Find the minimum velocity which should be imparted to the first bead such that n^{th} bead will fall in the tank after completing full circle in vertical plane as shown in the figure. Take all the collisions between the beads elastic ($e = 1$).



2. An elevator platform is going up at a speed 20 ms^{-1} and during its upward motion a small ball of 50 g mass falling in downward direction strikes the platform elastically at a speed 5 ms^{-1}. Find the speed (in ms^{-1}) with which the ball rebounds.



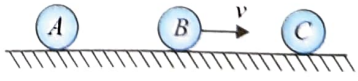
3. A ball of mass 1 kg moving with a velocity of 5 m/s collides elastically with rough ground at an angle θ with the vertical as shown in figure. What can be the minimum coefficient of friction if ball rebounds vertically after collision? (given $\tan\theta = 2$)



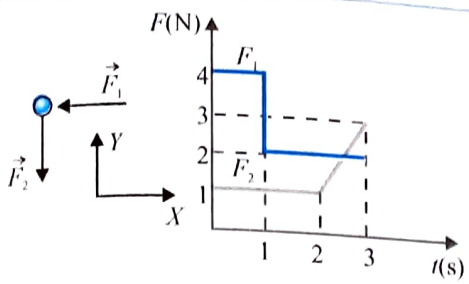
4. A small sphere of mass $m = 1$ kg is moving with a velocity $(4\hat{i} - \hat{j})$ m/s. It hits a fixed smooth wall and rebounds with velocity $(\hat{i} + 3\hat{j})$ m/s. The coefficient of restitution between the sphere and the wall is $n/16$. Find the value of n.

5. A bullet is fired on a fixed target. It penetrates inside the target through distance $d = 3.75$ cm and then stops. mass of the bullet is $m = 1$ kg and of the target is $M = 4$ kg. Now an identical bullet moving with the same velocity is fired on the identical target which is placed at rest on a frictionless horizontal surface. Then find the distance (in cm) to which the bullet will penetrate inside the target?

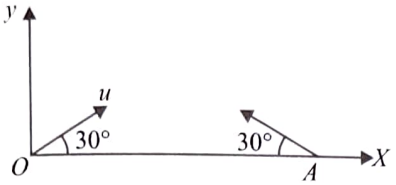
6. Three balls A, B and C ($m_A = m_C = 4m_B$) are placed on a smooth horizontal surface. Ball B collides with ball C with an initial velocity v as shown in figure. Find the total number of collisions between the balls (all collisions are elastic).



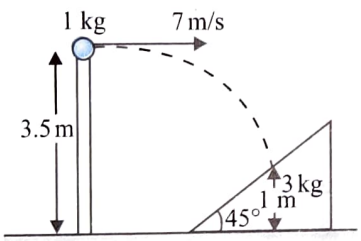
7. A particle with a mass of 1 kg is having a velocity of 10 m/s in +ve x-direction at $t = 0$. Forces $\vec{F}_1$ and $\vec{F}_2$ act on the particle whose magnitudes are changing with time according to the variation shown in figure. The magnitude of the velocity of the particle at $t = 3$ s (neglect gravity effect) is found to be $n\sqrt{5}$ m/s. Find the value of n.



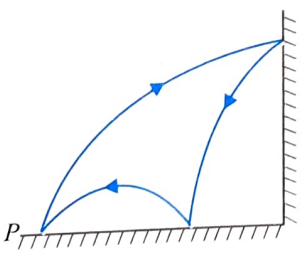
8. Two identical smooth balls are projected from points O and A on the horizontal ground with same speed of projection. The angle of projection in each case is 30° (see figure). The distance between O and A is 100 m. The balls collide in mid-air and return to their respective points of projection. If the coefficient of restitution is 0.7, find the speed of projection of either ball (in m/s) correct to nearest integer. (Take $g = 10\ \text{m s}^{-2}$ and $\sqrt{3} = 1.7$)



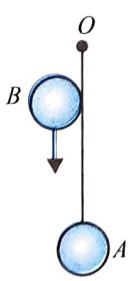
9. A smooth ball of mass 1 kg is projected with velocity 7 m/s horizontal from a tower of height 3.5 m. It collides elastically with a wedge of mass 3 kg and inclination of 45° kept on ground (figure). The ball collides with the wedge at a height of 1 m above the ground. Find the velocity of the wedge (in m/s) after collision. (Neglect friction at any contact.)



10. A small ball is projected from point P towards a vertical wall as shown in figure. It hits the wall when its velocity is horizontal. Ball reaches point P after one bounce on the floor. The coefficient of restitution assuming it to be same for two collisions is $n/2$. All surfaces are smooth. Find the value of n.



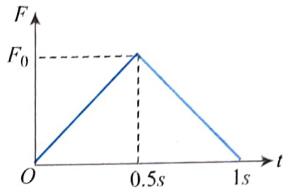
11. A small steel ball A is suspended by an inextensible thread of length $l = 1.5$ m from O. Another identical ball is thrown vertically downwards such that its surface remains just in contact with thread during downward motion and collides elastically with the suspended ball. If the suspended ball just completes vertical circle after collision, calculate the velocity (in cm/s) of the falling ball just before collision ($g = 10\ \text{m/s}^2$).



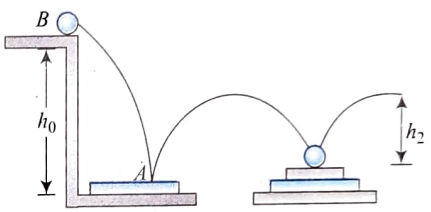
12. Two wooden plank of mass $M_1 = 1$ kg, $M_2 = 2.98$ kg smooth surface. A bullet of mass $m = 20$ g strikes the block M_1 and pierces through it, then strikes the plank B and sticks to its. Consequently both the planks move with equal velocities. Find the percentage change in speed of the bullet when it escapes from the first block.

13. A ball moving vertically downward with a speed of 10 m/s collides with a platform. The platform moves with a velocity of 5 m/s in downward direction. If $e = 0.8$, find the speed (in m/s) of the ball just after collision.

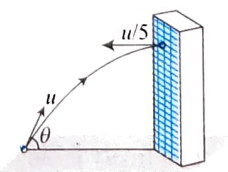
14. A body of mass 1 kg moving with velocity 1 m/s makes an elastic one dimensional collision with an identical stationary body. They are in contact for brief time 1 s. Their force of interaction increases from zero to F_0 linearly in time 0.5 s and decreases linearly to zero in further time 0.5 s as shown in figure. Find the magnitude of force F_0 in newton.



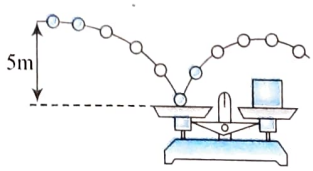
15. A 70 g ball B dropped from a height $h_0 = 9$ m reaches a height $h_2 = 0.25$ m after bouncing twice from identical 210 g plates. Plate A rests directly on hard ground, while plate C rests on a foam–rubber mat. Find the height h_1 (in m) of the ball's first bounce.



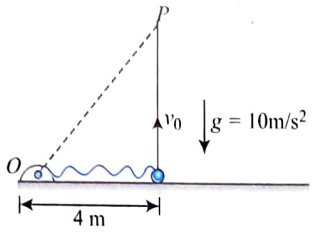
16. A small ball thrown at an initial velocity $u = 25$ m/s directed at an angle $\theta = 37°$ above the horizontal collides elastically with a vertical massive smooth wall moving with a uniform horizontal velocity $u/5$ towards the ball. After collision with the wall the ball returns to the point from where it was thrown. Determine the time t (in s) from the beginning of motion of the ball to the moment of its impact with the wall. (Take $g = 10\ \text{m/s}^2$)



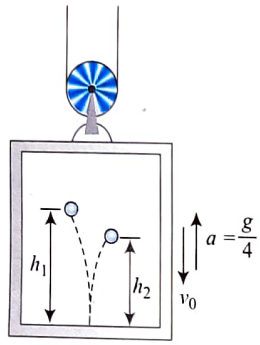
17. A stream of glass beads, each with a mass of 15 gram, comes out of a horizontal tube at a rate of 100 per second. The beads fall a distance of 5 m to a balance pan and bounce back to their original height. How much mass (in kg) must be placed in the other pan of the balance to keep the pointer at zero?



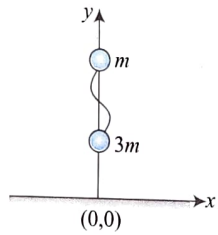
18. A 5 kg sphere is connected to a fixed point O by an inextensible string of length 5 m. The sphere is resting on a horizontal surface at a distance 4 m from O. Sphere is given a vertical velocity v_0, and it moves freely till it reaches point P, when string becomes taut. Find the maximum allowable velocity v_0 (in m/s), if impulse of tension in string is not to exceed 6 N s.



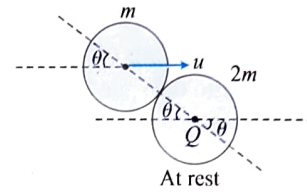
19. A ball is released from rest relative to the elevator at a distance $h_1 = 18$ m above the floor. The speed of the elevator at the time of ball release is v_0. Determine the bounce height h_2 (in m) of the ball. The elevator has an upward acceleration $a = g/4$. The coefficient of restitution for the impact $e = 2/3$.



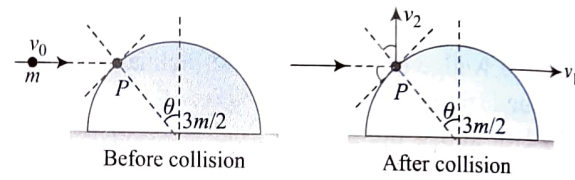
20. A coordinate axis system taking x-axis as horizontal smooth floor is shown in figure. Two small balls of masses m and $3m$ attached with a string are released from some heights on y-axis as shown in figure. The balls may collide head on or obliquely. After a certain time mass m is at (9 cm, 20 cm) while mass $3m$ is 25 cm above the x axis and the string is taut. The balls always remain in x-y plane. Find the length of string (in cm)



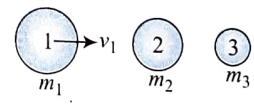
21. A boy is playing carom board. Strikers is moving with velocity u (= 10 m/s) and queen Q is at rest. See angle $\theta(=37°)$ in the figure. Mass of S and Q are m and $2m$. Friction is absent and coefficient of resistution is e. If after collision Q moves along SQ direction and S moves along perpendicular to SQ. Find the value of coefficient of restitution.



22. Mass of a particle and a hemisphere are m and $3m/2$ respectively. Particle hits the hemisphere (at rest on smooth surface) with a horizontal velocity v_0 and rebounds in vertical direction after collision (neglect friction), if the value of θ (in degree) (collision is elastic):



23. The centres of the spheres 1, 2 and 3 lie on a single straight line. Sphere 1 is moving with an (initial) velocity v_1 directed along this line and hits sphere 2. Sphere 2, acquiring after collision a velocity v_2, hits sphere 3. Both collisions are absolutely elastic. What must be the mass of sphere 2 (in kg) for the sphere 3 to acquire maximum velocity (The masses m_1 and m_3 of spheres 1 and 3 are 9 kg and 1 kg respectively)?



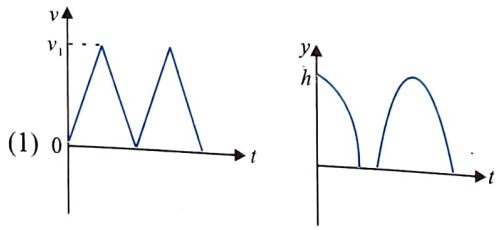
Archives

JEE Main

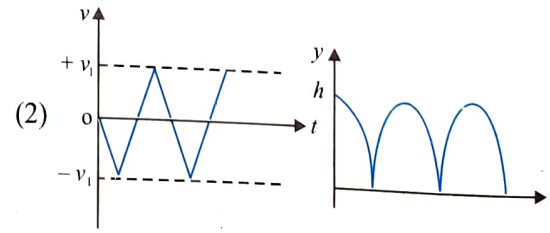
Single Correct Answer Type

1. Consider a rubber ball freely falling from a height $h = 4.9$ m on a horizontal elastic plate. Assume that the duration of collision is negligible and the collision with the plate is totally elastic. Then the velocity as a function of time and the height as a function of time will be

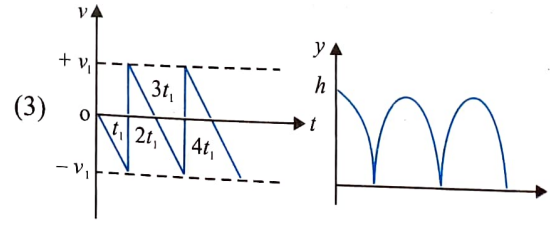
(1)

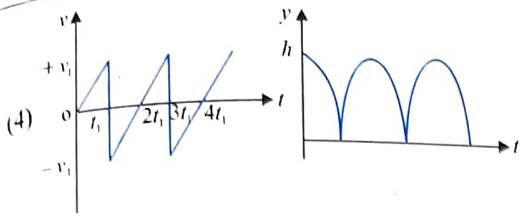


(2)



(3)



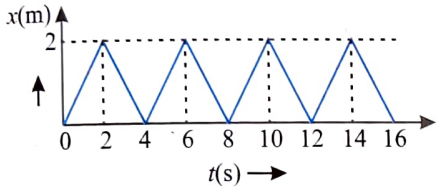
(4) 

(AIEEE 2009)

2. **Statement I:** Two particles moving in the same direction do not lose all their energy in a completely inelastic collision.
Statement II: The principle of conservation of momentum holds true for all kinds of collisions.
(1) Statement I is true, statement II is true; statement II is a correct explanation for statement I.
(2) Statement I is true, statement II is true; statement II is not a correct explanation for statement I.
(3) Statement I is false, statement II is true.
(4) Statement I is true, statement II is false.

(AIEEE 2010)

3. The figure shows the position–time (x–t) graph of one-dimensional motion of a body of mass 0.4 kg. The magnitude of each impulse is



(1) 0.4 Ns (2) 0.8 Ns
(3) 1.6 Ns (4) 0.2 Ns **(AIEEE 2010)**

4. This question has Statement I and Statement II. Of the four choices given after the Statements, choose the one that best describes the two Statements.
Statement I: A point particle of mass m moving with speed v collides with stationary point particle of mass M. If the maximum energy loss possible is given as $F=\left(\frac{1}{2}mv^2\right)$ then $f=\left(\frac{m}{M+m}\right)$.
Statement II: Maximum energy loss occurs when the particles get stuck together as a result of the collision.
(1) Statement I is true, Statement II is true, Statement II is not a correct explanation of Statement I.
(2) Statement I is true, Statement II is false.
(3) Statement I is false, Statement II is true
(4) Statement I is true, Statement II is true, Statement II is a correct explanation of Statement I.

(JEE Main 2013)

5. A particle of mass m moving in the x direction with speed $2v$ is hit by another particle of mass $2m$ moving in the y direction with speed v. If the collision is perfectly inelastic, the percentage loss in the energy during the collision is close to
(1) 44% (2) 50%
(3) 56% (4) 62% **(JEE Main 2015)**

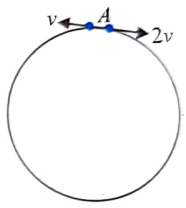
6. In a collinear collision, a particle with an initial speed v_0 strikes a stationary particle of the same mass. If the final total kinetic energy is 50 % greater than the original kinetic energy, the magnitude of the relative velocity between the two particles, after collision, is
(1) $\frac{v_0}{\sqrt{2}}$ (2) $\frac{v_0}{4}$
(3) $\sqrt{2}v_0$ (4) $\frac{v_0}{2}$ **(JEE Main 2018)**

7. It is found that if a neutron suffers an elastic collinear collision with deuterium at rest, fractional loss of its energy is P_d, while for its similar collision with carbon nucleus at rest, fractional loss of energy is P_c. The value of P_d and P_c are respectively.
(1) (0, 1) (2) (0.89, 0.28)
(3) (0.28, 0.89) (4) (0, 0) **(JEE Main 2018)**

JEE ADVANCED

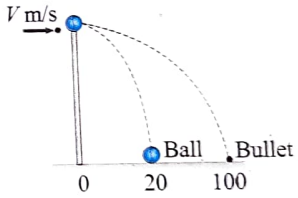
Single Correct Answer Type

1. Two small particles of equal masses start moving in opposite directions from a point A in a horizontal circular orbit. Their tangential velocities are v and $2v$, respectively, as shown in the figure. Between collisions, the particles move with constant speeds. After making how many elastic collisions, other than that at A, these two particles will again reach the point A?



(1) 4 (2) 3
(3) 2 (4) 1 **(IIT-JEE 2009)**

2. A ball of mass 0.2 kg rests on a vertical post of height 5 m. A bullet of mass 0.01 kg, travelling with a velocity V m/s in a horizontal direction, hits the centre of the ball.



After the collision, the ball and the bullet travel independently. The ball hits the ground at a distance of 20 m and the bullet at a distance of 100 m from the foot of the post. The initial velocity V of the bullet is
(1) 250 m/s (2) $250\sqrt{2}$ m/s
(3) 400 m/s (4) 500 m/s

(IIT-JEE 2011)

Multiple Correct Answers Type

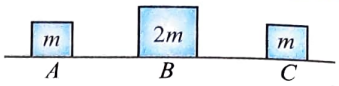
1. A point mass of 1 kg collides elastically with a stationary point mass of 5 kg. After their collision, the 1 kg mass reverses its direction and moves with a speed of 2 ms^{-1}. Which of the following statement(s) is (are) correct for the system of these two masses?
(1) Total momentum of the system is 3 kg ms^{-1}
(2) Momentum of 5 kg mass after collision is 4 kg ms^{-1}
(3) Kinetic energy of the centre of mass is 0.75 J
(4) Total kinetic energy of the system is 4 J

(IIT-JEE 2010)

2. A flat plate is moving normal to its plane through a gas under the action of a constant force F. The gas is kept at a very low pressure. The speed of the plate v is much less than the average speed u of the gas molecules. Which of the following options is/are true?
   (1) The pressure difference between the leading and trailing faces of the plate is proportional to uv
   (2) The plate will continue to move with constant non-zero acceleration, at all times
   (3) At a later time external force F balances the resistive force
   (4) The resistive force experienced by the plate is proportional to v **(JEE Advanced 2017)**

Numerical Value Type

1. Three objects A, B and C are kept in a straight line on a frictionless horizontal surface.



These have masses m, $2m$ and m, respectively. The object A moves towards B with a speed 9 m/s and makes an elastic collision with it. Thereafter, B makes completely inelastic collision with C. All motions occur on the same straight line. Find the final speed (in m/s) of object C. **(IIT-JEE 2009)**

2. A solid horizontal surface is covered with a thin layer of oil. A rectangular block of mass $m = 0.4$ kg is at rest on this surface. An impulse of 1.0 N s is applied to the block at time to $t = 0$ so that it starts moving along the x-axis with a velocity $v(t) = v_0 e^{-t/\tau}$, where v_0 is a constant and $\tau = 4$ s. The displacement of the block, in metres, at $t = \tau$ is..... . Take $e^{-1} = 0.37$. **(JEE Advanced 2018)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (4)	3. (1)	4. (3)	5. (3)
6. (4)	7. (2)	8. (2)	9. (4)	10. (3)
11. (2)	12. (2)	13. (1)	14. (1)	15. (3)
16. (2)	17. (4)	18. (3)	19. (1)	20. (3)
21. (3)	22. (2)	23. (3)	24. (2)	25. (1)
26. (2)	27. (3)	28. (3)	29. (3)	30. (4)
31. (2)	32. (1)	33. (2)	34. (3)	35. (4)
36. (1)	37. (2)	38. (4)	39. (1)	40. (2)
41. (3)	42. (4)	43. (4)	44. (1)	45. (4)
46. (1)	47. (1)	48. (3)	49. (3)	50. (3)
51. (4)	52. (1)	53. (2)	54. (2)	55. (3)
56. (2)	57. (3)	58. (3)	59. (2)	

Multiple Correct Answers Type

1. (2),(3),(4)	2. (1),(3),(4)	3. (2),(3)
4. (2),(3)	5. (1),(3)	6. (1),(3)
7. (2),(4)	8. (2),(3)	9. (1),(2),(4)
10. (1),(2),(4)	11. (1),(3)	12. (3),(4)
13. (1),(3)	14. (1),(2)	15. (1),(4)
16. (1),(2),(4)	17. (1),(2),(3)	18. (1),(3)
19. (1),(4)	20. (3),(4)	21. (1),(2),(3),(4)
22. (1),(3)	23. (2),(3)	24. (1),(2),(4)
25. (2),(3)	26. (1),(2),(3)	27. (1),(3),(4)
28. (3),(4)	29. (2),(3),(4)	30. (1),(2)

Linked Comprehension Type

1. (4)	2. (1)	3. (1)	4. (2)	5. (3)
6. (4)	7. (2)	8. (4)	9. (2)	10. (1)
11. (2)	12. (1)	13. (1)	14. (2)	15. (4)
16. (4)	17. (1)	18. (2)	19. (3)	20. (2)
21. (2)	22. (3)	23. (4)	24. (1)	25. (1)
26. (3)	27. (2)	28. (3)	29. (4)	30. (2)
31. (1)	32. (3)	33. (2)	34. (2)	35. (3)
36. (2)	37. (3)	38. (1)	39. (4)	40. (1)
41. (4)	42. (2)	43. (4)	44. (3)	45. (1)
46. (4)	47. (4)	48. (3)	49. (2)	50. (1)

Matrix Match Type

1. i. → d.; ii. → a.; iii. → c.; iv. → b.
2. i. → a., d.; ii. → c., d.; iii. → c.; iv. → a., b., c.
3. i. → a., d.; ii. → c., d.; iii. → c., d.; iv. → c., d.
4. i. → a., b.; c.; ii. → a., b., c.; iii. → a., b.; iv. → a., b.
5. i. → c., d.; ii. → c.; iii. → a.; iv. → a., b., c., d.
6. i. → c.; ii. → d.; iii. → a.; iv. → b.
7. i. → a., b.; ii. → a., b.; iii. → d.; iv. → c.
8. (2) 9. (1)

Numerical Value Type

1. (5)	2. (45)	3. (1)	4. (9)	5. (3)
6. (2)	7. (2)	8. (38)	9. (4)	10. (1)
11. (1250)	12. (33.3)	13. (1)	14. (2)	15. (4)
16. (1.8)	17. (3)	18. (8)	19. (8)	20. (13)
21. (0.50)	22. (60)	23. (3)		

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3)	2. (1)	3. (2)	4. (3)	5. (3)
6. (3)	7. (2)			

JEE Advanced

Single Correct Answer Type

1. (3) 2. (4)

Multiple Correct Answers Type

1. (1),(3) 2. (1),(3),(4)

Numerical Value Type

1. (4) 2. (6.30)

3 Rigid Body Dynamics: Part 1

INTRODUCTION

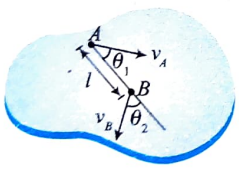
In previous sections we had learnt about particle dynamics and introduced some basic concepts of circular motion. In this chapter we will learn about rigid body dynamics. Rigid body dynamics studies the movement of systems of interconnected bodies under the action of external forces.

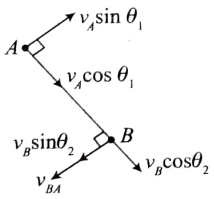
RIGID BODY

Generally, we call an object **rigid** when we cannot deform (squeeze and twist) it. This mean, the shape and size of an ideal rigid body does not change. However, in practice, any object can be compressed or elongated by certain extent. In nature nothing is perfectly rigid. When the deformation of an object is insignificant compared to its dimension, practically we call it a **rigid body**. (Here, we have used the word 'deformation' for compression, elongation and shearing (twisting or bending).) We can define a rigid body as a body which undergoes no deformation compared to its size, under the action of an external force or torque. In consequence, the distance between any two particles of rigid body is fixed.

For every pair of particles in a rigid body, there is no velocity of separation or approach between the particles i.e., any relative motion of a point B on a rigid body with respect to another point A on the rigid body will be perpendicular to line joining A to B. Hence the motion of any particle B with respect to any other particle A of a rigid body is circular motion.

Let velocities of A and B with respect to ground be $\vec{v}_A$ and $\vec{v}_B$ respectively in the figure.





$v_A \cos\theta_1 = v_B \cos\theta_2$ (velocity of approach/separation is zero)

v_{BA} = Velocity of B with respect to A.

$v_{BA} = v_A \sin\theta_1 + v_B \sin\theta_2$ (which is perpendicular to line AB)

B will appear to move in a circle to an observer fixed at A.

The angular velocity of B with respect to A, $\omega_{B,A} = \frac{v_{B,A}}{l}$, where l is distance between A and B.

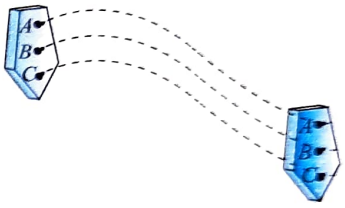
This is also angular velocity of rigid body about its centre of mass.

GENERAL RIGID BODY MOTION

Translation

If a body is moving such that a line drawn between any two internal points remains parallel to itself, the motion is said to be translation. In translation, all the particles of the body move along parallel paths. If these parallel paths are straight lines, the motion is said to be a rectilinear; if the paths are along curved lines, the motion is a curvilinear. For a rigid body in translation, all the points of the body have the same velocity and same acceleration at any given instant.

Rectilinear translation motion

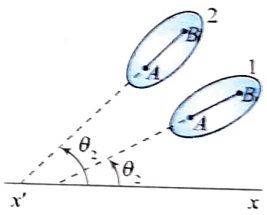


Curvilinear translational motion

Rotation

The simplest case of rotation is rotational motion in a plane, i.e., the two dimensional.

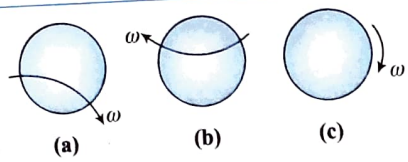
Let us consider a plate like (laminar) rigid body and draw a straight line connecting any two points A and B, say, on the plate. Then produce the line AB towards a fixed reference line $X'X$. Let the line AB make and angle θ_1 with $X'X$. We call this angle as 'angular position' of the straight-line AB. As the plate moves from position 1 to position 2, we can see that the angular position of the line AB changes from θ_1 to θ_2 That is what we call **angular motion** of the straight line AB. More popularly angular motion is known as 'rotational motion or rotation'.



The straight-line AB changes its angle of orientation with $X'X$ line from θ_1 to θ_2 as the plate moves from position 1 to position 2. From the above discussions we conclude that:

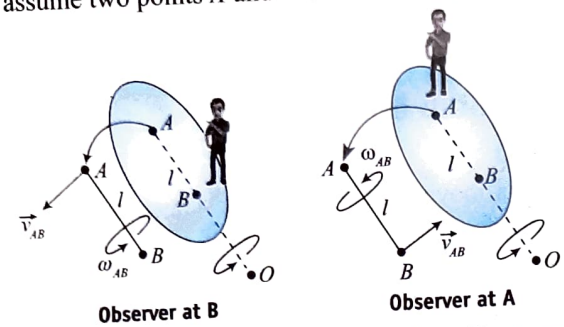
"Whenever the angle of orientation of a straight line drawn inside a rigid body changes with respect to a fixed reference line, we say that the rigid body undergoes angular motion or rotational motion or rotation."

A rigid body does not require any axis to rotate in an unconstructed rotation. In this case you can mention the angular velocity in any way you like, keeping its sense constant as shown in Figs. (a), (b) and (c).



Relative Angular Velocity in Case of a Rigid Body

In a rigid body the distance between two points remains constant. Let us assume two points A and B on a rotating plate.



If we observe point A from point B (see Fig. (a)), A appears to move perpendicular to the line AB with velocity $\vec{v}_{AB}$. Here we can say that point A rotates about point B with angular velocity ω_{AB}, which can be expressed as

$$\omega_{AB} = \frac{v_{AB}}{l} \quad \text{(in anticlockwise sense)}$$

In vector form, we can write $\vec{\omega}_{AB} = \frac{v_{AB}}{l}(\hat{k})$ ...(i)

Now if we place our observer at A and observe point B [see Fig. (b)], we find B appear to move with velocity $\vec{v}_{BA}$ perpendicular to line AB. It means point B rotates about A with angular velocity ω_{BA} which can be expressed as in previous case.

$$\omega_{BA} = \frac{v_{BA}}{l} \quad \text{(in anticlockwise sense)}$$

In vector form $\vec{\omega}_{BA} = \frac{v_{BA}}{l}(\hat{k})$ ...(ii)

Comparing Eqs. (i) and (ii), we find $\vec{\omega}_{AB} = \vec{\omega}_{BA}$

"Each point in a rigid body turns relative to the other points of the rigid body with same angular velocity at a given instant." The angular velocity of any point of a rigid body at any instant relative to any other point of rigid body is constant which is defined as the angular velocity of the rigid body, whereas at any instant the angular velocities of different points of the rigid body relative to any point outside the rigid body are different.

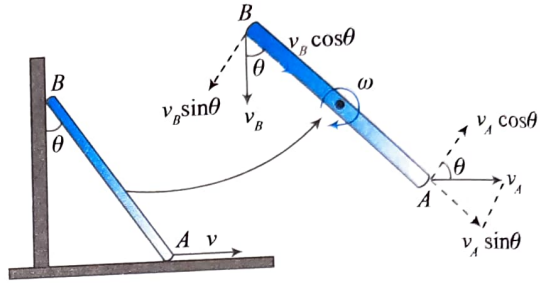
Important Points:

- With respect to any particle of the rigid body, the motion of any other particle of that rigid body is circular.
- In case of a rotating body we can say angular velocity of one point w.r.t. to other point of rigid body will be equal, i.e., $\vec{\omega}_{12} = \vec{\omega}_{21}$, but their relative linear velocities will not be equal $\vec{v}_{12} \neq \vec{v}_{21}$. Here we can say $\vec{v}_{12} = -\vec{v}_{21}$.
- In case of rotation of a rigid body, the statement that the relative angular velocity of point w.r.t. to other point is equal, i.e., $\vec{\omega}_{12} = \vec{\omega}_1 - \vec{\omega}_2$ is vague, but for relative linear velocity $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ is valid.
- When any two points A and B (say) of a rigid body have velocities $\vec{v}_A$ and $\vec{v}_B$, respectively, the angular velocity of rigid body ω can be expressed as $\omega = \frac{|\vec{v}_A - \vec{v}_B|}{l}$.

Illustration 3.1

A rod of length l resting on a wall and the floor. Its lower end A is pulled towards right with a constant velocity v. Find the velocity of the other end B downward when the rod makes an angle θ with the vertical. Also, find the angular velocity of the rod.

Sol. As rod is a rigid body, the distance between point A and point B should remain unchanged. Hence the relative velocity of ends of rod along its length should be zero.



It means the velocities of ends of rod along its length must be equal.

$$v_A \sin\theta = v_B \cos\theta \Rightarrow v_B = v_A \frac{\sin\theta}{\cos\theta} = v\tan\theta$$

or $v_B = v\tan\theta$

Then, $\omega_{BA} = \frac{v_{BA_\perp}}{l} = \frac{|v_A\cos\theta - (-v_B\sin\theta)|}{l}$

$$\Rightarrow \omega_{BA} = \frac{v_A\cos\theta + v_B\sin\theta}{l}$$

Using Eqs. (i) and (ii),

$$\omega_{BA} = \frac{v}{l\cos\theta}$$

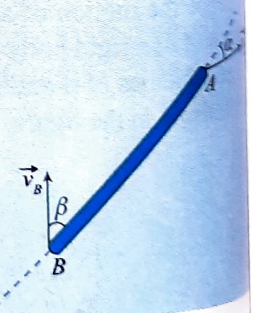
This is also angular velocity of rigid body about its centre of mass.

Illustration 3.2

The ends A and B of a rod of length l have velocities of magnitudes $|\vec{v}_A| = v$ and $|\vec{v}_B| = 2v$, respectively. If the inclination of $\vec{v}_A$ relative to the rod is α, find the:

(a) inclination β of $\vec{v}_B$ relative to the rod,

(b) angular velocity of the rod.



Sol.

(a) We know in a rigid body the velocities of two points along the line joining two should remain same.

i.e., $(v_A)_{\parallel} = (v_B)_{\parallel}$

Hence $v\cos\alpha = 2v\cos\beta$

$$\cos\beta = \frac{1}{2}\cos\alpha \text{ or } \beta = \cos^{-1}\left(\frac{1}{2}\cos\alpha\right)$$

(b) Angular velocity of the rod, $\omega = \frac{|(\vec{v}_{AB})_\perp|}{l} = \frac{|(\vec{v}_A - \vec{v}_B)_\perp|}{l}$

$$|\vec{v}_{AB}| = |\vec{v}_A - \vec{v}_B| = v\sin\alpha - (-2v\sin\beta)$$
$$= v(\sin\alpha + 2\sin\beta)$$

$$\Rightarrow \omega = \frac{(\sin\alpha + 2\sin\beta)v}{l} \text{ (in clockwise direction)}$$

We have calculated $\cos\beta = \frac{\cos\alpha}{2}$. Thus,

$$\sin\beta = \sqrt{1-\cos^2\beta} \Rightarrow \sin\beta = \frac{\sqrt{3+\sin^2\alpha}}{2}$$

Hence $\omega = \frac{\left(\sin\alpha + \sqrt{3+\sin^2\alpha}\right)v}{l}$ (in clockwise direction)

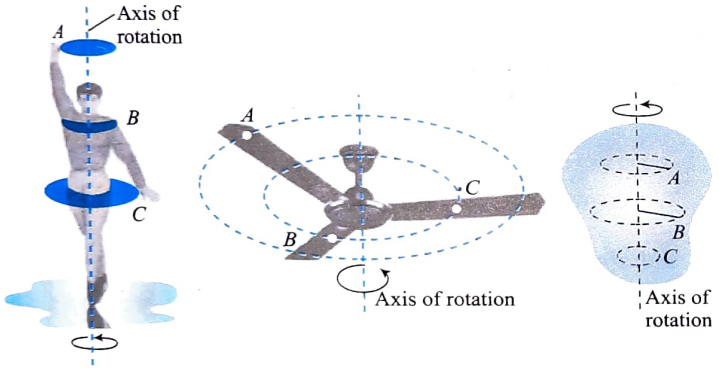
TYPES OF ROTATIONAL MOTION

Motion of body involving rotation can be classified into the following two categories.

1. Rotation about a fixed axis or pure rotation.
2. Rotation about an axis in translation or combined rotation and translation.

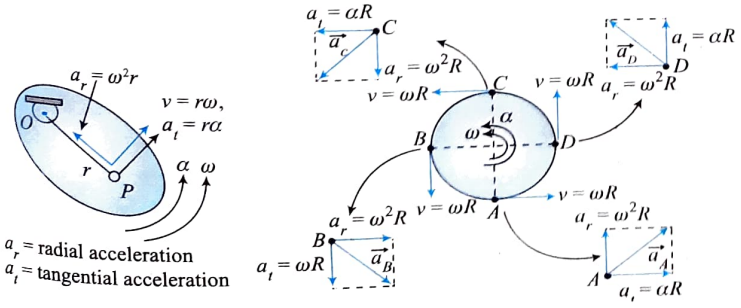
ROTATION ABOUT A FIXED AXIS OR PURE ROTATION

When a rigid object rotates, points on the object, such as A, B, or C, move in circular paths. An imaginary line perpendicular to plane of circular paths of particles of a rigid body in rotation and containing the centers of all these circular paths is known as **axis of rotation**. Rotation of ceiling fan, potter's wheel, opening and closing of doors and hands of a wall clock etc. come into this category.



Velocity and Acceleration of a Point in Rotating Rigid Body

For pure translation, angular velocity and angular acceleration is zero and linear velocity is constant for all points. For pure rotation (fixed axis rotation) the angular velocity is constant for all particles about the axis of rotation (see figure).



Velocity and Acceleration in case of pure rotation

We can write the speed of point P [see Fig. (a)] as $v = \omega r$. In vector form, $\vec{v} = \vec{\omega} \times \vec{r}$. In Fig. (b), we can observe the velocities and accelerations of points A, B, C and on the disc. We can make two components of acceleration of point P, the tangential acceleration and radial acceleration.

The tangential acceleration of point P, $a_t = \alpha r$.

In vector form, it can be expressed as $\vec{a}_t = \vec{\alpha} \times \vec{r}$

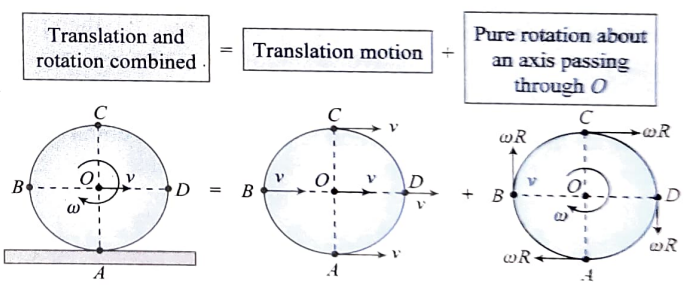
The radial acceleration of point P, $a_r = \omega^2 r$

Important Point:

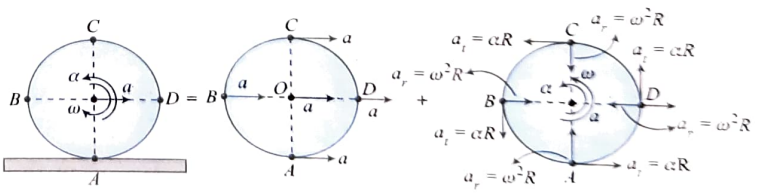
The direction of $\vec{\omega}$ can be found by using right hand rule. In vector form it can be expressed as $\vec{v}_{AB} = \vec{\omega}_{AB} \times \vec{r}_{AB}$, where $\vec{r}_{AB}$ = position vector of A relative to B.

COMBINED ROTATION AND TRANSLATION

The most general motion of a rigid body is the combination of translational and rotational motions. We have studied pure rotation and pure translation motions but what happens when the two are combined? The most familiar example of combined rotational and translational motion is a rolling wheel. Consider rolling of wheels of a vehicle, moving on straight level road. Relative to a reference frame the axel fixed with the vehicle, the wheel appears rotating about its axel. The rotation of the wheel from this frame is pure rotation about fixed axis. But relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axel, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions–rotation about the fixed axel and translation of the wheel along the level road.



Velocity of different points



Acceleration of different points

Velocity and Acceleration of different points in case translation and rotation combined

Point A	Velocity	$A \xrightarrow{v}$ + $\xleftarrow[\omega R]{} A$	=	$A \xrightarrow{v-\omega R}$
	Acceleration	$A \xrightarrow{a}$ + $a_r=\omega^2 R$, $a_t=\alpha R$	=	$\omega^2 R$, a_A, $a+\alpha R$

Point B	Velocity	$B \xrightarrow{v} + B \uparrow \omega R = \overset{B}{\bullet}\!\nearrow v_B,\ \overset{B}{\bullet}\!\rightarrow v,\ \uparrow \omega R$ $\lvert\vec{v}_B\rvert = \sqrt{v^2+\omega^2R^2}$
	Acceleration	$B \xrightarrow{a} + B \downarrow \omega^2R,\ \alpha R = \overset{B}{\bullet}\!\nearrow a+\omega^2R,\ \overset{B}{\bullet}\!\rightarrow \vec{a}_B,\ \downarrow \alpha R$
Point C	Velocity	$C \xrightarrow{v} + C \xrightarrow{\omega R} = \rightarrow v+\omega R$
	Acceleration	$C \xrightarrow{a} + \alpha R \leftarrow C \downarrow \omega^2R = C\!\nearrow a-\alpha R,\ C\!\rightarrow \vec{a}_C,\ \downarrow \omega^2R$
Point D	Velocity	$D \xrightarrow{v} + D \downarrow \omega R = D\!\rightarrow v,\ D\!\nearrow \vec{v}_D,\ \downarrow \omega R$ $\lvert\vec{v}_B\rvert = \sqrt{v^2+\omega^2R^2}$
	Acceleration	$D \xrightarrow{a} + \alpha R \uparrow D \leftarrow \omega^2R = D\!\rightarrow \alpha R,\ D\!\nearrow \vec{a}_D,\ \downarrow a-\omega^2R$

Expressing Translation and Rotation Combined in Vector Form

In this type of motion, axis of rotation is not stationary. If a body rotates about an axis with an angular velocity ω, then with respect to the axis of rotation linear velocity of any particle 'P' in the body at a distance r from the axis of rotation is equal to

$$\vec{v} = \vec{\omega}\times\vec{r}$$

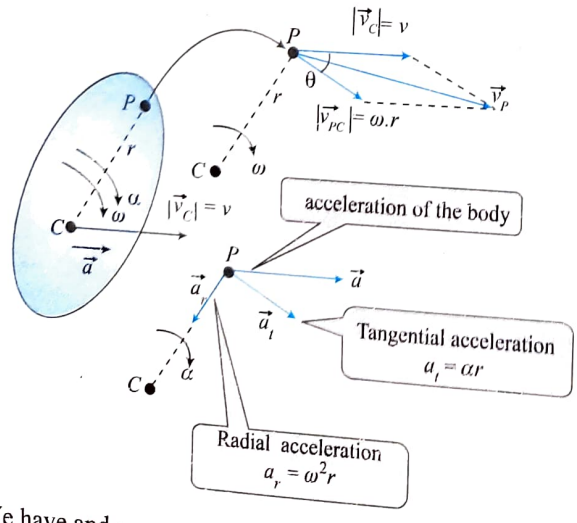
If the axis of rotation also moves with velocity $\vec{v}_0$, then the net velocity of the particle relative to stationary frame will be

$$\vec{v} = \vec{\omega}\times\vec{r} + \vec{v}_0$$

Now let us take a body which is moving with velocity $\vec{v}$ and rotating about center of mass with angular velocity ω. Let us analyze a point P on the body. The velocity of point P in the body is

$$\vec{v}_P = \vec{v}_{PC} + \vec{v}_C$$

$$v_P = \sqrt{v_{PC}^2 + v_C^2 + 2v_{PC}\cdot v_C\cdot\cos\theta}$$



We have and $v_{PC} = r\omega$ and $v_C = v$. Thus

$$v_P = \sqrt{(r\omega)^2 + v^2 + 2(r\omega)v\cdot\cos\theta}$$

In same way we can write the acceleration of point p

$$\vec{a}_P = \vec{a}_{P,C} + \vec{a}_C$$

$\vec{a}_{P,C}$ has both tangential as well as radial acceleration components. Hence, we can express, $\vec{a}_{P,C}$ as

$$\vec{a}_{P,C} = (\vec{a}_{P,C})_{\text{tangential}} + (\vec{a}_{P,C})_{\text{radial}}$$

$$(\vec{a}_{P,C})_{\text{tangential}} = r\alpha \text{ (acts perpendicular to line } CP)$$

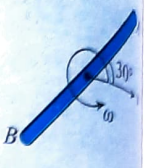
$$(\vec{a}_{P,C})_{\text{radial}} = \omega^2 r \text{ (acts along the line } PC)$$

Hence net acceleration of point P can be express as

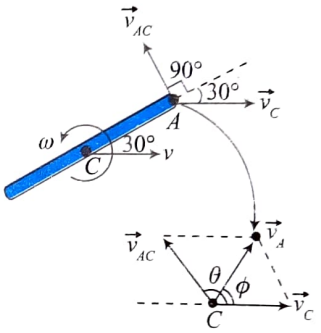
$$\vec{a}_P = [(\vec{a}_{P,C})_{\text{tangential}} + (\vec{a}_{P,C})_{\text{radial}}] + \vec{a}_C$$

ILLUSTRATION 3.3

A uniform rod of length l is spinning with an angular velocity $\omega = 2v/l$ while its centre of mass moves with a velocity v. Find the velocity of the end A of the rod.



Sol. The motion of point A is translation and rotation combined the velocity of end A can be expressed by, $\vec{v}_A = \vec{v}_{A,C} + \vec{v}_C$



Hence the velocity of A is

$$v_A = \sqrt{v_{AC}^2 + v_C^2 + 2v_{AC}v_C\cos\theta}$$

We know, $v_C = v, v_{AC} = \frac{l}{2}\cdot\omega$

and $\theta = 90° + 30° = 120°$

Hence, $v_A = \sqrt{v^2 + \frac{l^2\omega^2}{4} + lv\,\omega\cos 120°}$

But, $\omega = \frac{2v}{l}$, $v_{AC} = \frac{l}{2}\omega = \frac{l}{2}\left(\frac{2v}{l}\right) = v$

which gives $v_A = v$

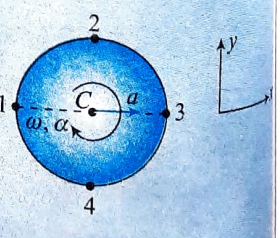
Let $\vec{v}_A$ make angle ϕ with the direction of $\vec{v}_C$

Then $\phi = \tan^{-1}\left(\frac{v_{AC}\sin\theta}{v_{AC}\cos\theta + v_C}\right)$

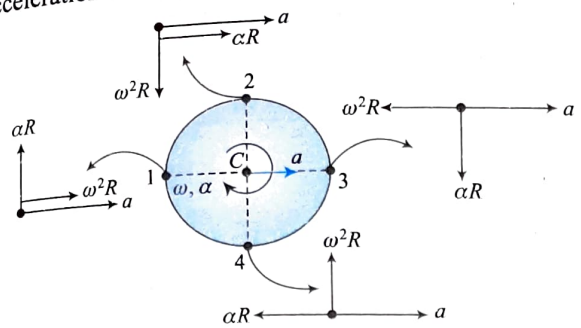
We get $\phi = \tan^{-1}\left(\frac{v\sin 120°}{v\cos 120° + v}\right) = \frac{\pi}{3}$

ILLUSTRATION 3.4

A uniform disc of radius r spins with angular velocity ω and angular acceleration α. If the centre of mass of the disc has linear acceleration a, find the magnitude and direction of acceleration of the points 1, 2, 3 and 4.



Sol. The motion of disc is translation and rotation combined, the acceleration of any point 'P' on the disc can be expressed by:



$$\vec{a}_P = \vec{a}_{PC} + \vec{a}_C$$

The acceleration of point '1':

$$\vec{a}_1 = \vec{a}_{1,C} + \vec{a}_C$$

But $\vec{a}_{1,C} = (\vec{a}_{1,C})_{\text{tangential}} + (\vec{a}_{1,C})_{\text{radial}}$

$\Rightarrow \vec{a}_{1,C} = \alpha R\,\hat{j} + \omega^2 R\,\hat{i}$ and $\vec{a}_C = a\hat{i}$

Hence $\vec{a}_1 = (\omega^2 R + a)\hat{i} + \alpha R\,\hat{j}$

Similarly, the acceleration of '2':

$$\vec{a}_2 = \vec{a}_{2,C} + \vec{a}_C$$

Here $\vec{a}_{2,C} = (\alpha R\hat{i} - \omega^2 R\hat{j})$ and $\vec{a}_C = a\hat{i}$

Hence $\vec{a}_2 = (a + \alpha R)\hat{i} - \omega^2 R\hat{j}$

Also, acceleration of '3':

$$\vec{a}_3 = \vec{a}_{3,C} + \vec{a}_C$$

Here $\vec{a}_{3,C} = -\omega^2 R\hat{i} - \alpha R\hat{j}$ and $\vec{a}_C = a\hat{i}$

Hence $\vec{a}_3 = (a - \omega^2 R)\hat{i} - \alpha R\hat{j}$

The acceleration of point '4':

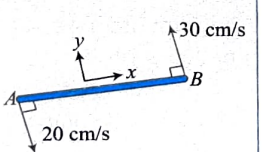
$$\vec{a}_4 = \vec{a}_{4,C} + \vec{a}_C$$

$\Rightarrow \vec{a}_{4,C} = -\alpha R\hat{i} + \omega^2 R\,\hat{j}$ and $\vec{a}_C = a\hat{i}$

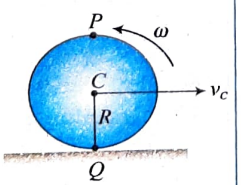
Hence $\vec{a}_4 = (a - \alpha R)\hat{i} + \omega^2 R\hat{j}$

CONCEPT APPLICATION EXERCISE 3.1

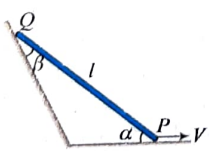
1. A 100 cm rod is moving on a horizontal surface. At an instant, when it is parallel to the x-axis its ends A and B have velocities 20 cm/s and 30 cm/s as shown in the figure. Find its angular velocity of the rod.



2. A ball of radius $R = 20$ cm is moving with combined translation and rotation. If the velocity of centre of mass of ball is $v_C = 10$ m/s and spins with an angular speed $\omega = 10$ rad/s, find the speeds of P and Q.

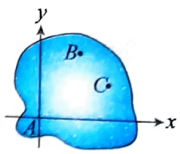


3. A rod of length l has a velocity v at P. Find the velocity at Q and the angular velocity of the rod.



4. If the ends of a rod of length l have velocities $\vec{v}_A$ and $\vec{v}_B$, find the velocity of mid-point of the rod.

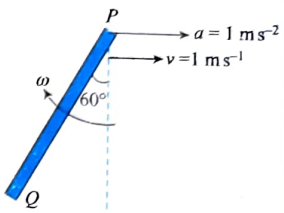
5. A flat rigid body is moving in x-y plane on a table. The body lies in the x-y plane. At an instant it was found that some of the velocity components of its three particles A, B and C were $v_{Ax} = 4$ m/s, $v_{Bx} = 3$ m/s and $v_{Cy} = -2$ m/s, respectively. At the instant the three particles A, B and C were located at (0,0), (3,4) and (4,3) (all in metre) respectively in a co-ordinate system attached to the table.



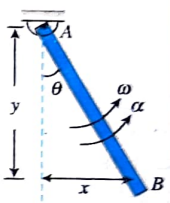
(a) Find the velocity of A, B and C.

(b) Find the angular velocity of the body.

6. The rod of length $l = 1$ m rotates with an angular velocity $\omega = 2$ rad s^{-1} and the point P moves with velocity $v = 1$ m s^{-1} and acceleration $a = 1$ m s^{-2}. Find the velocity and acceleration of Q.



7. The angular velocity and angular acceleration of the pivoted rod are given as ω and α respectively. Find the x and y components of acceleration of B.



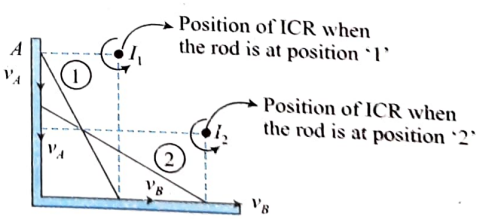
ANSWERS

1. $0.50\,\hat{k}$ rad / s
2. 8 m/s, 12 m/s
3. $\dfrac{v\cos\alpha}{\cos\beta}, \dfrac{v\sin(\alpha+\beta)}{l\cos\beta}$
4. $\vec{v}_C = \dfrac{\vec{v}_A + \vec{v}_B}{2}$
6. $\sqrt{3}$ m/s, $\sqrt{17 + 4\sqrt{3}}$ m/s^2

INSTANTANEOUS AXIS OF ROTATION

In combined translational and rotational motion, we cannot have a permanent axis of rotation. However, there may be a point inside the given body (or extended body) which can be momentarily at rest. This is what we call **instantaneous axis of rotation.**

Consider a rod which remains always in the same vertical plane with its ends in contact with smooth wall and floor as shown in figure. As the rod slides, its centre of mass translates and the rod rotates about a horizontal axis passing through centre of mass. The motion of such an object can be analyzed by making equations for translation and rotation separately, which involves lengthy calculations.



But the motion of such an object may be looked as pure rotation about a point that has zero velocity to simplify the study. Such a

point is called **instantaneous centre of rotation (ICR)** and the axis passing through this point and perpendicular to the plane of motion is called **instantaneous axis of rotation**.

In the figure shown I_1 and I_2 are the instantaneous centers of rotation of rod at two different instants.

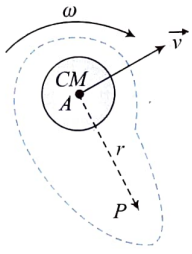
Important Points:

- The instantaneous centre of rotation is stationary with respect to ground.
- It may lie within or outside the body.
- The angular velocity of the body about this point is the same as that about its centre of mass.

FINDING INSTANTANEOUS AXIS OF ROTATION IN DIFFERENT SITUATIONS

When Velocity of a Point on Rigid Body (or Velocity of Centre of Mass) and Its Angular Velocity is Given

The **instantaneous axis of rotation** of the body is the point P of the given (shaded) body or extended body which remains at rest at that instant.

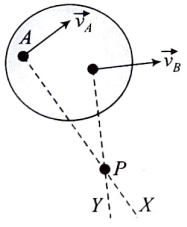


To locate **instantaneous axis of rotation**, drop a perpendicular at a point moving with velocity vector $\vec{v}$. Take point P (ICR) at a distance $r = v/\omega$ in the proper direction along the perpendicular.

Velocities of Any Two Points in the Body are Given

Consider a rigid body moving in a plane. Let A and B are two points having velocities $\vec{v}_A$ and $\vec{v}_B$ respectively at the instant considered.

AX and BY are the perpendicular to the velocities $\vec{v}_A$ and $\vec{v}_B$ respectively. The point of intersection of the two normal AX and BY, P will be the center of zero velocity or instantaneous centre of rotation of the rigid body.

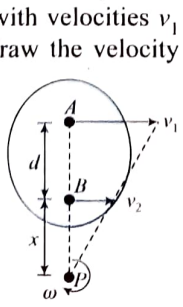


The instantaneous axis of rotation passes through instantaneous centre of rotation and perpendicular to the plane of rotation.

Now we can calculate the angular velocity of the rigid body, we have $v_A = (AP)\omega$. Then, the angular velocity of the rigid body is given as: $\omega = \frac{v_A}{AP} = \frac{v_B}{BP}$

Two Points in a Body Move Along Parallel Lines

Case I: Let two points A and B of a rigid body which are separated by a distance d are moving in same direction with velocities v_1 and v_2. For finding the position of ICR, we draw the velocity vectors at positions A and B and join their tails and heads and produce the straight lines to meet at the ICR 'P'.



This is the point of zero velocity as shown in figure. The location of instantaneous axis of rotation is given as $\frac{v_1}{d+x} = \frac{v_2}{x} = \omega$

where ω is the angular velocity of the body.

Case II: Join the tails and heads of the velocity vectors $\vec{v}_A$ and $\vec{v}_B$. The meeting point P is the ICR. If antiparallel velocities are given, location of instantaneous axis of rotation is given, as $\frac{v_1}{d-x} = \frac{v_2}{x} = \omega$

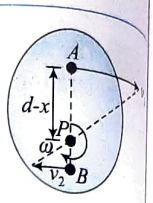
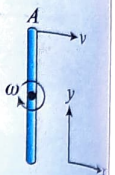


ILLUSTRATION 3.5

Find the instantaneous centre of rotation (ICR) of a rod of length l when its end A moves with a velocity $\vec{v}_A = v\hat{i}$ and the rod rotates with an angular velocity $\vec{\omega} = -\frac{v}{2l}\hat{k}$



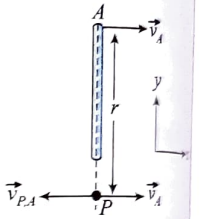
Sol. **Method 1:** For locating the position of instantaneous centre of rotation (ICR), let us select a point P on the extended rod such that it has zero velocity as ICR is a point of zero velocity so we can write, $\vec{v}_P = \vec{v}_{P,A} + \vec{v}_A$

We have $\vec{v}_P = 0$; hence $\vec{v}_{P,A} + \vec{v}_A = 0$

Here $\vec{v}_A = v\hat{i}$ and $\vec{v}_{P,A} = -\omega r\hat{i}$

Hence $-\omega r\hat{i} + v\hat{i} = 0 \Rightarrow v = \omega r$

or $r = \frac{v}{\omega} = \frac{v}{(v/2l)} = 2l$



Hence ICR will be located at a distance $2l$ from A.

Method 2: The instantaneous centre of rotation will lie on the perpendicular to $\vec{v}_A$ at point A.

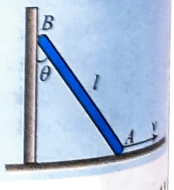
The ICR will lie at distance $AP = \frac{|\vec{v}_A|}{\omega} = \frac{v}{(v/2l)} = 2l$

or at a distance $2l$ from point A.

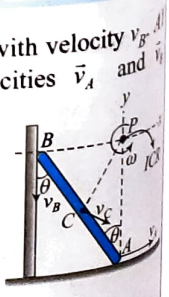
Note: When we get the IAR outside the given (real) body, we think that the point lying on IAR is fixed with the hypothetically extended body. It may mean that, the rod had been sufficiently long, the IAR would have been situated on the real rod at the distance of $2l$ from the end A of the rod.

ILLUSTRATION 3.6

A rod 'AB' of length l leaning against a smooth vertical wall and smooth horizontal surface. Now lower end 'A' of the rod is pulled with a constant velocity v (say). In consequence, the upper end start sliding against the wall and the rod rotates in the vertical plane. Locate the position of instantaneous center of rotation (ICR) of the rod when it makes an angle θ with vertical and find the angular velocity of the rod.



Sol. Let end B slide on the vertical wall with velocity v_B. AY and BX are the perpendicular to the velocities $\vec{v}_A$ and $\vec{v}_B$ respectively. The point of intersection of the two normal AY and BX, P will be the center of zero velocity or instantaneous centre of rotation of the rod. Each and every point of the rod rotate about P with same angular velocity (say ω) in counter clockwise sense.



We have $\omega_{BP} = \omega_{AP} = (\omega)$,

where $\omega_{AP} = \dfrac{v_A}{PA}$

Substituting $PA = l\cos\theta$ and $v_A = v$, we have

$$\omega = \frac{v}{l\cos\theta}$$

ROLLING MOTION

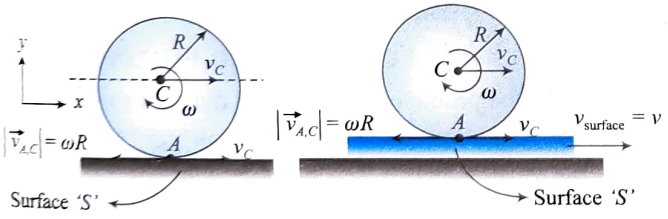
The term rolling means that there is no slipping at the point of contact with the ground—no skidding takes place. The wheel of a car that is moving forward while its tyres are spinning at high speed, leaving behind black strips on the road, is an example of rolling motion with slipping.

A body in combined rotation and translation motion said to be rolling over a surface if the surfaces in contact do not slide relative to each other, it means that the relative velocity between the points of contact of the body with the surface is zero.

Let us take a uniform disc of radius R (say). Place it vertically on a smooth horizontal surface. Spin the disc and simultaneously push it. In consequence, the disc will rotate with an angular velocity ω and its centre of mass C moves with a velocity v_c. The velocity of lowest point A of the disc can be given as: $\vec{v}_A = \vec{v}_{AC} + \vec{v}_C$, where $\vec{v}_{AC} = -R\omega\hat{i}$ and $\vec{v}_C = v_C\hat{i}$. Then velocity of point A, $\vec{v}_A = (v_C - R\omega)\hat{i}$.

The above expression tells us that for different values of v_c and ω, the point A will move with different speeds. Let us assume that A moves with a velocity which is equal to the velocity of the horizontal surface S; then $\vec{v}_A = \vec{v}_S$. It means, relative velocity between A and S is zero.

Then you can say that P does not slide on the surface S. In other words, the body is said to be rolling without sliding on the horizontal surface.



(a) (b)

In the given Fig. (a), the surface on which the disc is rolling, is fixed, then the velocity of point of contact should be zero ($v_A = 0$). Thus, the rolling condition is equivalent to the following relation between the translational speed of the centre of mass and the angular speed.

$$v_C = R\omega \qquad ...(i)$$

Differentiation of Eq. (i) relates the magnitude of the linear acceleration of the centre of the wheel; a_c, to the magnitude of the angular acceleration α about the axis through the centre of the wheel. Similarly,

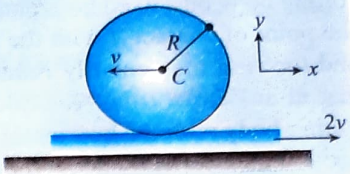
$$a_C = R\alpha \qquad ...(ii)$$

In the given Fig. (b), the surface on which the disc is rolling, is moving with velocity v, then for no sliding case the velocity of point of contact should be same as velocity of the surface

$$v = v_C - R\omega$$

ILLUSTRATION 3.7

A uniform disc is spun with an angular velocity $\vec{\omega}$ and simultaneously projected with a linear velocity v towards left on a plank, while the plank moves towards right with a constant velocity $2v$. If the disc rolls without sliding on the plank just after its spinning, find the magnitude of $\vec{\omega}$.



Sol. For pure rolling of the disc on the plank, $\vec{v}_P = \vec{v}_P$

The velocity of plank is in rightward direction and disc is moving in leftward direction. As disc is rolling, there should not be relative sliding between plank and disc at point of contact. For this, velocity of point P should be rightwards. It means the rotation of disc should be in counter clockwise sense.

$$\vec{v}_{P,C} + \vec{v}_C = \vec{v}_P$$

or $\omega R - v = 2v$

$\Rightarrow \omega = \dfrac{3v}{R}$ in counter clockwise sense.

In vector form, $\vec{\omega} = \dfrac{3v}{R}(\hat{k})$

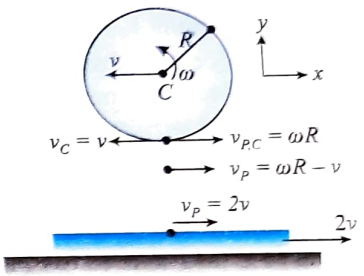
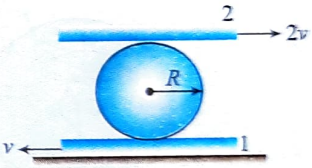


ILLUSTRATION 3.8

A cylinder of radius R is rolling without sliding between two horizontal planks (surfaces) 1 and 2 as shown in figure. If the velocities of the planks A and B are v and $2v$ respectively, find the:

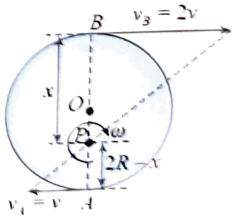
(a) position of ICR.

(b) angular velocity of the cylinder.



Sol.

(a) Since, there is no relative sliding between the cylinder and the planks 1 and 2, the points A and B of the disc will move with velocities equal to the velocities of the respective plank. For finding the position of ICR, we draw the velocity vectors at positions A and B. Joining A and B and A' and B', we find the point 'P' as ICR



$$\omega = \frac{2v}{x} = \frac{v}{2R - x}$$

This gives, $x = \dfrac{4R}{3}$

Hence, ICR is located $4R/3$ below the top of the disc.

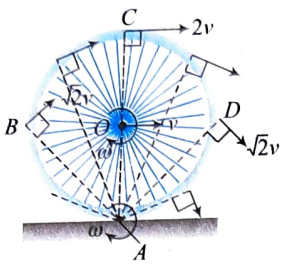
(b) Angular velocity of the cylinder $\omega = \dfrac{2v}{x} = \dfrac{2v}{4R/3}$

This gives $\omega = \dfrac{3v}{2R}$ (clockwise)

Rolling as Pure Rotation About Point of Contact

Consider a ring rolling on a smooth fixed horizontal surface. A rolling body on a fixed horizontal surface, does not slip, therefore the point of contact A with the surface is instantaneously at rest and the ring momentarily rotates about this point. We can say that at any instant the "lowest point" (point of contact) of the rolling body with the horizontal surface is the ICR. It means, the body rotates perfectly about the point A at a given instant.

It means the rolling can be viewed as pure rotation, with angular speed ω, about an axis that always extends through the point where the wheel contacts the ground as the wheel moves. The vectors show the instantaneous linear velocities of selected points on the rolling wheel.



Velocity of point A, $v_A = 0$

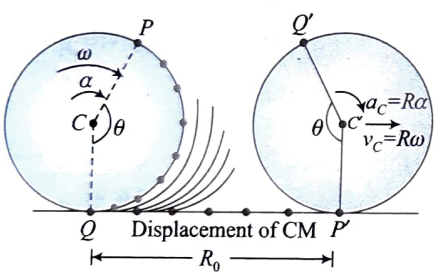
Velocity of centre of mass O, $v_O = \omega R$

Velocity of topmost point C,

$v_C = \omega(2R) = 2\omega R = 2v_O$

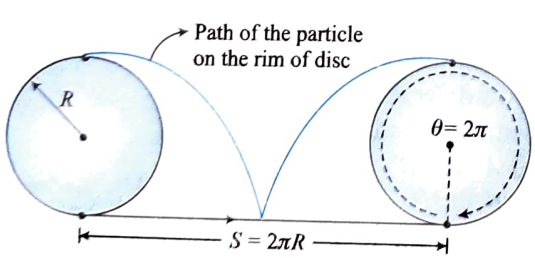
Velocity of points B or D, $v_B = v_D = \omega\left(\sqrt{2}R\right) = \sqrt{2}\,\omega R = \sqrt{2}v_O$

The disc in figure rolls to the right and point C on the axis moves to C' as P moves to P' and Q moves to Q'. The arc length from P to Q equals x just as length $QP' = CC' = x$. Thus, the linear distance travelled by the centre of mass is $x_{CM} = R\theta$

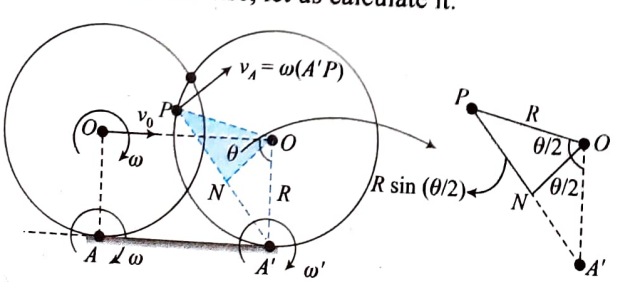


If the disc rotates through an angle θ (in radians), as in figure, then the centre moves through a distance

$x_{CM} = R\theta$



One complete revolution of a rolling disc translates the centre of mass by a distance equal to the circumference of the circular cross-section. But what about the length of the path covered by a point on the rim of the disc, let us calculate it.



Magnitude of the velocity of any point on the disc

$$|\vec{v}_P| = \omega\left(2R\sin\frac{\theta}{2}\right) = 2\omega R\sin\frac{\omega t}{2} \qquad (\text{As } \theta = \omega t)$$

The length of the arc covered by any point on the rim of the disc is one complete rotation, $s = \int_0^T |\vec{v}_P|\,dt$,

where T = time for one complete rotation $= \frac{2\pi}{\omega}$

Hence $s = \int_0^{\frac{2\pi}{\omega}} 2\omega R\sin\frac{\omega t}{2}\cdot dt = 2\omega R\left[\frac{-\cos\frac{\omega t}{2}}{\frac{\omega}{2}}\right]_0^{\frac{2\pi}{\omega}}$

$$\Rightarrow \quad s = -4R\left[\cos\frac{\omega}{2}\cdot\frac{2\pi}{\omega} - \cos 0\right] = 8R$$

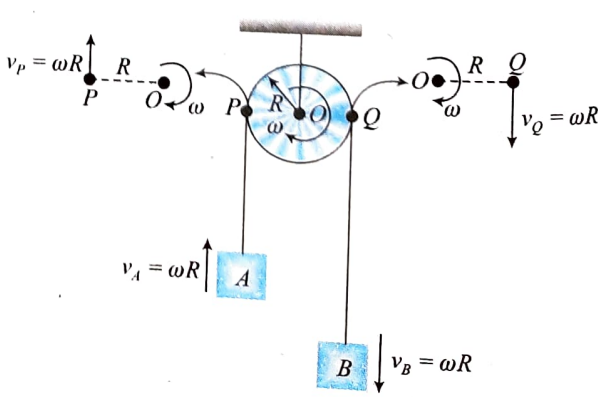
Constraint Relations in Case of Rotation and Translation Combined

In rotational dynamics all the particles of a body move with different velocities and acceleration. For analyzing the motion of a body under combined rotation and translation, we need to relate the velocity and acceleration of any particle in the body with the velocity and acceleration of some other particle of the same body.

In that case we need to write a constraint equation in the same way as we already done in the chapter 'Newton's Laws of Motion-I' in book 'Mechanics I'. Here we are going to learn writing constraint relations in combined rotation and translation through some selected illustrations.

Constraints Due to Ropes and Pulleys

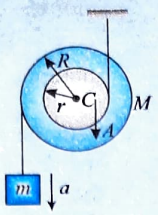
While solving the problems of rotational dynamics we encounter situations where pulleys are connected via strings or belts to other objects. We assume that the string turns on the pulley without slipping. Due to this constraint the string's speed v_{string} must be equal to the speed of the rim of the pulley which has magnitude $|\vec{v}_{rim}| = |\vec{\omega}|R$. If the pulley has angular acceleration α, the string acceleration a_{string} must be equal to the tangential acceleration of the rim of the pulley $|\vec{a}_t| = |\vec{\alpha}|R$



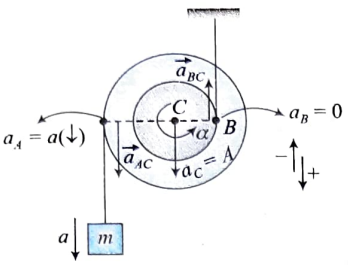
In nutshell we can write $\vec{v}_{block} = |\vec{\omega}|R$; $\vec{a}_{block} = |\vec{\alpha}|R$.

ILLUSTRATION 3.9

A bobbin with inner radius r and outer radius R is arranged with light strings and a block as shown in the figure. The string is not sliding over the cylinder. The system is released from rest. The block and bobbin move down with accelerations a and A respectively. Find the ratio a/A.



Sol. Let the angular acceleration of the bobbin is α as shown. Point B is directly connected with roof through string. It means net acceleration of point B should be zero.



$$\vec{a}_B = \vec{a}_{B,C} + \vec{a}_C$$

$$0 = -\alpha r + A \text{ or } \alpha = \frac{A}{r} \quad \ldots(i)$$

Point A on the bobbin is directly connected with block, it means the acceleration of point A should be same as the acceleration of the block.

$$\vec{a}_A = \vec{a}_{A,C} + \vec{a}_C$$

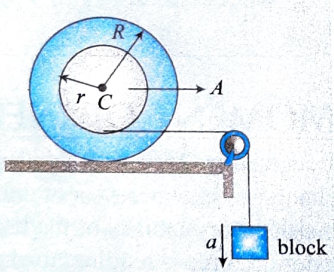
$$a = \alpha r + A \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

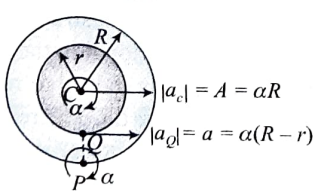
$$a = \left(\frac{A}{r}\right)R + A \text{ or } \frac{a}{A} = \frac{R+r}{r}$$

ILLUSTRATION 3.10

A bobbin has inner radius r and outer radius R is placed on a rough horizontal surface. A light is string wrapped over inner core, connects a block with bobbin as shown in the figure. Now system is released from rest and bobbin moves on the horizontal surface without sliding. If the string does not slide from bobbin, find the ratio of the acceleration of the block and bobbin.



Sol. As the motion of the bobbin is pure rolling, point P should be instantaneous centre of rotation (ICR) of the bobbin. Hence, we can consider pure rotation of the bobbin about P. The acceleration of point C; $a_C = \alpha R$.



But the bobbin is moving with acceleration A,

Hence $A = \alpha R$ ...(i)

The acceleration of point Q; $a_Q = \alpha(R-r)$

But point Q is directly connected with block, it means the magnitude of acceleration of the point Q should be equal to the acceleration of the block

$$a_Q = a = \alpha(R-r) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

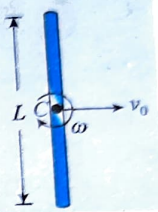
$$\frac{a}{A} = \frac{R-r}{R} = \left(1 - \frac{r}{R}\right)$$

CONSTRAINTS ON A ROD HAVING COMBINED TRANSLATIONAL AND ROTATIONAL MOTION

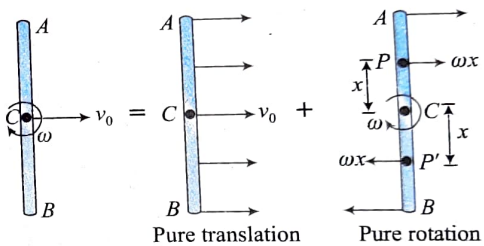
In many situations in rotational dynamics we need to analyze the situations where a rod moves in combined translational and rotational motion. We need to relate the velocity and acceleration of a particle on the rod with the velocity and acceleration of centre of mass of the rod. Let us learn this through following illustrations.

ILLUSTRATION 3.11

A rod of length L is placed on a smooth horizontal surface. An impulse is given to rod, as a result the rod translates and rotates about its centre of mass as shown in figure. Find the point on the rod which has zero velocity.



Sol. The motion of the rod can be considered as superposition of pure translation and pure rotation.



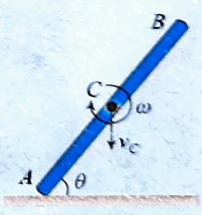
From figure, the point of zero velocity on the rod should be between C and B. Let point P' is the point of zero velocity.

$$\vec{v}_{P'} = \vec{v}_{P',C} + \vec{v}_C$$

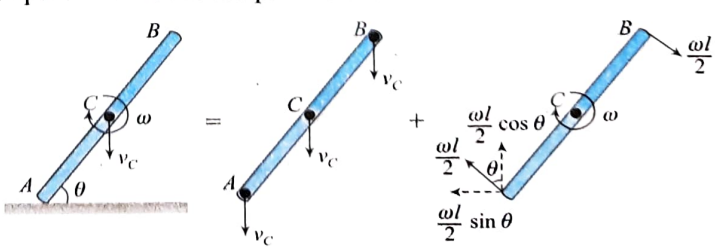
$$0 = -\omega x + v_0 \Rightarrow x = \frac{v_0}{\omega}$$

ILLUSTRATION 3.12

A rod AB of length L is supported against smooth horizontal surface. The rod is inclined at angle θ with horizontal. Now it is released so that end 'A' starts sliding on horizontal surface. Make a relation between velocity of centre of mass of the rod and its angular velocity.



Sol. The motion of the rod can be considered as superposition of pure translation and pure rotation.

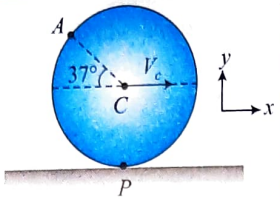


As end A, the rod slide along horizontal surface, the component of its velocity normal to ground should be zero.

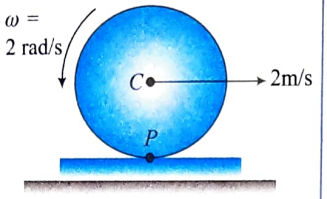
i.e., $-v_C + \frac{\omega l}{2}\cos\theta = 0$ or $v_C = \frac{\omega l}{2}\cos\theta$

CONCEPT APPLICATION EXERCISE 3.2

1. A cylinder of radius 5 m rolls on a horizontal surface. If the velocity of its center is 25 m/s. Find its angular velocity and velocity of the point A.



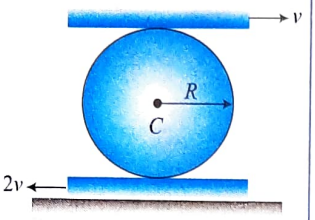
2. A disc of radius $R = \frac{1}{2}$ m is rolling on a plank. If the disc is moving on the plank with angular velocity ω = 2 rad/s and speed v_c = 2 m/s. Find the speed of the plank for pure rolling.



3. A uniform disc of radius R rolls perfectly over two horizontal plank A and B moving with velocities v and $2v$, respectively. Find the

(a) velocity of CM of the disc.

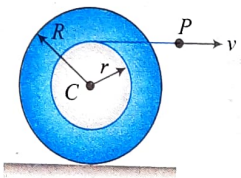
(b) angular velocity of the disc.



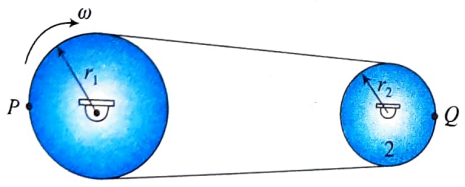
4. A cotton reel rolls without sliding such that the point P of the string has velocity v. Find the:

(i) velocity of its centre C

(ii) angular velocity of the cotton reel.



5. Two wheels 1 and 2 of radii r_1 and r_2 are connected by a belt. If the belt does not slide over the wheels, find the:

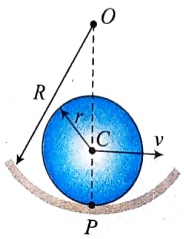


(i) speed of a point on the belt.

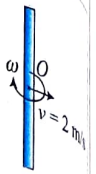
(ii) angular velocity of the wheel 2.

(iii) acceleration of P relative to Q.

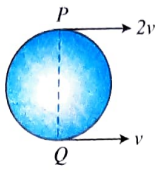
6. A cylinder of radius r rolls perfectly with a speed v on a concave surface of radius R. Find the acceleration of the bottom of the rolling body.



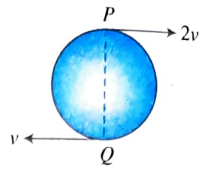
7. Shown in the figure is a rod which moves with $v = 2\ \text{m s}^{-1}$ and rotates with $\omega = 2\pi$ rad s^{-1}. Find the instantaneous axis of rotation.



8. Find the position of instantaneous centre of rotation and angular velocity of the disc in the following cases as shown. Radius of disc is R in each case.

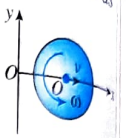


Case (a)



Case (b)

9. A rotating disc moves in the positive direction of x-axis as shown. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation if at the initial moment the centre C of the disc was located at origin after which



(a) it moved with constant acceleration 'a' (initial velocity zero) while the disc rotating anticlockwise with constant angular velocity ω.

(b) it moved with constant velocity v while the disc started rotating anticlockwise with a constant angular acceleration α (with initial angular velocity zero).

ANSWERS

1. 5 rad/s, $(40\hat{i} + 20\hat{j})$ m/s **2.** 3 m/s **3.** (a) $v/2$ (b) $3v/2R$

4. $vR/(R+r)$ **5.** (i) ωr_1 (ii) $\frac{\omega r_1}{r_2}$ (iii) $\omega r_1\left(1+\frac{r_1}{r_2}\right)$

6. $v^2\left[\frac{1}{r}+\frac{1}{R-r}\right]\hat{j}$ **9.** (a) $y^2 = \left(\frac{2a}{\omega^2}\right)x$ (b) xy = constant

MOMENT OF INERTIA

According to Newton's first law of motion, no body, by itself, can change its state of rest or of uniform motion in a straight line. This inertness or inability of the body is called inertia. Obviously, an external force is required to change the state of rest or of uniform motion in a straight line of the body. Larger the force required to change the state of the body, greater is its inertia. Therefore, to produce or to destroy a given acceleration, force required is directly proportional to the mass of the body. Clearly, heavier the body, greater is the force required to change its state and hence greater is its inertia. The reverse is also true. Mass of the body is, therefore, a measure of its inertia in translatory motion.

Similarly, a body rotating about an axis is unable to produce a change in its rotational motion by itself and this inertness in case of rotational motion is known as moment of inertia. It is the resistance to change in rotational motion.

The moment of inertia of a body about a given axis plays the same role in rotational motion about that axis as the mass of a body does in translational motion. However, the two are not the same whereas the inertia of a body depends only upon its mass, the moment of inertia of a body about a given axis depends upon:

(i) its mass
(ii) position and direction of the axis of rotation.

It accounts the fact that two bodies of the same mass and shape but rotating about different axes of rotation, will have different moments of inertia.

(iii) shape of the body (i.e., distribution of the mass of the body about the said axis).

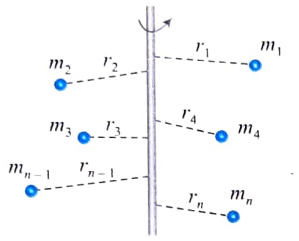
It is due to the reason that two bodies having same mass, but different shapes will have different moments of inertia about a given axis of rotation.

Quantitatively, moment of inertia of a body about a given axis is equal to the sum of the products of the masses of the constituent particles and squares of their respective perpendicular distances from the axis of rotation.

If $m_1, m_2, m_3, \ldots$ are the masses of the particles distant $r_1, r_2, r_3, \ldots$ respectively from the axis of rotation, then moment of inertia of the body,

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots$$

i.e., $I = \Sigma mr^2$



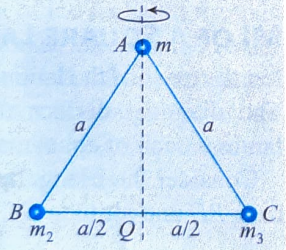
where Σ denotes the summation of the said products for all the particles constituting the body.

The unit of moment of inertia is g cm^2 in cgs system and kg m^2 in SI.

As moment of inertia = mass × (distance)^2, the dimensional formula for moment of inertia is $[ML^2]$. The moment of inertia is scalar quantity.

ILLUSTRATION 3.13

Three mass points m_1, m_2, m_3 are located at the vertices of an equilateral triangle of length a as shown in figure. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1?



Sol. Let the mass points m_1, m_2 and m_3 be situated at vertices A, B and C of the equilateral triangle of each side a. AQ is altitude of the triangle passing through mass m_1. Then, moment of inertia of the system about the altitude AQ,

$I = m_1 \times$ (distance of m_1 from $AQ)^2 + m_2 \times$ (distance of m_2 from $AQ)^2 + m_3 \times$ (distance of m_3 from $AQ)^2$

$$I = m_1(0)^2 + m_2\left(\frac{a}{2}\right)^2 + m_3\left(\frac{a}{2}\right)^2 = \frac{1}{4}(m_2 + m_3)a^2$$

MOMENT OF INERTIA OF A RIGID BODY

The moment of inertia (MI) of a symmetrical body rotating about its axis of symmetry can be found by finding the moment of inertia of a small element of it about the axis of rotation and then integrating it within the proper limits for whole of the body.

Consider that a small elementary portion of mass dm of the rigid body rotates about the axis of rotation at a distance r from it. Then, moment of inertia of the small elementary portion about the axis of rotation,

$$dI = dmr^2 \quad \ldots(i)$$

The moment of inertia of the whole rigid body about the axis of rotation,

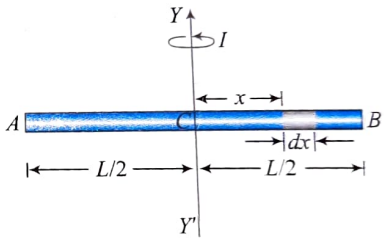
$$I = \int dI = \int dmr^2 \quad \ldots(ii)$$

Note:

- MI is not constant for a body. It depends on the axis of rotation.
- MI depends on the mass of the body. The higher the mass, the higher is the MI.
- MI depends on the distribution of the mass about an axis. The farther the mass is distributed from the axis, the higher will be the MI.
- Moment of inertia does not change if the mass
  (a) is shifted parallel to the axis of the rotation.
  (b) is rotated with constant radius about axis of rotation clockwise or anticlockwise (i.e., the MI does not depends upon sense of rotation.

MOMENT OF INERTIA OF A STRAIGHT ROD ABOUT AN AXIS THROUGH ITS CENTRE

Consider a uniform straight rod AB of length L, mass M and centre C. Then, mass per unit length of the rod $= \frac{M}{L}$



Let the moment of inertia of the rod about the axis YY' passing through its centre C and perpendicular to its length be I. Consider a small element of length dx of the rod at a distance x from the point C.

The mass of elementary portion of the rod, $dm = \frac{M}{L}dx$

The moment of inertia of the elementary portion of the rod about the axis YY',

$dI =$ mass of the elementary portion × (distance from the axis $YY')^2$

$$\Rightarrow \quad dI = \left(\frac{M}{L}dx\right) \times x^2 = \frac{M}{L}x^2\,dx$$

The moment of inertia (I) of the rod AB about the axis YY' can be found by integrating the above for all such elementary portions between the points A and B i.e., between the limits $x = -L/2$ to $x = +L/2$.

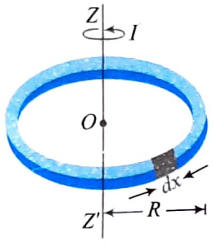
$$I = \int_{-L/2}^{+L/2} dI = \int_{-L/2}^{+L/2} \frac{M}{L}x^2\,dx = \frac{M}{L}\int_{-L/2}^{+L/2} x^2\,dx$$

$$= \frac{M}{L}\left|\frac{x^3}{3}\right|_{-L/2}^{+L/2} = \frac{M}{3L}\left|\left(\frac{L}{2}\right)^3 - \left(\frac{-L}{2}\right)^3\right| = \frac{M}{3L} \times \frac{2L^3}{8}$$

or $$I = \frac{1}{12}ML^2$$

Moment of Inertia of a Circular Ring About an Axis Passing Through Its Centre and Perpendicular to Its Plane

Consider a circular ring of mass M, radius R and centre O. Let the moment of inertia of the circular ring about the axis ZZ' passing through its centre O and perpendicular to its plane be I (see figure). Then,



Mass per unit length of the circular ring $= \dfrac{M}{2\pi R}$

Consider an elementary portion of the ring of length dx.

Then, mass of the elementary portion $= \dfrac{M}{2\pi R}dx$

As the ring rotates about axis ZZ', the elementary portion always remains at a distance R from it. Therefore, moment of inertia of elementary portion about ZZ',

$$dI = \left(\frac{M}{2\pi R}dx\right)R^2 = \frac{M}{2\pi}R\,dx$$

The moment of inertia I of the ring about axis ZZ' can be found by integrating the above for whole circumference of the ring i.e., between limits $x = 0$ to $x = 2\pi R$.

$$\therefore \quad I = \int_0^{2\pi R} dI = \int_0^{2\pi R} \frac{M}{2\pi}R\,dx = \frac{M}{2\pi}R\int_0^{2\pi R} dx$$

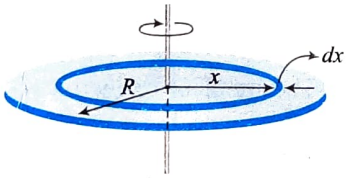
$$\Rightarrow \quad I = \frac{M}{2\pi}R|x|_0^{2\pi R} = \frac{M}{2\pi}R|2\pi R - 0| \text{ or } I = MR^2$$

Note: Answer will remain the same even if the mass is non-uniformly distributed because $\int dm = M$ always.

Moment of Inertia of a Disc

Consider a disc of mass M and radius R, shown in figure. To find the moment of inertia, we consider an elemental ring of radius x and width dx. Its mass dm is given by

$$dm = \frac{M}{\pi R^2}(2\pi x dx)$$



Now the moment of inertia of this elemental ring is given as

$$dI = dmx^2 = \frac{2M}{R^2}x^3 dx$$

Moment of inertia of whole disc is given by integration of above relation from 0 to R.

$$I = \frac{2M}{R^2}\int_0^R x^3\,dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

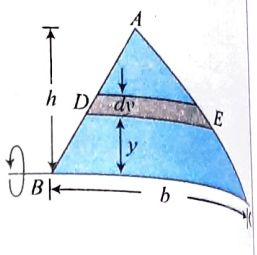
Hence, $I = \dfrac{1}{2}MR^2$

MI of a Triangular Lamina About Its Base

Consider a triangular lamina of base b, altitude h and mass M.

Surface mass density (mass per unit area of the lamina)

$$\sigma = \frac{M}{\left(\frac{1}{2}bh\right)} = \frac{2M}{bh}$$



Consider a rectangular differential strip parallel to the base of width dy, at a distance 'y' from the axis of rotation (base).

From similar triangles ΔADE and ΔABC, we have

$$\frac{DE}{BC} = \frac{h-y}{h} \Rightarrow DE = (h-y)\frac{b}{h}$$

Area of strip DE, $dA = (h-y)\dfrac{b}{h}dy$

MI of the strip DE about the axis of rotation BC is given by

dI_{B_C} = (Mass of strip DE) × (distance from axis)^2

$$= \left(\frac{2M}{bh}(h-y)\frac{b}{h}dy\right)y^2 = \frac{2M}{h^2}y^2(h-y)dy$$

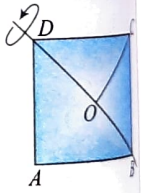
Integrating the above expression between the limits from $y = 0$ to $y = h$.

$$I_{BC} = \int_0^h \frac{2M}{h^2}(hy^2 - y^3)dy = \frac{2M}{h^2}\left[\left(\frac{hy^3}{3}\right) - \left(\frac{y^4}{4}\right)\right]_0^h$$

$$\Rightarrow \quad I_{BC} = \frac{Mh^2}{6}$$

MI of a Square Lamina About Its Diagonal

Let the mass of the lamina be M and each of its edge has length l. The diagonal divides the square lamina into two triangular laminas each of equal mass $M/2$.



Consider the triangular lamina BCD, with base BD and altitude OC.

$$OC = \frac{1}{2}(\text{diagonal}) = \frac{1}{2}\left(\frac{l}{\sqrt{2}}\right) = \frac{l}{\sqrt{2}}$$

MI of the lamina BCD about BD is given by

$$\frac{1}{6}\left(\frac{M}{2}\right)\left(\frac{l}{\sqrt{2}}\right)^2 = \frac{Ml^2}{24}$$

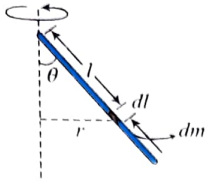
MI of the square lamina about the diagonal BD algebraic sum of MI of laminas ABD and BCD each about their respective base BD. Thus,

$$I_{\text{total}} = 2\left(\frac{Ml^2}{24}\right) = \frac{Ml^2}{12}$$

Illustration 3.14

A uniform rod of mass m and length l_0 is rotating with a constant angular speed ω about a vertical axis passing through its point of suspension. Find the moment of inertia of the rod about the axis of rotation if it makes an angle θ to the vertical (axis of rotation).

Sol. We can observe that every element of rod is rotating with different radius about the axis of rotation



Take an elementary mass dm of the rod.

$$dm = \frac{m}{l_0} dl$$

The moment of inertia of the elementary mass is given as $dI = (dm)r^2$

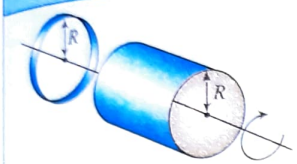
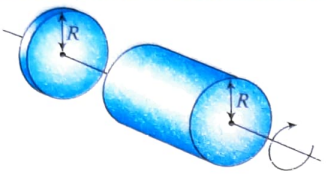
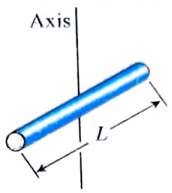
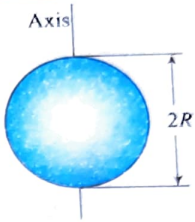
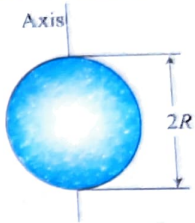
The moment of inertia of the rod $= I = \int dI \Rightarrow I = \int r^2\, dm$

Substituting $r = l \sin\theta$ and $dm = \frac{m}{l_0} \cdot dl$ we obtain

$$I = \int (l^2 \sin^2\theta)\frac{m}{l_0} dl = \frac{m \sin^2\theta}{l_0}\int_0^{l_0} l^2\, dl = \frac{ml_0^3}{3l_0}\sin^2\theta$$

$$\Rightarrow \quad I = \frac{ml_0^2 \sin^2\theta}{3}$$

Some important results of moment of inertia

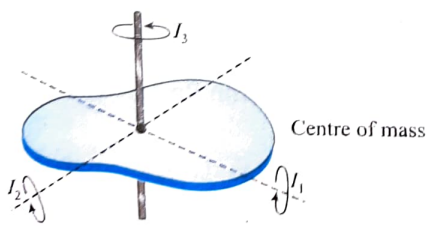
Hoop and annular cylinder	Disc and solid cylinder	Thin rod	Solid sphere	Thin spherical shell
				
$I = MR^2$	$I = \frac{MR^2}{2}$	$I = \frac{1}{12}ML^2$	$I = \frac{2}{5}MR^2$	$I = \frac{2}{3}MR^2$

MOMENT OF INERTIA ABOUT A GENERAL AXIS OF ROTATION

Up to this point we have learnt to calculate the moment of inertia of different objects using the method of integration. In most of the cases we have evaluated the moment of inertia about the axis of symmetry of the objects, generally passing through the center of mass. For evaluation of moment of inertia about any other axis of rotation of body, we use two theorems. Through these two theorems we can easily calculate the moment of inertia about any randomly selected axis of rotation of a given body.

PERPENDICULAR AXIS THEOREM

This theorem is valid only for two dimensional objects or laminar (thin sheet kind of) object, which are rotating about an axis passing through their centre of mass. Let us consider a two-dimensional object as shown in figure.



If the body is rotated about x-axis, lying in the plane on body and passing through its centre of mass, the moment of inertia about this axis be I_1 (say). If the body is rotated about y-axis, which is also in its plane and passing through its centre of mass, moment of inertia about this axis be I_2 (say). Now, if the body is rotated about a third axis, which is perpendicular to both previous axes and perpendicular to the plane of the body, its moment of inertia I_3 is given by the sum of I_1 and I_2.

$$I_3 = I_1 + I_2 \qquad \ldots(i)$$

Thus, the sum of moments of inertia of a laminar object about two mutually perpendicular axes lying in the plane of lamina is equal to the moment of inertia about an axis normal to the plane of the lamina and passing through the two perpendicular axes. This equation is known as the **perpendicular axis theorem**.

We shall now prove it.

We consider an elemental mass dm in the body at a distance r from the main axis, which is perpendicular to the plane of body, as shown in figure. Let the distance of dm from x- and y- axes are a and b respectively. The moment of inertia of the whole body about main axis will be according definition of momentum of inertia.

$$I = \int dm r^2 = \int dm(a^2 + b^2)$$

$$I = \int dm a^2 + \int dm b^2 = I_x + I_y$$

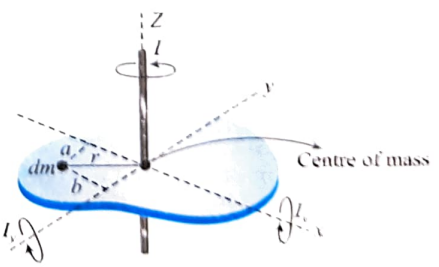


ILLUSTRATION 3.15

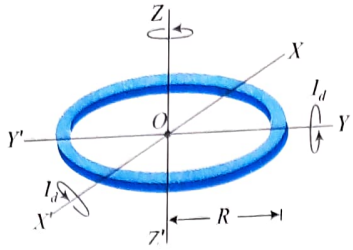
Calculate the moment of inertia of

(i) a ring of mass M and radius R about an axis coinciding with the diameter of the ring.

(ii) a thin disc about an axis coinciding with the diameter.

Sol.

(i) Let I_d be the moment of inertia of the circular ring about its diameter XX'. Let YY' be another diameter of the ring perpendicular to the diameter XX'.



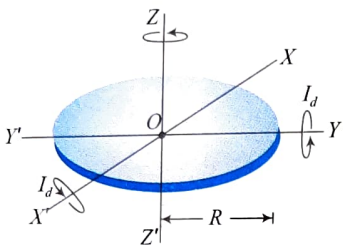
Since the ring is symmetrical about its any diameter, the moment of inertia of the ring about the diameter YY' is also equal to I_d. According to the theorem of perpendicular axes,

MI of the ring about ZZ' = MI about XX' + MI about YY'

i.e., $I = I_d + I_d$

or $I_d = \frac{1}{2}I = \frac{1}{2}MR^2$

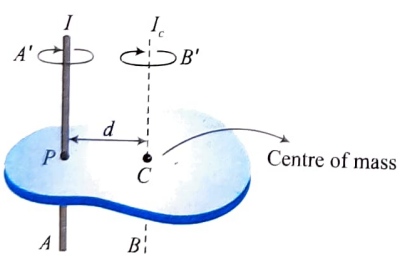
(ii) From previous part, it can be obtained that the moment of inertia of the circular disc about its diameter,



$$I_d = \frac{1}{2}I = \frac{1}{2}\times\left(\frac{1}{2}MR^2\right) = \frac{1}{4}MR^2$$

PARALLEL AXIS THEOREM

If we want to find the moment of inertia I of a body of mass M about a given axis. In principle, we can always find it by method of integration. However, there is a neat shortcut if we already know the moment of inertia I_c of the body about a parallel axis that extends through the body's centre of mass.

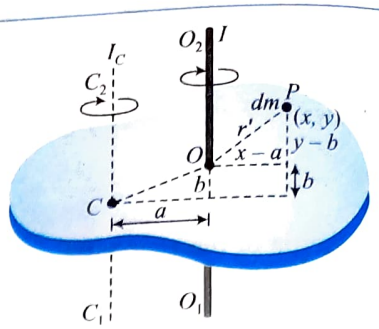


In the figure given, consider a body rotating about an axis AA' passing through it. To find moment of inertia about this axis of rotation, first we find an imaginary axis BB' which is parallel to AA' and passing through centre of mass of the body. Now we evaluate the moment of inertia of the body if it were rotated about BB', let it be I_c. The moment of inertia of the body about the axis AA' is given as

$$I = I_c + Md^2 \qquad ...(i)$$

where d is perpendicular distance between the two axes AA' and BB'. This equation is known as the **parallel axis theorem**. We shall now prove it.

Let us take a body as shown in figure. C is the centre of mass of the body and O is the point in x-y plane on the axis about which we are required to evaluate the moment of inertia.



Consider an elemental mass dm at coordinate (x, y) with respect to C and let (a, b) be the coordinate of the point O. The rotational inertia of the body about the axis through O (about the axis O_1O_2) is given as, $I = \int dm r_1^2$

We have $r_1^2 = (x-a)^2 + (y-b)^2$

$$\Rightarrow I = \int r_1^2 .dm = \int\left[(x-a)^2 + (y-b)^2\right] dm,$$

which we can rearrange as

$$I = \int (x^2 + y^2)dm - 2a\int x\, dm - 2b\int y\, dm + \int (a^2 + b^2)dm$$

From the definition of the centre of mass, the middle two integrals of Eq. (ii) give the coordinates of the center of mass (multiplied by a constant) and thus must each be zero. Because $x^2 + y^2$ is equal to r^2, where r is the distance from O to dm, the first integral is simply I_c, the rotational inertia of the body about an axis through its center of mass. Figure shows that the last term in Eq. (ii) is Md^2, where M is the body's total mass.
So, we have

$$I = I_c + Md^2$$

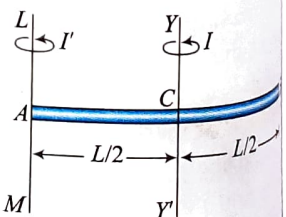
Thus, Eq. (ii) reduces to Eq. (i), which is the required relation.

Important Points:

- Theorem of parallel axes is applicable for any type of rigid body whether it is two dimensional or three dimensional while the theorem of perpendicular axes is applicable for laminar type or two-dimensional bodies only.
- In theorem of perpendicular axes, the point of intersection of the three axes (x, y and z) may be any point on the plane of the body (it may even lie outside the body). This point may or may not be the centre of mass of the body.

Moment of Inertia of a Thin Rod About an Axis Through One End

Let I' be the moment of inertia of the rod about an axis LM passing through the end A and perpendicular to its length (see figure). The axis LM is at a distance $L/2$ from the axis YY' through the centre of mass of the rod. Therefore, according to the theorem of parallel axis, the moment of inertia of the rod about the axis LM is given by

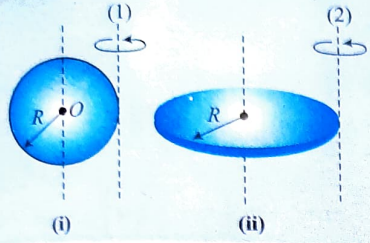


$$I' = I + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2$$

or $I' = \frac{1}{3}ML^2$

ILLUSTRATION 3.16

Find out the moment of inertia of a disc having uniform mass distribution (mass M and radius R) about an axis which is tangent to the disc if
(a) the tangent is in the plane of the disc (Fig. (i)),
(b) the tangent is perpendicular to the plane of the disc (Fig. (ii)).



Sol. The moment of inertia of a disc about an axis passing through centre and perpendicular to plane of ring is $I_0 = MR^2/2$ because of symmetry we can say $I_x = I_y$ and using perpendicular axis theorem,

$$I_z = I_x + I_y \Rightarrow \frac{MR^2}{2} = 2I_y$$

$$\Rightarrow \quad I_y = \frac{MR^2}{4} = \text{M.I. about diameter axis [in Fig. (i)]}$$

For case (a): Using parallel axis theorem,

$$I_1 = I_c + MR^2$$

As the diameter axis passes through the centre of mass, so here we can write $I_y = I_c = \frac{MR^2}{4}$

Hence, $I_1 = I_c + MR^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2$

For case (b): Using parallel axis theorem,

$$I_2 = I_z + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

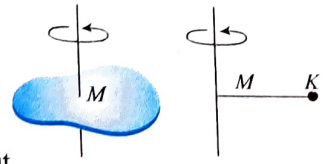
Note: Similarly, the moment of inertia of a solid sphere about a tangent is

$$I_{\text{tangent}} = I = I_c + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

RADIUS OF GYRATION

Radius of gyration (K) of a body about an axis is the effective distance from the axis where the whole mass can be assumed to be concentrated so that the moment of inertia remains the same. Thus,

$$I = MK^2 \text{ or } K = \sqrt{\frac{I}{M}}$$



Radius of gyration of a disc about an axis perpendicular to its plane and passing through its centre of mass is

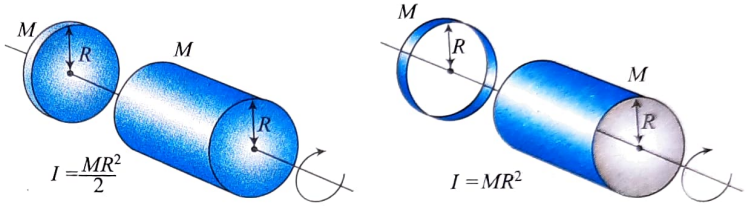
$$K = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \frac{R}{\sqrt{2}}$$

- Radius of gyration of a solid sphere, $K = \sqrt{\frac{2}{5}}R$
- Radius of gyration of a hollow sphere, $K = \sqrt{\frac{2}{3}}R$

MASS DISTRIBUTION AND MOMENT OF INERTIA

We have read that moment of inertia of an object depends on its mass as well as its mass distribution about the axis of rotation. There are several objects which can be derived from the basic objects by changing their dimensions.

For example, if we increase the thickness of a disc it becomes a cylinder. The expression for moment of inertia for the derived object will also remain same as shown in Fig. (a).

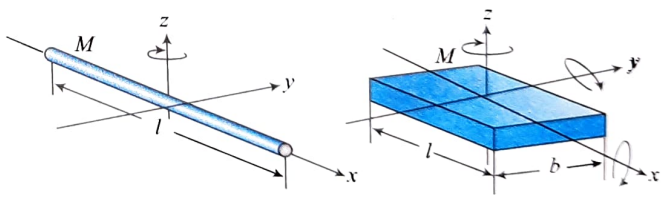


(a) **(b)**

Similarly, if we think about a hollow cylinder shown in figure, it is derived from a ring. Its moment of inertia can be given directly as MR^2 as shown in Fig. (b). It means in changing the dimension parallel to axis of rotation, the expression for moment of inertia will not change.

For example, consider a thin rod of length l, shown in Fig. (c). The moment of inertia of the rectangular plate can also be given by the expression used for rod as

$$I = \frac{mL^2}{12}$$



(c) **(d)**

If we increase the thickness of this rod parallel to x-axis without changing its mass, it becomes a rectangular plate. Now consider a rectangular plate shown in Fig. (d), with dimension $l \times b$. If this plate is rotated about x-axis and y-axis, the respective moment of inertias can be given as

$$I_x = \frac{Mb^2}{12} \text{ and } I_y = \frac{Ml^2}{12}$$

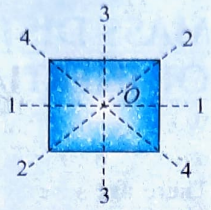
Now if this plate is rotated about the axis passing through its centre and perpendicular to its plane as shown in figure. The moment of inertia can be given by the sum of the above two moment of inertias according to perpendicular axes theorem. So, the moment of inertia of plate about z-axis is given by

$$I_z = I_x + I_y = \frac{Ml^2}{12} + \frac{Mb^2}{12} = \frac{M}{12}(l^2 + b^2)$$

Now we can say:
"If size of a body in any or all dimensions is increased in such a way that its mass distribution about given axis of rotation remain same then expression of moment of inertia also remain same."

ILLUSTRATION 3.17

Find the moment of inertia of a square lamina of mass M and side l about following axis:
(i) $I_{1,1}$ **(ii)** $I_{2,2}$ **(iii)** $I_{3,3}$ **(iv)** $I_{4,4}$ and **(v)** an axis passing through O and perpendicular to the plane of the square (I_{oz}) (as shown in figure).



Sol. We know the moment of inertia of a body is independent of its size parallel to axis of rotation. Hence

$$I_{1,1} = I_{3,3} = \frac{Ml^2}{12}$$

Using perpendicular axis theorem

$$I_{oz} = I_{1,1} + I_{3,3} = 2 \times \frac{Ml^2}{12} \Rightarrow I_{oz} = \frac{Ml^2}{6}$$

As mass distribution is symmetrical about diagonal axis, hence

$$I_{2,2} = I_{4,4}$$

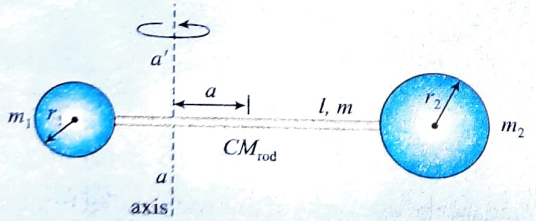
Again, from perpendicular axis theorem, $I_{oz} = I_{2,2} + I_{4,4}$

or $$\frac{Ml^2}{6} = 2I_{2,2} \Rightarrow I_{2,2} = \frac{Ml^2}{12}$$

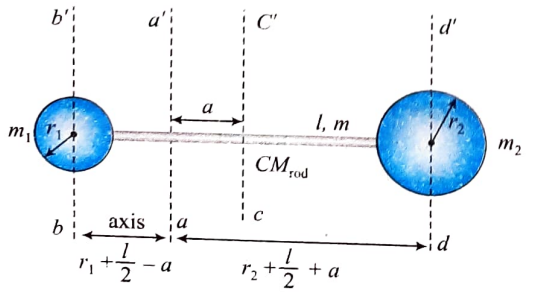
It means $$I_{2,2} = I_{4,4} = \frac{Ml^2}{12}$$

ILLUSTRATION 3.18

Two uniform solid spheres of masses m_1 and m_2 and radii r_1 and r_2, respectively, are connected at the ends of a uniform rod of length l and mass m. Find the moment of inertia of the system about an axis perpendicular to the rod and passing through a point at a distance a from the centre of mass of the rod as shown in figure.



Sol. Draw the axes bb', dd' and cc' passing through the centers of the spheres and the centre of mass of the rod as shown in figure.



For m_1: $I_{bb'} = \frac{2}{5}m_1r_1^2$

and $(I_{aa'})_1 = I_{bb'} + m_1\left(r_1 + \frac{l}{2} - a\right)^2 = \frac{2}{5}m_1r_1^2 + m_1\left(r_1 + \frac{l}{2} - a\right)^2$

[From parallel axis theorem]

For m: $I_{cc'} = \frac{ml^2}{12}$; $(I_{aa'})_2 = I_{cc'} + ma^2 = \frac{ml^2}{12} + ma^2$

For m_2: $I_{dd'} = \frac{2}{5}m_2r_2^2$

$$\Rightarrow \quad (I_{aa'})_3 = I_{dd'} + m_2\left(r_2 + \frac{l}{2} + a\right)^2$$

$$= \frac{2}{5}m_2r_2^2 + m_2\left(r_2 + \frac{l}{2} + a\right)^2$$

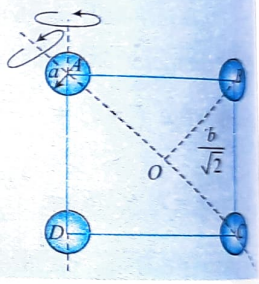
The moment of inertia of the given system about aa',

$$I_{aa'} = (I_{aa'})_1 + (I_{aa'})_2 + (I_{aa'})_3$$

$$= \left\{\frac{2}{5}m_1r_1^2 + m_1\left(r_1 + \frac{l}{2} - a\right)^2\right\} + \frac{ml^2}{12} + ma^2 + \left\{\frac{2}{5}m_2r_2^2 + m_2\left(r_2 + \frac{l}{2} - a\right)^2\right\}$$

ILLUSTRATION 3.19

Four spheres, each of radius a and mass m, are placed with their centres on four corners of a square of side b. Calculate the moment of inertia of the arrangement about any
(a) diagonal of the square and
(b) any side of the square.



Sol.

(a) Moment of inertia of the arrangement about the diagonal AC:
The moment of inertia of each of spheres A and C about their common diameter $AC = (2/5)ma^2$. The moment of inertia of each of spheres B and D about an axis passing through their centres and parallel to AC is $(2/5)\,ma^2$. The distance between the axis and AC is $b/\sqrt{2}$, b being side of the square. Therefore, the moment of inertia of each spheres B and D about AC by the theorem of parallel axis is

$$\frac{2}{5}ma^2 + m\left(\frac{b}{\sqrt{2}}\right)^2 = \frac{2}{5}ma^2 + \frac{mb^2}{2}$$

Therefore, the moment of inertia of all the four spheres about diagonal AC is

$$I = 2\left(\frac{2}{5}ma^2\right) + 2\left[\frac{2}{5}ma^2 + \frac{mb^2}{2}\right] = \frac{8}{5}ma^2 + mb^2$$

(b) The moment of inertia of each of spheres A and D about side AD is $(2/5)\,ma^2$.
The moment of inertia of each of spheres B and C about AD is $(2/5)\,ma^2 + mb^2$.

Therefore, the moment of inertia of the whole arrangement about side AD is

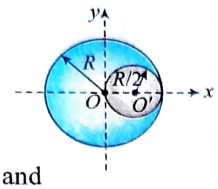
$$2\left[\frac{2}{5}ma^2+\left(\frac{2}{5}ma^2+mb^2\right)\right]=\frac{2m}{5}[4a^2+5b^2]$$

ILLUSTRATION 3.20

A circular hole of radius $R/2$ is cut from a circular disc of radius R. The disc lies in the x-y plane and its centre coincides with the origin. If the remaining mass of the disc is M, then

(a) determine the initial mass of the disc, and

(b) determine its moment of inertia about the z-axis.



Sol.

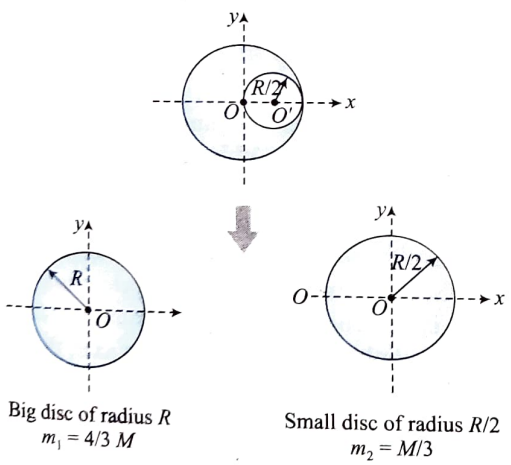
(a) Let σ be the surface mass density of the plate given by

$$\sigma=\frac{M}{\pi R^2-\pi\left(\frac{R}{2}\right)^2}=\frac{4}{3}\frac{M}{\pi R^2}$$

Now, initial mass of the plate, $m_1=\sigma(\pi R^2)=\frac{4}{3}M$

Mass of the plate removed, $m_2=\sigma\left(\frac{\pi R^2}{4}\right)=\frac{M}{3}$

(b) The given arrangement may be considered as a combination of a circular plate of mass $(4/3)\,M$ and circular plate of negative mass $(M/3)$.



The moment of inertia of the big disc of radius R and mass m_1 about O and parallel to the z-axis is

$$I_{1o}=\frac{m_1R^2}{2}$$

The moment of inertia of the small disc of mass m_2 and radius $R/2$ about O' and parallel to the z-axis is

$$I=\frac{m_2(R/2)^2}{2}=\frac{m_2R^2}{8}$$

The moment of inertia of the small disc about O (using parallel axis theorem)

$$I_{2o}=\frac{m_2R^2}{8}+\frac{m_2R^2}{4}=\frac{3m_2R^2}{8}$$

As the part of the disc of mass m_2 is removed from disc m_1, hence the net moment of inertia of structure is

$$I_O=I_{1o}-I_{2o}\Rightarrow I_O=\frac{m_1R^2}{2}-\frac{3m_2R^2}{8}$$

Substituting the values of m_1 and m_2, we get

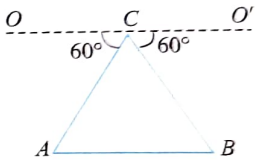
$$I_0=\frac{13}{24}MR^2$$

ILLUSTRATION 3.21

Three identical thin rods, each of mass m and length l, are joined to form an equilateral triangular frame. Find the moment of inertia of the frame about an axis parallel to its one side and passing through the opposite vertex. Also find its radius of gyration about the given axis.

Sol. Moment of inertia of each of rod AC and BC about the given axis OO' is

$$I_{AC}=I_{BC}=\frac{ml^2}{3}\sin^2 60^\circ=\frac{ml^2}{4}$$



and MI of rod AB about the given axis OO' is

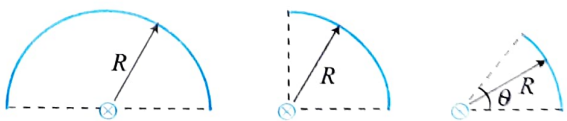
$$I_{AB}=m\left(\frac{l\sqrt{3}}{2}\right)^2=\frac{3}{4}ml^2$$

Hence, $I=I_{AC}+I_{BC}+I_{AB}=\frac{ml^2}{4}+\frac{ml^2}{4}+\frac{3}{4}ml^2=\frac{5}{4}ml^2$

Now, $I=MK^2\Rightarrow K=\sqrt{\frac{I}{M}}=\sqrt{\frac{\frac{5}{4}ml^2}{3m}}=l\sqrt{\frac{5}{12}}$

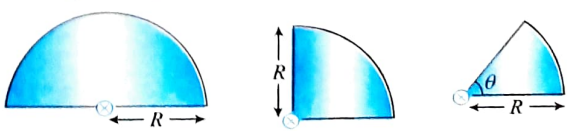
CONCEPT APPLICATION EXERCISE 3.3

1. Find out the moment of inertia of the circular arcs shown, each having mass M, radius R and having uniform mass distribution about an axis passing through the centre and perpendicular to the plane.



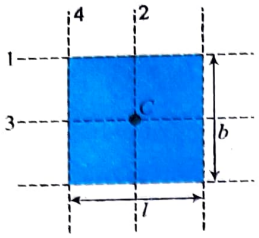
(a) (b) (c)

2. Calculate the moments of inertia of the figures shown, each having mass M, radius R and having uniform mass distribution about an axis perpendicular to the plane and passing through the centre.

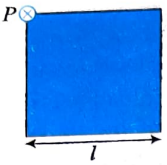


(a) (b) (c)

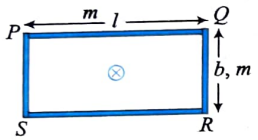
3. In figure, find moment of inertia of a plate having mass M, length l and width b about axes 1, 2, 3 and 4. Assume that C is the centre and mass is uniformly distributed.



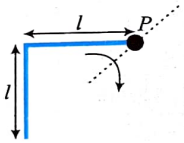
4. Find the moment of inertia of a uniform square plate of mass M and edge of length 'l' about its axis passing through P and perpendicular to it.



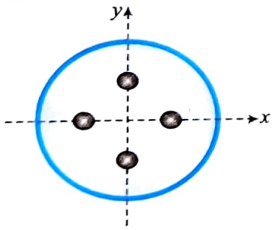
5. Calculate the moment of inertia of a rectangular frame formed by uniform rods having mass m each as shown in figure about an axis passing through its centre and perpendicular to the plane of frame. Also find moment of inertia about an axis passing through PQ ?



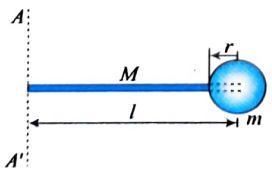
6. Find the moment of inertia of the two uniform joint rods about point P as shown in figure. Use parallel axis theorem. Mass of each rod is M.



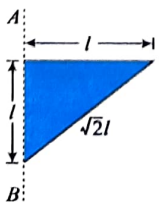
7. A uniform disc of mass m and radius R has an additional rim of mass m as well as four symmetrically placed masses, each of mass $m/4$, tied at positions $R/2$ from the centre as shown in figure. What is the total moment of inertia of the disc about an axis perpendicular to the disc through its centre?



8. Find the moment of inertia of a spherical ball of mass m and radius r attached at the end of a straight rod of mass M and length l, if this system is free to rotate about an axis passing through the end of the rod (end of the rod opposite to sphere).

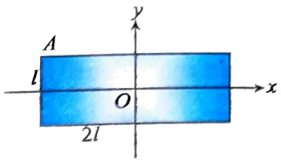


9. Find MI of a triangular lamina of mass M about the axis of rotation AB shown in figure.



10. A uniform circular disc of radius R has a sector of angle 90° removed from it. Mass of the remaining disc is M. Write the moment of inertia of the remaining disc about the axis xx shown in figure.

11. A uniform rectangular plate has side length l and $2l$. The plate lies in x-y plane with its centre at origin and sides parallel to x- and y-axes. The moment of inertia of the plate about an axis passing through a vertex (say A) perpendicular to the plane of the figure is I_0. Now the axis is shifted parallel to itself so that moment of inertia about it remains I_0. Write the locus of point of intersection of the axis with x-y plane.



ANSWERS

3. $\frac{Mb^2}{3}, \frac{Ml^2}{12}, \frac{Ml^2}{12}, \frac{Mb^2}{3}$ 4. $\frac{2Ml^2}{3}$

5. $\frac{2}{3}m(l^2+b^2), \frac{5}{3}mb^2$ 6. $\frac{5}{3}Ml^2$ 7. $\frac{7mR^2}{4}$

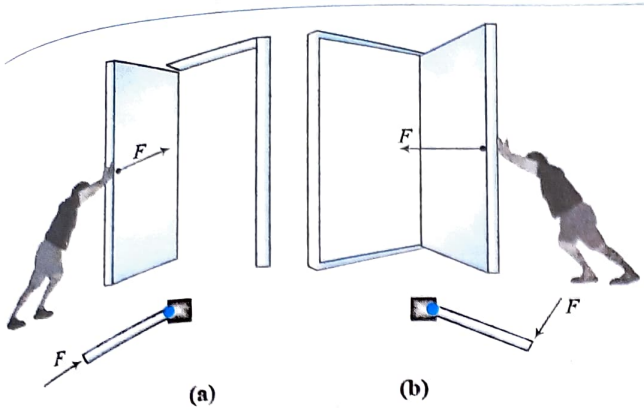
8. $\frac{Ml^2}{3}+\left(\frac{2}{5}mr^2+ml^2\right)$ 9. $\frac{Ml^2}{6}$ 10. $\frac{MR^2}{4}$

11. $x^2+y^2=5l^2/4$

TORQUE

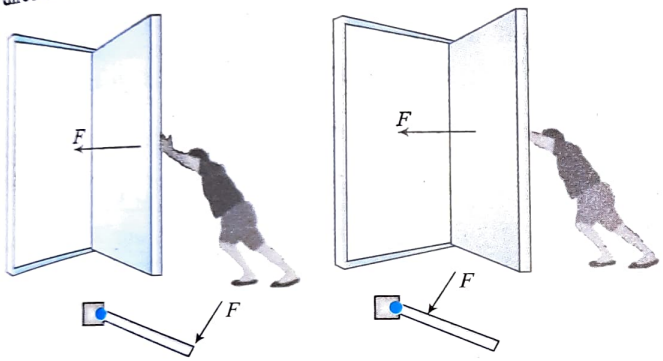
We have learned in particle dynamics, how a net force affects linear motion by causing an object to accelerate. Now in this section let us now analyze how a rigid object can also have an angular acceleration. A net external force causes linear motion to change, but what causes rotational motion to change? Is it simply the net force which produces angular acceleration? No, it is the torque which is responsible for angular motion. When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a quantity called **torque** $\vec{\tau}$ (Greek letter tau). The word 'Torque', which comes from the Latin word meaning "to twist," may be loosely identified as the turning or twisting action of the force $\vec{F}$. When you apply a force to an object—such as a screwdriver or torque wrench—with the purpose of turning that object, you are applying a torque.

If you want to open a heavy door, you must certainly apply a force, but that is not enough. Where you apply that force and in what direction you push are also important. When you push on a door with a force $\vec{F}$, as in Fig. (a), it is not possible to rotate the door, you will have a hard time in opening the door, because the torque is nearly zero. If you push door perpendicular to the surface of door as shown in Fig. (b), now you can rotate the door with ease.



(a) (b)

Now apply the force of larger magnitude Fig. (c), the door opens more quickly. In case of Fig. (c), other things being same as in case of Fig. (b), a larger force generates a larger torque. However, the door does not open as quickly if you apply the same force at a point closer to the hinge, as shown in Fig. (d), because the force now produces less torque. In summary, the torque depends on the magnitude of the force, on the point where the force is applied relative to the axis of rotation and on the direction of the force.

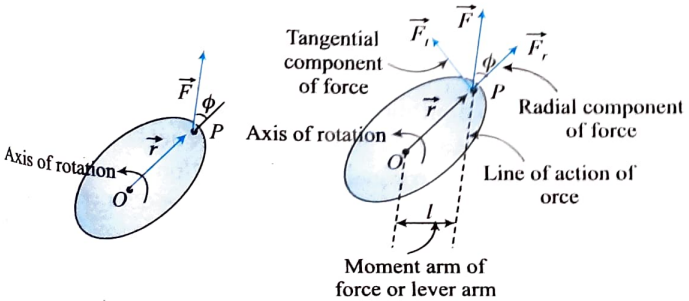


(c) (d)

Two concepts that are important in the definition of torque, line of action of the force and lever arm. The **line of action** is an extended line drawn collinear with the force. The **lever arm** is the distance l between the line of action and the axis of rotation, measured on a line that is perpendicular to both.

CALCULATION OF TORQUE

Let us take a body which is free to rotate about an axis passing through O. Considering the cross-section of the body as shown in Fig. (a). A force $\vec{F}$ is applied at point P, whose position relative to O is defined by a position vector $\vec{r}$. The force $\vec{F}$ is in the plane of the page, the directions of vectors $\vec{F}$ and $\vec{r}$ make an angle ϕ with each other.



To determine how force ($\vec{F}$) results in a rotation of the body around the rotation axis, we resolve the force ($\vec{F}$) into two components as shown in Fig. (b). One component which acts along a line that extends through O and points along $\vec{r}$ is called the radial component has magnitude $F_r = F\cos\phi$. This component does not cause rotation. The other component of force ($\vec{F}$), called the *tangential component* F_t, is perpendicular to $\vec{r}$ and has magnitude $F_t = F\sin\phi$. This component *does* cause rotation.

The ability of $\vec{F}$ to rotate the body depends not only on the magnitude of its tangential component F_t, but also on how far from O the force is applied. To include both these factors, we define a quantity called **torque** τ as the product of the two factors and write it as

$$\tau = (r)(F\sin\phi) \quad \ldots(i)$$

Two equivalent ways of computing the torque are:

$$\tau = (r)(F\sin\phi) = rF_t \quad \ldots(ii)$$

and $$\tau = (r\sin\phi)(F) = lF = Fl \quad \ldots(iii)$$

where l is the perpendicular distance between the rotation axis at O and an extended line running through the vector $\vec{F}$ as shown in Fig.(b), this extended line is called the **line of action** of $\vec{F}$, and l is called the **moment arm or lever arm** of $\vec{F}$.

Thus, Torque = Force × Perpendicular distance of line of action of force from the axis of rotation = Force × Moment arm

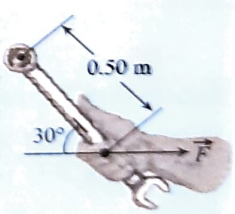
Torque is a vector quantity (rotational or axial vector) and its unit is newton-metre (N m) in SI. Its dimensional formula is $[L]\times[MLT^{-2}] = [ML^2T^{-2}]$, this dimensional formula is the same as for work. But the torque and work both are different physical quantities.

Though torque or moment of force is a vector quantity, but when we deal with rotation around a single axis, we can use only an algebraic sign. The torque or moment of the force is taken as positive, if it tends to rotate in anticlockwise direction and as negative, if it tends to rotate in clockwise direction.

Torques obey the superposition principle, when several torques act on a body, the net torque (or resultant torque) is the sum of the individual torques. The symbol for net torque is τ_{net}.

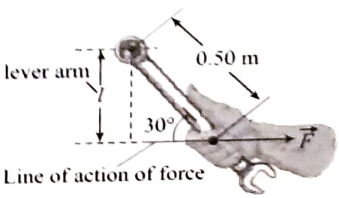
ILLUSTRATION 3.22

You are installing a new spark plug in your car, and the manual specifies that it will be tightened to a torque that has a magnitude of 50 N m. Using the data in the drawing, determine the magnitude F of the force that you must exert on the wrench.



Sol. The magnitude of torque applied to spark plug, $\tau = Fl$, where F is the magnitude of the applied force and l is the lever arm.

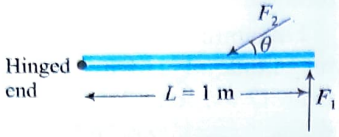
From the figure given, the lever arm is, $l = (0.50\text{ m})\sin 30°$. Since both the magnitude of the torque and l are known, now solving for F.



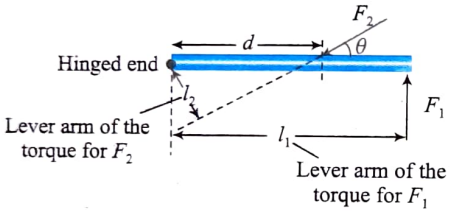
$$F = \frac{\text{Magnitude of torque}}{l} = \frac{50\text{ N}\cdot\text{m}}{(0.50\text{ m})\sin 30°} = 200\text{ N}$$

ILLUSTRATION 3.23

A meter stick is placed on smooth table. One end of the meter stick is hinged to a table, so the stick can rotate freely in a plane parallel to the tabletop. Two forces are applied to the stick in such a way that the net torque is zero. The first force has a magnitude of $F_1 = 2$ N and is applied perpendicular to the length of the stick at the free end. The second force has a magnitude of $F_2 = 5$ N and acts at a 30° angle with respect to the length of the stick. Where along the stick is the 5 N force applied? Express this distance with respect to the end of the stick that is hinged.



Sol. If the torques generated by the two applied forces sum to zero, for this one of the forces must tend to produce a clockwise rotation about the hinged end of the meter stick, and the other must tend to produce a counterclockwise rotation about the hinged end. We can observe the first force, applied perpendicular to the length of the meter stick at its free end, tends to produce a counterclockwise rotation. We take counterclockwise torque as positive.



For net torque to be zero:

$$F_1 l_1 - F_2 l_2 = 0 \quad \text{or} \quad F_1 l_1 = F_2 l_2 \qquad \ldots(i)$$

From the figure that the lever arm l_1 of the force F_1 is equal to the length of the meter stick: $L = 1$ m. The force F_2 is applied a distance d from the hinged end of the meter stick.

The lever arm l_2 of the second force is,

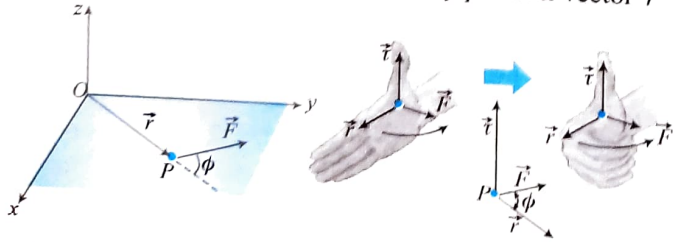
$$l_2 = d \sin\theta \qquad \ldots(ii)$$

Substituting Eq. (ii) into Eq. (i) and solving for the distance d, we obtain

$$F_1 l_1 = F_2 (d \sin\theta) \quad \text{or} \quad d = \frac{F_1 l_1}{F_2 \sin\theta} = \frac{(2\text{N})(1\text{m})}{(5\text{N})\sin 30^\circ} = 0.8 \text{ m}$$

Expressing Torque in Vector Form

Figure (a) shows a particle at point P in an x-y plane. A single force $\vec{F}$ in that plane acts on the particle, and the particle's position relative to the origin O is given by position vector $\vec{r}$



The torque $\vec{\tau}$ acting on the particle relative to the fixed-point O is a vector quantity defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \ldots(i)$$

To find the direction of $\vec{\tau}$, we use the right-hand rule. Fig. (b), sweeping the fingers of the right hand from $\vec{r}$ (the first vector in the product) into $\vec{F}$ (the second vector). The outstretched right thumb then gives the direction of $\vec{\tau}$. In Fig. (b), it is in the positive direction of the z axis.

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

The cross product of any two vectors $\vec{A}$ and $\vec{B}$ can be expressed in the following determinant form:

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} y & z \\ F_y & F_z \end{vmatrix}\hat{i} - \begin{vmatrix} x & z \\ F_x & F_z \end{vmatrix}\hat{j} + \begin{vmatrix} x & y \\ F_x & F_y \end{vmatrix}\hat{k}$$

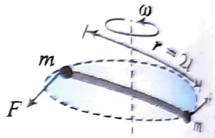
Expanding these determinants gives the result,

$$\vec{\tau} = \vec{r} \times \vec{F} = (yF_z - zF_y)\hat{i} - (xF_z - zF_x)\hat{j} + (xF_y - yF_x)\hat{k}$$

COUPLE

Two equal and opposite forces whose lines of action do not coincide make a couple.

Moment of a couple is given by $\tau = r_\perp F$, where $r_\perp$ is the separation of the lines of action of the forces and F is the magnitude of either of the forces.



Two balls each of mass m are fixed at the ends of a rigid body of length 2l

EQUILIBRIUM OF RIGID BODIES

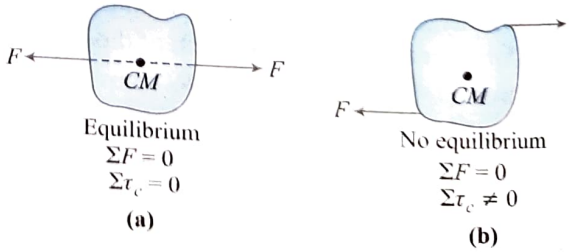
The conditions of equilibrium for a rigid body are different from that of a particle. Unlike particles, rigid bodies have tendency to rotate, therefore, one must consider the rotational equilibrium also, in addition to the translational equilibrium.

The equations of equilibrium (in two dimensions) for a rigid body are stated as:

Translational equilibrium: $\sum \vec{F} = 0$

Rotational equilibrium: $\sum \vec{\tau}_C = 0$

Note that for rotational equilibrium of a rigid body, the net torque about its centre of mass must be zero. The body have both translational and rotational equilibria.



Action of two forces on a particle and a rigid body is shown in Figs. (a) and (b).

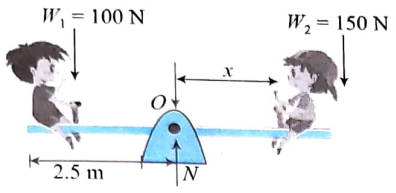
A particle is in equilibrium under the action of two equal and opposite forces, but a rigid body may or may not be in equilibrium under the action of two equal and opposite forces.

The body is in translational equilibrium but not necessarily in rotational equilibrium.

ILLUSTRATION 3.24

Two small kids weighing 10 kg and 15 kg, respectively, are trying to balance a seesaw of total length 5.0 m, with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit?

Sol. If the kid of mass 15 kg sits at the end, he will produce maximum torque about fulcrum. The seesaw will not be balance for any position of kid of mass 10 kg. The 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is x.



As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are as follows:

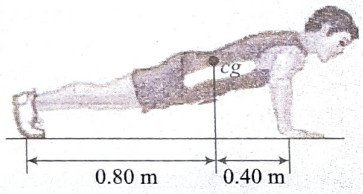
(a) (15 kg) g = 150 N downwards by the 15 kg kid,
(b) (10 kg) g = 100 N downwards by the 10 kg kid,
(c) weight of the seesaw, and
(d) the normal force by the fulcrum.

Taking torques about the fulcrum,

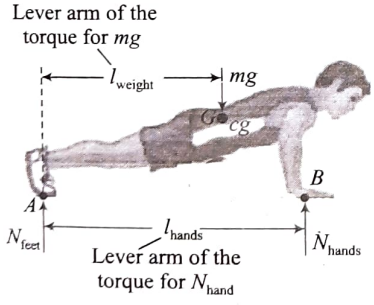
$150 \times x = 100 \times 2.5$ or $x = 1.7$ m

ILLUSTRATION 3.25

In the given figure, a man of mass 60 kg is doing push-ups. Find the normal force exerted by the floor on each hand and each foot, if the man holds this position.



Sol. The man is in equilibrium, the sum of the forces acting on the man must be zero.



From the free body diagram of man, we have,

$$N_{Feet} + N_{Hands} = mg \quad ...(i)$$

Let us consider a rotational axis perpendicular to floor and passing through point A where the man's toes touch the floor. The lever arms for the torques are shown in figure. As the man is in equilibrium, we know that the sum of the torques about any axis should also be zero.

We have, $mg \times l_{Weight} - N_{Hands} l_{Hands} = 0$

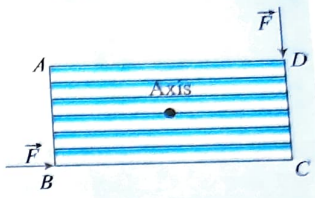
$$\Rightarrow N_{Hands} = \frac{mgl_{Weight}}{l_{Hands}} = \frac{(60 \times 10 \text{ N})(0.80 \text{ m})}{(0.80 \text{ m} + 0.40 \text{ m})} = 400 \text{ N}$$

Hence, $N_{Feet} = mg - N_{Hands} = 600 \text{ N} - 400 \text{ N} = 200 \text{ N}$

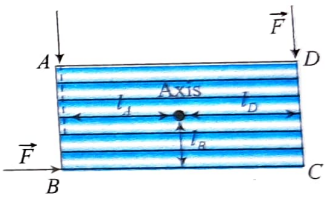
The force on each hand is half the value calculated above, i.e., 200 N. Likewise, the force on each foot is half the value calculated above, i.e., 100 N.

ILLUSTRATION 3.26

In the figure shown a rectangular piece of wood is pinned at its centre. The forces applied to corners B and D have the same magnitude of 20 N and are directed parallel to the long and short sides of the rectangle. The long side of the rectangle is twice the short side. A third force (not shown in the figure) is applied to corner A, directed along the short side of the rectangle (either toward B or away from B), such that the rectangle is at equilibrium. Find the magnitude and direction of the force applied to corner A.



Sol. The net torque acting on rectangle is the sum of the torques produced by the three forces acting at points A, B and D. We are considering the torque that tends to produce a counterclockwise rotation about the axis as positive torque. As the piece of wood is at equilibrium, the net torque is equal to zero.



Let us consider L be the length of the short side of the rectangle, so that the length of the long side is $2L$. The lever arms for the forces acting at A, B and D are shown in figure. The counterclockwise torque produced by the force at corner B is $+Fl_B$, and the clockwise torque produced by the force at corner D is $-Fl_D$. Assume the force at A (directed along the short side of the rectangle) points toward corner B, the counterclockwise torque produced by this force is $+F_A l_A$. As the net torque about the axis shown should be equal to zero:

$$\Sigma\tau = F_A l_A + Fl_B - Fl_D = 0$$

$$\Rightarrow \quad F_A L + (20 \text{ N})\left(\frac{1}{2}L\right) - (20 \text{ N})L = 0$$

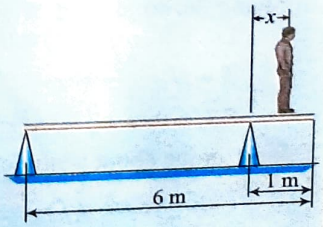
The length L can be eliminated algebraically from this result, which can then be solved for F_A:

$$F_A = -(20 \text{ N})\left(\frac{1}{2}\right) + 20 \text{ N} = 10 \text{ N}$$

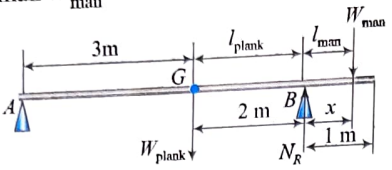
Here the value calculated for F_A is positive, our assumption that F_A points toward corner B must have been correct. Otherwise, the result for F_A would have been negative.

ILLUSTRATION 3.27

In the given figure, a uniform plank of length 6 m and weight 200 N rests horizontally on two supports, with 1 m of the plank hanging over the right support. To what distance can a person who weighs 800 N walk on the overhanging part of the plank before it just begins to tip?



Sol. At every instant before the plank begins to tip, it is in equilibrium, and the net torque on it is zero: $\Sigma\tau = 0$. When the person reaches the maximum distance x along the overhanging part, the plank is just about to rotate about the right support. At that instant, the plank loses contact with the left support, the normal reaction at this point will become zero. This leaves only three vertical forces acting on the plank: the weight W_{plank} of the plank, the normal reaction N_R due to the right support, and the weight of the man W_{man}.



Let us take torque about point B

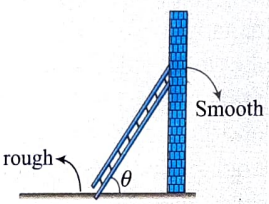
$$\Sigma\tau = W_{plank}\,l_{plank} - W_{man}\,l_{man} = 0 \quad \text{or} \quad l_{man} = x = \frac{W_{plank}\,l_{plank}}{W_{man}} \qquad ...(i)$$

Here, $l_{plank} + 1 = \frac{6}{2}\text{m}$ or $l_{plank} = 2\,\text{m}$

Hence from Eq. (i) $x = \frac{(200\ \text{N})(2\ \text{m})}{800\ \text{N}} = 0.50\ \text{m}$

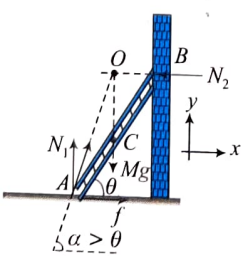
ILLUSTRATION 3.28

A ladder of mass M and length L is supported in equilibrium against a smooth vertical wall and a rough horizontal surface, as shown in figure. If θ be the angle of inclination of the rod with the horizontal, then calculate:



(a) normal reaction of the wall on the ladder,
(b) normal reaction of the ground on the ladder,
(c) friction force between the ground and the ladder and
(d) net force applied by the ground on the ladder.

Sol. The forces acting on the ladder are its weight, contact force of wall (normal reaction) and contact force of ground (normal reaction and friction). The free-body diagram of the ladder is shown in figure.



These forces acting on the ladder should pass through a common point O, otherwise it cannot be in equilibrium.

The total reaction force F from the horizontal surface is inclined at an angle $\alpha\ (> \theta)$ to the horizontal. The horizontal component of this force is friction force and its vertical component is the normal reaction applied by the ground on the ladder.

Applying the conditions of equilibrium, we get

$$\Sigma F_x = 0 \Rightarrow f - N_2 = 0$$

$$f = N_2$$

$$\Sigma F_y = 0 \Rightarrow N_1 - Mg = 0 \Rightarrow N_1 = Mg \qquad ...(i)$$

$$...(ii)$$

Taking torque about centre of the rod 'C',

$$(N_1)\frac{L}{2}\cos\theta - (f)\frac{L}{2}\sin\theta - N_2\frac{L}{2}\sin\theta = 0 \qquad ...(iii)$$

Substituting the value of N_1 and f from Eqs. (i) and (ii), we get

$$Mg\frac{L}{2}\cos\theta - N_2 L\sin\theta = 0$$

$$\Rightarrow \quad N_2 = \frac{Mg}{2\tan\theta} = \frac{1}{2}Mg\cot\theta = f \qquad ...(ii)$$

It is not compulsory to take torque about centre of mass, we can get the same result if we take torque about A.

As ladder is in equilibrium, the net torques about A will be zero. $\sum\tau_A = 0$, taking torque about end A of the rod.

$$Mg\left(\frac{L}{2}\cos\theta\right) - N_2(L\sin\theta) = 0$$

$$N_2 = \frac{Mg}{2\tan\theta} = \frac{1}{2}Mg\cot\theta = f$$

It is the same result as we get by taking torque about C.

Thus, the net force applied by ground on the ladder is

$$F = \sqrt{N_1^2 + f^2}$$

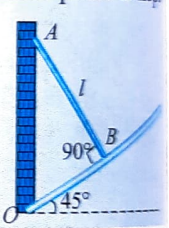
$$F = \frac{1}{2}Mg\sqrt{4 + \cot^2\theta}$$

Results:

$$N_2 = f = \frac{1}{2}Mg\cot\theta, \quad N_1 = Mg, \quad F = \frac{1}{2}Mg\sqrt{4+\cot^2\theta}$$

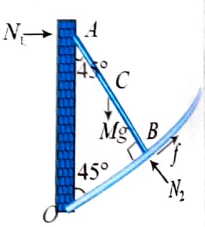
ILLUSTRATION 3.29

At the bottom edge of a smooth wall, an inclined plane is kept at an angle of 45°. A uniform rod of length l and mass M rests on the inclined plane against the wall such that it is perpendicular to the incline. What is the minimum coefficient of friction necessary so that the rod does not slip on the incline?



Sol. For finding the direction of frictional force, let us assume there is no friction anywhere. If we observe from FBD of the rod the normal reaction of the wall has tendency to rotate the rod in clockwise sense. It means end 'B' of the rod has tendency to slide in downward direction on the inclined plane. Hence friction should act in upward direction.

The free-body diagram of the rod is shown in figure.



Balancing torque about point A,

$$fl = Mg\left(\frac{l}{2}\sin 45°\right)$$

$$\therefore \quad f = \frac{Mg}{2\sqrt{2}} \qquad ...(i)$$

Balancing forces in the vertical direction,

$$Mg = N_2\cos 45° + f\sin 45° \qquad ...(ii)$$

From Eqs. (i) and (ii), we get

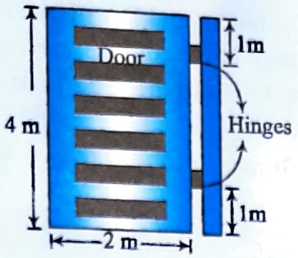
$$N_2 = \frac{3Mg}{2\sqrt{2}} \qquad ...(iii)$$

From Eqs. (i) and (iii), $f \le \mu N_2$

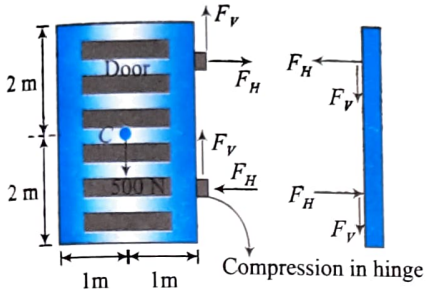
or $$\frac{Mg}{2\sqrt{2}} = \mu_{min}\left(\frac{3Mg}{2\sqrt{2}}\right) \text{ or } \mu_{min} = \frac{1}{3}$$

ILLUSTRATION 3.30

The door of an almirah is 4 m high, 2 m wide and weighs 50 kg. The door is supported by two hinges situated 1.0 m from the ends. If the magnitude of forces exerted by the hinges on the door is equal, find this force.



Sol. The free-body diagram of the door is shown in figure.



The forces exerted on the hinges are equal in magnitude. The door is in equilibrium in horizontal as well as in vertical direction, i.e., the horizontal and vertical components of the forces will be equal. For vertical equilibrium of the door, we have

$$2F_v = 500 \Rightarrow F_v = 250\,\text{N}$$

For rotational equilibrium of the door, we have $\sum \tau = 0$

Taking moment of all forces acting on the door about lower hinge (may be upper hinge or any other point), we get

$$500 \times 1.0 - F_H \times 2 = 0$$

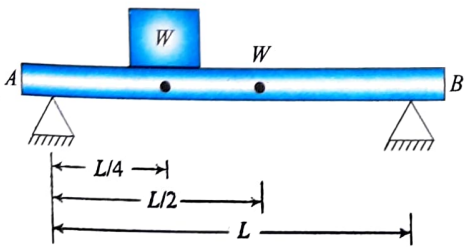
which gives, $F_H = 250\,\text{N}$

The force on the door exerted by either hinge, in magnitude, is

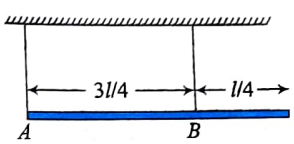
$$F = \sqrt{F_H^2 + F_v^2} = \sqrt{(250)^2 \times (250)^2} = 250\sqrt{2}\,\text{N}$$

CONCEPT APPLICATION EXERCISE 3.4

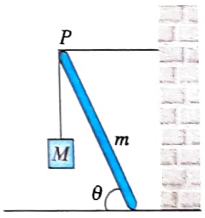
1. A beam of weight W supports a block of weight W. The length of the beam is L and weight is at a distance $L/4$ from the left end of the beam. The beam rests on two rigid supports at its ends. Find the reactions of the supports.



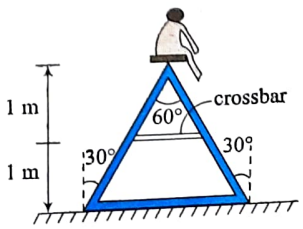
2. A uniform rod of length l and mass m is hung from two strings of equal length from a ceiling as shown in figure. Determine the tensions in the strings.



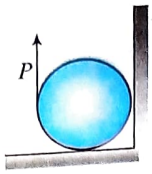
3. A uniform rod is made to lean between a rough vertical wall and the ground. The coefficient of friction between the rod and the ground is μ_1 and between the rod and the wall is μ_2. Find the angle at which the rod can be leaned without slipping.
4. A uniform ladder of mass 10 kg leans against a smooth vertical wall making an angle of 53° with it. The other end rests on a rough horizontal floor. Find the normal force and the friction force that the floor exerts on the ladder.
5. A uniform ladder of length 10.0 m and mass 16.0 kg is resting against a vertical wall making an angle of 37° with it. The vertical wall is frictionless but the ground is rough. An electrician weighing 60.0 kg climbs up the ladder.
   (a) If he stays on the ladder at a point 8.00 m from the lower end, what will be the normal force and the force of friction on the ladder by the ground?
   (b) What should be the minimum coefficient of friction for the electrician to work safely?
6. A uniform ladder of length L and mass m_1 rests against a frictionless wall. The ladder makes an angle θ with the horizontal.
   (a) Find the horizontal and vertical forces the ground exerts on the base of the ladder when a firefighter of mass m_2 is at a distance x from the bottom.
   (b) If the ladder is just on the verge of slipping when the firefighter is a distance d from the bottom, what is the coefficient of static friction between ladder and ground?
7. A uniform beam of mass m is inclined at an angle θ to the horizontal. Its upper end produces a ninety degree bend in a very rough rope tied to a wall, and its lower end rests on a rough floor (figure).



   (a) If the coefficient of static friction between beam and floor is μ_s, determine an expression for the maximum mass M that can be suspended from the top before the beam slips.
   (b) Determine the magnitude of the reaction force at the floor and the magnitude of the force exerted by the beam on the rope at P in terms of m, M, and μ_s.
8. The ladder shown in figure has negligible mass and rests on a frictionless floor. The crossbar connects the two legs of the ladder at the middle. The angle between the two legs is 60°. The fat person sitting on the ladder has a mass of 80 kg. Find the contact forces exerted by the floor on each leg and the tension in the crossbar.



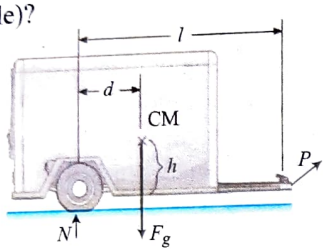
9. Figure shows a vertical force applied tangentially to a uniform cylinder of weight F_g. The coefficient of static friction between the cylinder and all surfaces is 0.500. In terms of F_g, find the maximum force P that can be applied that does not cause the cylinder to rotate.



10. A trailer with loaded weight F_g is being pulled by a vehicle with a force P, as in figure. The trailer is loaded such that its centre of mass is located as shown. Neglect the force of rolling friction and let a represent the x component of the acceleration of the trailer.

(a) Find the vertical component of P in terms of the given parameters.

(b) If $a = 2.00\ \text{ms}^{-2}$ and $h = 1.50$ m, what must be the value of d in order that $P_y = 0$ (no vertical load on the vehicle)?



ANSWERS

1. $\frac{5W}{4}; \frac{3W}{4}$ **2.** $\frac{mg}{3}; \frac{2mg}{3}$ **3.** $\tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_1}\right)$

4. 100 N, $\frac{200}{3}$ N **5.** (a) 760 N, 420 N (b) 0.55

6. (a) $\left[\frac{1}{2}m_1 g + \left(\frac{x}{L}\right)m_2 g\right]\cot\theta, (m_1+m_2)g$

(b) $\frac{\left(\frac{m_1}{2}+\frac{m_2 d}{L}\right)\cot\theta}{m_1+m_2}$

7. (a) $\frac{m}{2}\left(\frac{2\mu_s\sin\theta-\cos\theta}{\cos\theta-\mu_s\sin\theta}\right)$

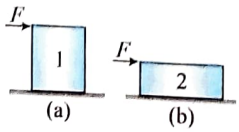
(b) $(M+m)g\sqrt{1+\mu_s^2}, g\sqrt{M^2+\mu_s^2(M+m)^2}$

8. 400 N, $\frac{80}{\sqrt{3}}$ N **9.** $\frac{3}{8}F_g$

10. (a) $\frac{F_g}{L}\left(d-\frac{ah}{g}\right)$ (b) 0.30 m

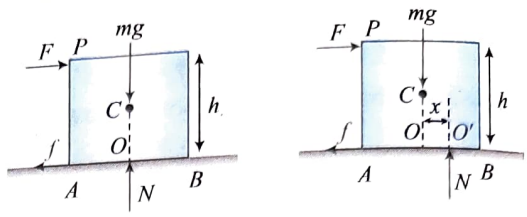
TOPPLING AND SHIFTING OF NORMAL REACTION

Let us take two blocks of equal masses. The block '1' shown in Fig. (a) has greater height than the block '2' shown in Fig. (b).



If same force F is applied on both blocks as shown in the figures. Block '1' is more likely to topple down before sliding. For block '2' of broader base, chances of its sliding are more compared to its toppling. The question arises why it happens? To understand this now consider a block of mass m is placed on a rough horizontal surface and a horizontal force F is applied on it as shown figure. The friction is sufficient so that the block is not moving.

It means: $\Sigma\vec{F}=0$ and $\Sigma\vec{\tau}=0$



(i) (ii)

If we consider the normal reaction is passing through centre of mass as shown in Fig. (i), in this case there is unbalanced torque about C in clockwise sense. To make the block in rotational equilibrium an anticlockwise torque is required, which is possible only if the normal reaction shifts towards O' as shown in Fig. (ii).

If the block is in rotational equilibrium, i.e., it is not going to topple then torque about C is zero.

$$Nx = f\frac{h}{2} \Rightarrow x = \frac{fh}{2N} = \frac{fh}{2mg} \quad \ldots(i)$$

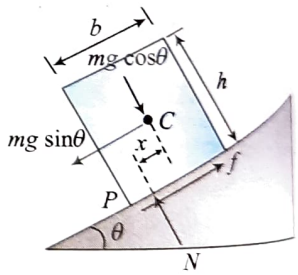
As the block is at rest, $f = F \Rightarrow x = \frac{Fh}{2mg}$ ...(ii)

The maximum value of x can be $\frac{b}{2}$ only because then the normal force will be at the extreme end. For toppling,

$$\frac{b}{2} = \frac{Fh}{2mg} \Rightarrow F = \frac{bmg}{h}$$

Now let us consider the block is kept on rough inclined plane then its FBD is as shown.

Again, if the block is in rotational equilibrium, i.e., it is not going to topple then torque about C is zero,



$$\vec{\tau}_C = Nx - f\frac{h}{2} = 0$$

$$\Rightarrow \quad x = \frac{fh}{2N} = \frac{fh}{2mg\cos\theta}$$

If block is at rest, $f = mg\sin\theta$

$$x = \frac{mg\sin\theta h}{2mg\cos\theta} = \frac{h\tan\theta}{2} \quad \ldots(i)$$

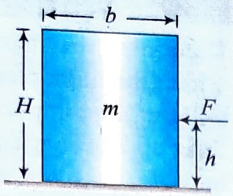
The maximum value of x can be $b/2$ only. So, for a given inclined plane and with a fixed base of the body the maximum height of a body which can be placed on it without topping is given by putting $x = \frac{b}{2}$ in Eq. (i),

$$h = 2\frac{b}{2\tan\theta} = \frac{b}{\tan\theta} = b\cot\theta$$

To understand this concept in better way let us take a few illustrations.

ILLUSTRATION 3.31

A horizontal force F is applied to a homogeneous rectangular block of mass m, width b and height H. The block moves with constant velocity; the coefficient of friction is μ_k.



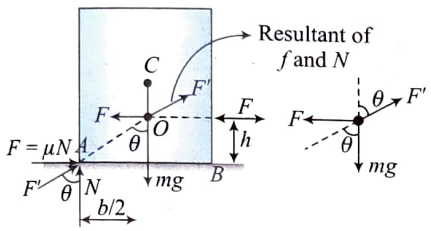
(a) What is the greater height h at which the force F can be applied so that the block will slide without tipping over?

(b) Through which point on the bottom face of the block will the resultant of the friction and normal forces act if $h = H/2$?

(c) If the block is at rest and coefficient of static friction is μ_s, what are the various criteria for which sliding, or tipping occurs?

Sol.

(a) In the absence of any external force in horizontal direction the normal reaction N passes through the centre of mass of the block; when force F is applied, normal reaction shifts in the direction of applied force F (in leftward direction i.e., towards A)



Since right part of body is having tendency to lift from surface, at the instant of tipping over about the edge the normal reaction passes through edge. From the conditions of equilibrium.

In horizontal direction: $f = F = \mu_k mg$

In vertical direction: $N = mg$

Balancing torque about edge A,

$$Fh - mg\frac{b}{2} = 0 \text{ or } h = \frac{mgb}{2F} = \frac{mgb}{2\mu_k mg} = \frac{b}{2\mu_k}$$

We can solve the problems by another approach also as the resultant of friction force ($\mu_k N$) and the normal reaction must pass through the same point through which F passes, since three coplanar forces keeping a body in equilibrium pass through a common point i.e., they should be concurrent.

From Lami's theorem:

$$\frac{F}{\sin 90°} = \frac{F}{\sin(180° - \theta)} = \frac{mg}{\sin(90° + \theta)}$$

$$\Rightarrow \frac{F}{\sin\theta} = \frac{F}{\sin(180° - \theta)} = \frac{mg}{\sin(90° + \theta)}$$

From figure it is clear $\tan\theta = \dfrac{\mu_k N}{N} = \dfrac{b/2}{h}$

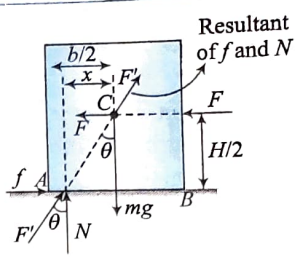
or $\dfrac{(b/2)}{h} = \mu_k \Rightarrow h = \dfrac{b}{2\mu_k}$

For the block slide without toppling over about edge A, h should be less than $\dfrac{b}{2\mu_k}$.

(b) In this case we cannot take normal reaction at the edge.

Let normal reaction acts at a distance x from the line of action of mg.

Torque of all the forces about C must be zero.

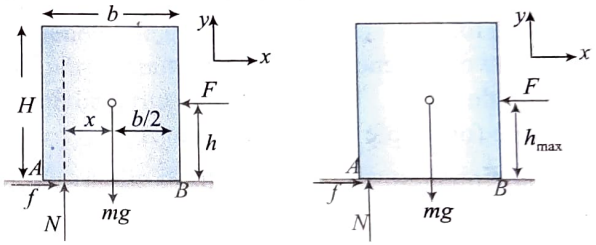


$$\left(\mu_k N \times \frac{H}{2}\right) = Nx \Rightarrow x = \frac{\mu_k H}{2}$$

Alternately: We can proceed as in part (a). Resultant of $\mu_k N$, N passes through C the interaction of F and mg. Therefore, from geometry of the figure.

$$\frac{x}{H/2} = \tan\theta = \mu_k;\ x = \frac{\mu_k H}{2}$$

(c) As the point of application of force is raised higher, the location of the line of action of the normal reaction N moves towards edge A the left Fig. (a). In the limiting case (when the block is about to tip over), $x = b/2$. The normal reaction passes through edge as shown in Fig. (a).



(a) **(b)**

If the block topples about edge A, then

$$Fh_{max} = mg\frac{b}{2} \text{ or } h_{max} = \frac{mgb}{2F} \quad \ldots(i)$$

For sliding of the block $F = \mu N$

In vertical direction: $N = mg$

Hence, $F = \mu mg$ ...(ii)

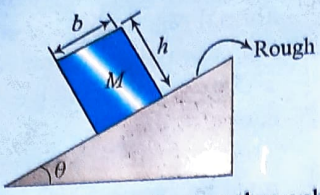
If sliding and tipping are equally likely to occur, we can eliminate F from Eqs. (i) and (ii) to get $h_{max} = \dfrac{b}{2\mu}$ which is independent of weight mg, height H of the body and applied force F.

Important conclusions from above discussion:

- If $h < h_{max}$, $F < \mu mg$, neither toppling nor sliding occurs.
- If $h < h_{max}$, $F = \mu mg$, no toppling, sliding motion impending.
- If $h < h_{max}$, $F > \mu mg$, no toppling, sliding occurs.
- If $h = h_{max}$, $F = \mu mg$, both sliding, and toppling are impending.
- If $h > h_{max}$, the body will tip over for any value of weight of the body.
- If $\mu \geq \dfrac{F}{mg}$, onset of toppling occurs, and toppling edge remains stationary.
- If $\mu < \dfrac{F}{mg}$, onset of toppling occurs with sliding edge on surface.

ILLUSTRATION 3.32

A heavy block of length b and height h is placed at rest on a rough inclined plane of inclination θ with the horizontal, as shown in figure.

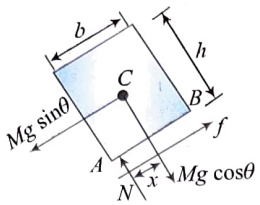


(a) The normal reaction from the surface cannot pass through the centre mass. Why?

(b) If μ is the coefficient of friction between the block and the inclined plane, then discuss the conditions of translational and rotational equilibrium of the block.

Sol.

(a) The free-body diagram of the block is shown in figure.



Note that the point of application of the normal reaction N is displaced through x in the downward direction. It must produce a clockwise torque about the centre of mass which can be balanced by the anticlockwise torque produced by the friction force.

(b) There are two tendencies of the block:

- to slide down
- to rotate (or topple) about the point A

For translational equation $\sum \vec{F} = 0$

$$f = Mg\sin\theta \quad \ldots(i)$$

$$N = Mg\cos\theta \quad \ldots(ii)$$

For rotational equilibrium, $\sum \vec{\tau}_C = 0$

$$N \times x = f \times \frac{h}{2} \Rightarrow Mg\cos\theta \times x = \frac{fh}{2}$$

$$\Rightarrow \quad f = \frac{2Mgx\cos\theta}{h}$$

If the block topples over about A, $x = \frac{b}{2}$

Hence, (iii) $f = \frac{b\,Mg\cos\theta}{h}$ $\quad \ldots(iii)$

From Eq. (i) if block do not slide down, $f < \mu N$

$$Mg\sin\theta < \mu \times Mg\cos\theta \Rightarrow \tan\theta \le \mu$$

or we can say if the block slides $\mu < \tan\theta$

From Eq. (iii) if the block topples before sliding,

$$f \le \mu(Mg\cos\theta)$$

$$\frac{b\,Mg\cos\theta}{h} \le \mu Mg\cos\theta \Rightarrow \frac{b}{h} \le \mu$$

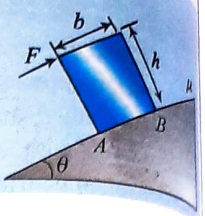
Hence, if the block topples before sliding, $\mu \ge \frac{b}{h}$

Important Points:

- The block has tendency to topple before sliding if $\mu > b/h$.
- The block slides down if $\mu < \tan\theta$, where θ is the angle of inclination of the incline with respect to the horizontal.
- The block topples over without sliding if $\mu > \tan\theta$ and $\mu > b/h$.

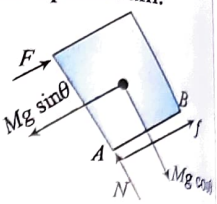
ILLUSTRATION 3.33

Determine the maximum ratio h/b for which the homogenous block will slide without toppling under the action of force F. The coefficient of static friction between the block and the incline is μ.



Sol. Block is sliding without toppling, which implies that block is in translational as well as rotational equilibrium.

When the block is on the verge of tipping about edge A, the normal reaction will pass through the edge of the block.



From conditions of equilibrium along sloping surface:

$$F + \mu N = mg\sin\theta \quad \ldots(i)$$

Perpendicular to the sloping surface:

$$N = mg\cos\theta \quad \ldots(ii)$$

Balancing torque about point A:

$$F \times h = (mg\cos\theta)\frac{b}{2} \quad \ldots(iii)$$

From Eqs. (i), (ii) and (iii),

$$F = mg\sin\theta - \mu mg\cos\theta \quad \ldots(iv)$$

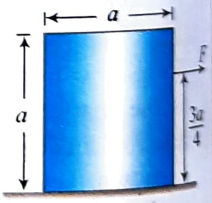
On substituting F in Eq. (iii), we obtain

$$(mg\sin\theta - \mu mg\cos\theta)h - mg\cos\theta\frac{b}{2} = 0$$

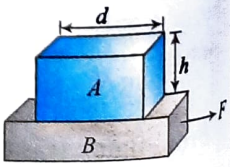
which on solving for h/b yields $\frac{h}{b} = \frac{1}{2(\tan\theta - \mu)}$

CONCEPT APPLICATION EXERCISE 3.5

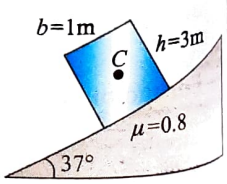
1. A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height $3a/4$ above the base. What is the minimum value of F for which the cube begins to tip about an edge?



2. A rectangular block 'A', having height h and width b has been placed on another block B as shown in the figure. Both blocks have equal mass and there is no friction between the block 'B' and ground. A horizontal time dependent force; $F = kt$ is applied on the block 'B'. At what time will block 'A' topple? Assume that friction between the two blocks is large enough to prevent the block 'A' from slipping.

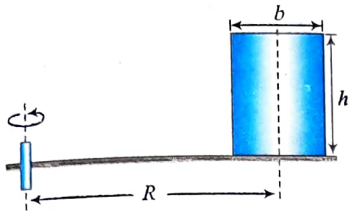


3. A tall block of mass $M = 50$ kg and base width $b = 1$ m and height $h = 3$ m is kept on a rough inclined surface with coefficient of friction $\mu = 0.8$ as shown in figure. The

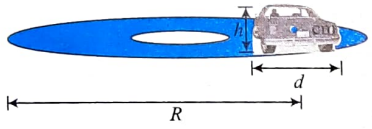


angle of inclination with the horizontal is 37°. Determine whether the block slides down or topples over.

4. A rectangular block rests on a horizontal rotating disc, as shown in the figure. Find at what angular velocity ω, the block falls of the disc, if the distance between the axis of the disc and block is R and the coefficient of friction $\mu > \frac{b}{h}$, where b is the width of the block and h is its height.



5. A car moves with speed v on a horizontal circular track of radius R. A head-on view of the car is shown in figure. The height of the car's centre of mass above the ground is h, and the separation between its inner and outer wheels is d. The road is dry, and the car does not skid. Show that the maximum speed the car can have without overturning is given by $v_{max} = \sqrt{\frac{gRd}{2h}}$. To reduce the risk of rollover, should one increase or decrease h? Should one increase or decrease the width d of the wheel base?



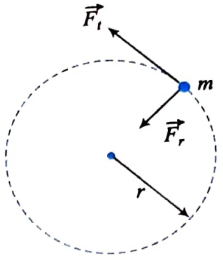
ANSWERS

1. $\frac{2mg}{3}$ 2. $\frac{2dgm}{kh}$

NEWTON'S SECOND LAW FOR ROTATIONAL MOTION

We know a net force on an object causes an acceleration of the object and that the acceleration is proportional to the net force which can be verified by Newton's second law. Now we will learn the rotational analog of Newton's second law: the angular acceleration of a rigid object rotating about a fixed axis is proportional to the net torque acting about that axis. Before discussing the more complex case of rigid-object rotation, however, it is instructive first to discuss the case of a particle moving in a circular path about some fixed point under the influence of an external force.

Consider a particle of mass m rotating in a circle of radius r about some fixed point under the influence of an external force. The components of the force in tangential direction and radial direction are $\vec{F}_t$ and $\vec{F}_r$ respectively are shown in figure.



The radial net force causes the particle to move in the circular path with a centripetal acceleration. The tangential force provides a tangential acceleration $\vec{a}_t$, and we can write

$$F_t = ma_t \quad \text{...(i)}$$

The magnitude of the net torque due to $\vec{F}_t$ on the particle about an axis perpendicular to the page through the center of the circle is

$$\tau = F_t r = (ma_t)r \quad \text{...(ii)}$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$, the net torque can be expressed as

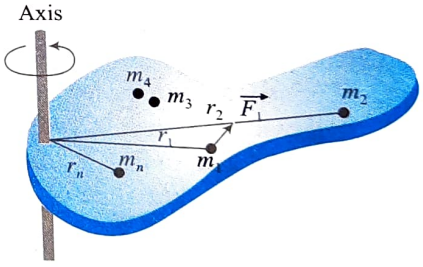
$$\tau = (mr\alpha)r = (mr^2)\alpha \quad \text{...(iii)}$$

Let us denote the quantity mr^2 with the symbol I for now. We will say more about this quantity below. Using this notation, Eq. (iii) can be written as

$$\tau = I\alpha \quad \text{...(iv)}$$

Equation (iv) is the form of Newton's second law we have been seeking. It indicates that the net external torque τ is directly proportional to the angular acceleration α. The constant of proportionality is $I = mr^2$, which is called the **moment of inertia of the particle.** The SI unit for moment of inertia is kg m^2.

Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis passing through a point O as in figure.



The object can be regarded as a collection of particles of masses $m_1, m_2, ..., m_n$. If we impose a Cartesian coordinate system on the object, each particle rotates in a circle about the origin with some tangential acceleration produced by external tangential force. For any given particle, we know from Newton's second law that,

$$\vec{F}_i = m_i a_i$$

The external torque $\vec{\tau}_i$ associated with the force $\vec{F}_i$ acts about the origin and its magnitude is given by,

$$\tau_i = r_i F_i = r_i m_i a_i$$

Because $a_i = r_i\alpha$, the expression for τ_i becomes, $\tau_i = m_i r_i^2 \alpha$

Although each particle in the rigid object may have a different translational acceleration a_i, they all have the same angular acceleration α. With that in mind, we can add the torques on all the particles making up the rigid object to obtain the net torque on the object about an axis through O due to all external forces:

$$\Sigma\tau_{ext} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha = \left(\sum_i m_i r_i^2\right)\alpha \quad ...(v)$$

where α can be taken outside the summation because it is common to all particles and $\Sigma mr^2 = m_1 r_1^2 + m_2 r_2 + \cdots + m_n r_n^2$ represents the sum of the individual moments of inertia. The latter quantity is the **moment of inertia I of the body**. The expression for $\Sigma\tau_{ext}$ becomes

$$\Sigma\tau_{ext} = I\alpha \quad ...(vi)$$

Rotational analog of Newton's second law for a rigid body rotating about a fixed axis

Net external torque = moment of inertia × Angular acceleration

or $\tau_{ext} = I\alpha$

The above equation is called 'torque equation'.

The torque equation is, however, more complex than the force equation. Why?

- Force is an absolute value, whereas torque depends on the point about which it is being calculated.
- Mass is an absolute value for a body, whereas MI can have infinite values for a body. MI is defined only for a given axis of the body.

Note:

Remember the following points about the torque equation:

Torque equation can be applied only about two points in a body. These are as follows:

- Centre of mass.
- Point which has zero velocity/acceleration. This point is commonly referred to as the center of rotation or instantaneous center of rotation.

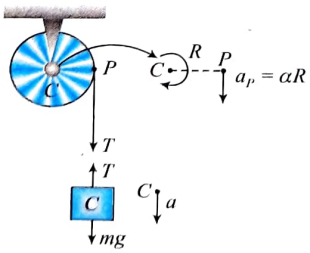
Now, remember that the axis of rotation passes through the point and is perpendicular to the surface (valid only for two dimensional bodies).

ILLUSTRATION 3.34

A block of mass m is attached at the end of an inextensible string which is wound over a rough pulley of mass M and radius R (see figure). Assume the string does not slide over the pulley. Find the acceleration of the block when released.

Sol. This problem is different from the problems which we have solved in laws of motion where we assumed the string is sliding over the pulley without friction. Here, the pulley is rough, and string is not sliding.

The free-body diagrams of the given situation are shown in figure.



The motion of the block is translational, here we can apply equation of motion of the block:

$$mg - T = ma \quad ...(i)$$

Motion of the pulley is pure rotational.

If we take a point P on the rim of the pulley, its tangential acceleration should be equal to αR. This acceleration should be equal to the acceleration of the block.

Hence $a = \alpha R$

Torque equation for the pulley: $\tau_c = I_c\alpha$ ...(ii)

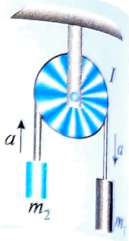
$$TR = \left(\frac{MR^2}{2}\right)\alpha \Rightarrow \alpha = \frac{2T}{MR} \quad ...(iii)$$

After solving Eqs. (i), (ii) and (iii), we get

$$a = \frac{2mg}{2m + M}$$

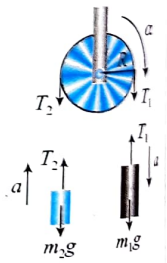
ILLUSTRATION 3.35

Find the acceleration of m_1 and m_2 in an Atwood's Machine if there is friction present between surface of pulley and the thread and thread does not slip over the surface of pulley. Moment of inertia of pulley is I.



Sol. Atwood's machine consists of two weights with masses m_1 and m_2 connected by a string or rope, which is guided over a pulley. The Atwood's machine we analyzed in the chapter laws of motion, was subject to the condition that the rope glides without friction over the pulley so that the pulley does not rotate (or, equivalently, that the pulley is massless). With the concepts of rotational dynamics, we can take another look at the Atwood's machine and consider the case where friction occurs between the rope and the pulley, causing the rope to turn the pulley without slipping.

In the chapter laws of motion, we saw that the magnitude of the tension, T, is the same everywhere in a string. However, now friction forces are involved to keep the string attached to the pulley, and we cannot assume that the tension is constant. Instead the string tension is separately determined in each segment of the string from which one of the two masses hang. Thus, there are two different string tensions, T_1 and T_2, in the free-body diagrams for the two masses (as shown in figure). As there is friction between pulley and thread, tension in thread throughout will not remain same. Let on two sides of the pulley tension in thread be T_1 and T_2 and the masses are going with acceleration a. As thread does not slip of the surface of pulley, pulley will rotate and the linear acceleration of the particles on pulley at its rim will also be a, thus its angular acceleration can be given as



$$\alpha = \frac{a}{R} \quad ...(i)$$

Now we write motion equations for masses m_1 and m_2 as

$$T_2 - m_2 g = m_2 a \quad ...(ii)$$

or $$m_1 g - T_1 = m_1 a \quad ...(iii)$$

A net torque does act on the pulley. Torques are applied by different tensions in string on both sides of pulley. The magnitude of the net torque on the pulley due to the string tensions,

$$\tau = \tau_1 - \tau_2 = T_1 R - T_2 R = R(T_1 - T_2) \quad ...(iv)$$

These two torques have opposite signs, because one is acting clockwise and the other counterclockwise. The net torque is

related to the moment of inertia of the pulley and its angular acceleration by the relation

$$\tau = I\alpha = I\frac{a}{R} \qquad ...(v)$$

Substituting this expression for the torque into Eq. (iv) we find,

$$R(T_1 - T_2) = \tau = I\frac{a}{R}$$

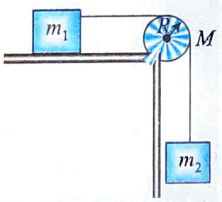
or $$T_1 - T_2 = I\frac{a}{R^2} \qquad ...(vi)$$

Adding Eqs. (ii), (iii) and (vi), we get

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right)a \text{ or } a = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$$

ILLUSTRATION 3.36

In figure, mass m_1 slides without friction on the horizontal surface, the frictionless pulley is in the form of a cylinder of mass M and radius R, and a string turns the pulley without slipping. Find the acceleration of each mass.



Sol. In this case the string rotates the pulley, i.e., the string does not slide. The pulley will have angular acceleration. Hence, the tension on the string on both the sides of the pulley will not be equal. The motion of m_1 and m_2 will be translational. Positive direction of angular acceleration α is taken along the corresponding acceleration a_1 which is taken positive in the direction of net force.

Let these be T_1 and T_2. The equation of motion respectively, for masses m_1 and m_2 is

$$T_1 = m_1 a \qquad ...(i)$$

and $$m_2 g - T_2 = m_2 a \qquad ...(ii)$$

Application of torque equation:
The torque equation for the pulley,

$$T_2 R - T_1 R = I\alpha \qquad ...(iii)$$

(N_2 constitutes no torque about the axis of rotation)
There is no slipping of the string over the pulley.
Constraint relation:

$$a = \alpha R \Rightarrow \alpha = \frac{a}{R} \qquad ...(iv)$$

From Eqs. (iii) and (iv), we get

$$\therefore \quad T_2 R - T_1 R = I\frac{a}{R}$$

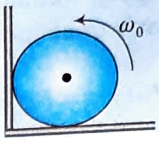
or $$T_2 - T_1 = \frac{Ia}{R^2} \qquad ...(v)$$

Solving the above equations, we get

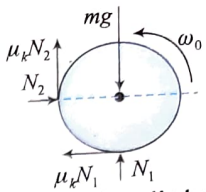
$$a = \frac{m_2 g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$$

ILLUSTRATION 3.37

A uniform cylinder of radius R is spun about its axis to the angular velocity ω_0 and then placed into a corner, (see figure). The coefficient of friction between the corner walls and the cylinder is μ_k. How many turns will the cylinder accomplish before it stops?



Sol. As the centre of mass of the cylinder does not accelerate, hence



FBD of the cylinder

$$\sum F = 0$$

$$\sum F_x = 0,\ N_2 - \mu_k N_1 = 0, \qquad ...(i)$$

$$\sum F_y = 0,\ N_1 + \mu_k N_2 - mg = 0 \qquad ...(ii)$$

Solving these equations:

$$N_1 = \frac{mg}{1+\mu_k^2},\ N_2 = \frac{\mu_k mg}{1+\mu_k^2}$$

The torque on the cylinder about the axis of rotation,

$$\tau_{CM} = mg \times 0 + N_1 \times 0 + N_2 \times 0 + \mu_k N_2 R + \mu_k N_1 R$$

$$= \frac{\mu_k(1+\mu_k)}{(1+\mu_k^2)} mgR$$

The moment of inertia about axis of rotation, $I_{CM} = \frac{1}{2}mR^2$

The torque equation, $\tau = I\alpha$

$$\frac{\mu_k(1+\mu_k)}{1+\mu_k^2} mgR = \left(\frac{1}{2}mR^2\right)\alpha \Rightarrow \alpha = \frac{2\mu_k(1+\mu_k)g}{(1+\mu_k^2)R}$$

Using equation $\omega^2 = \omega_0^2 + 2\alpha\theta$, calculate the angular displacement ω,

$$0^2 = \omega_0^2 - 2\left[\frac{2\mu_k(1+\mu_k)g}{(1+\mu_k^2)R}\right]\theta$$

(ω and α are in the opposite sense)

$$\Rightarrow \quad \theta = \frac{(1+\mu_k^2)\omega_0^2 R}{4\mu_k(1+\mu_k)g}$$

Revolution accomplished,

$$n = \frac{\theta}{2\pi} = \frac{(1+\mu_k^2)\omega_0^2 R}{2\pi \times 4\mu_k(1+\mu_k)g} = \frac{(1+\mu_k^2)\omega_0^2 R}{8\pi\mu_k(1+\mu_k)g}$$

ILLUSTRATION 3.38

A uniform rod of length L and mass M is pivoted freely at one end and placed in vertical position.

(a) What is the angular acceleration of the rod when it is at an angle θ with the vertical?

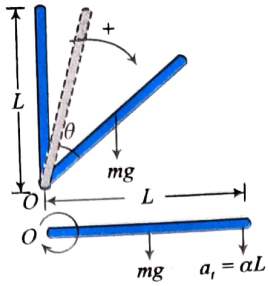
(b) What is the tangential linear acceleration of the free end when the rod is horizontal?

Sol. Figure shows the rod at an angle θ to the vertical. If we take torques about the pivot we need not be concerned with the force due to the pivot.

The torque due to the weight about O,

$$\tau_W = \frac{mgL}{2}\sin\theta$$

Using torque equation about O. Here we apply torque equation about fixed axis passing through O. We can apply torque equation about fixed axis or about centre of mass only.



(a) $\tau = I\alpha \Rightarrow \frac{mgL}{2}\sin\theta = \frac{ML^2}{3}\alpha$

Thus, $\alpha = \frac{3g\sin\theta}{2L}$

When the rod is horizontal, $\theta = \frac{\pi}{2}$ and $\alpha = \frac{3g}{2L}$

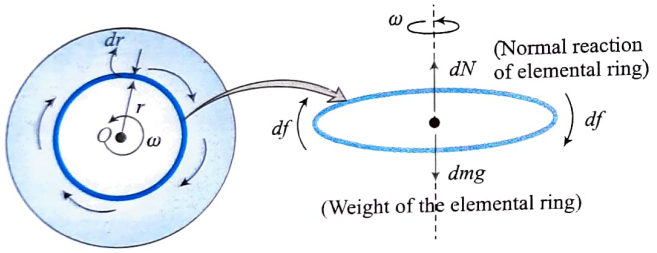
(b) The tangential linear acceleration of the free end is

$$a_t = \alpha L = \frac{3g}{2}$$

ILLUSTRATION 3.39

A uniform disc of radius R and mass M, spinning with angular velocity ω_0 is carefully lowered onto the rough horizontal surface with coefficient of friction μ. Find the time when the disc comes at rest.

Sol. The angular velocity of the disc changes due to torque of the friction force between the lower surface of the disc and floor. First, we need to calculate τ, let us take an elementary ring of radius r and thickness dr on lower surface of the disc.



The mass of the elemental ring will be distributed on its length will act in downward direction which will be equal to normal reaction of the ground.

$$dN = dmg$$

Hence magnitude of friction force on the ring

$$df = \mu dN = \mu dmg = \mu\left(\frac{M}{\pi R^2}\right) \times 2\pi r dr \times g$$

$$\Rightarrow \quad df = \frac{2\mu Mg}{R^2} r dr$$

This friction force is distributed on the length of the ring. The torque of friction force on the elemental ring

$$d\tau = df \times r = \left(\frac{2\mu Mg}{R^2}\right) r^2 dr \qquad ...(i)$$

To calculate net torque on the disc we need to integrate Eq. (i) between 0 and R.

$$\tau = \int d\tau = \frac{2\mu Mg}{R^2}\int_0^R r^2 dr \qquad ...(ii)$$

From Eq. (ii) we can observe that the torque acting on the disc is constant, hence angular acceleration will be contact.

Now applying torque equation

$$\tau = I\alpha$$

$$\alpha = \frac{\tau}{I} = \frac{\frac{2}{3}\mu MgR}{MR^2/2}$$

$$\Rightarrow \quad \alpha = \frac{4\mu g}{3R} \qquad ...(iii)$$

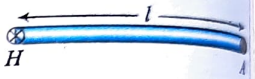
The initial angular velocity of the disc is ω_0 and finally it stops hence final angular velocity is zero.

Using, $\omega = \omega_0 + \alpha t$

$$0 = \omega_0 - \left(\frac{4\mu g}{3R}\right)t \quad \text{or} \quad t = \frac{3\omega_0 R}{4\mu g}$$

CONCEPT APPLICATION EXERCISE 3.6

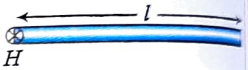
1. A uniform rod of mass m and length l can rotate in a vertical plane about a smooth horizontal axis hinged at point H.



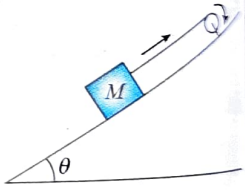
   (a) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest?

   (b) Calculate the acceleration (tangential and radial) of point A at this moment.

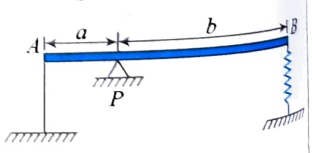
2. A uniform rod of mass m and length l can rotate in a vertical plane about a smooth horizontal axis hinged at point H. Find the force exerted by the hinge just after the rod is released from rest, from an initial horizontal position?



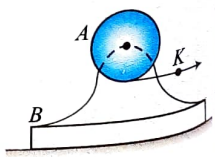
3. A wheel of radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in figure. A string is wrapped around the wheel and its free end supports a block of mass M which can slide on the plane. Initially, the wheel is rotating at a speed ω in a direction such that the block slides up the plane. How far will the block move before stopping?



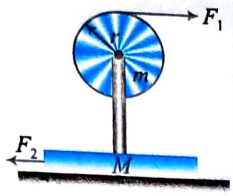
4. A uniform rod AB of mass $m = 2$ kg and length $l = 1.0$ m is placed on a sharp support P such that $a = 0.4$ m and $b = 0.6$ m. A spring of force constant $k = 600$ N/m is attached to end B as shown in figure. To keep the rod horizontal, its end A is tied with a thread such that the spring is elongated by 1 cm. Calculate reaction of support P when the thread is burnt.



5. A uniform solid cylinder A of mass m_1 can freely rotate about a horizontal axis fixed to a mount of mass m_2. A constant horizontal force F is applied to the end K of a light thread tightly wound on the cylinder. The friction between the mount and the supporting horizontal plane is assumed to be absent. Find the acceleration of the point K.



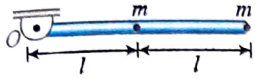
6. A disc of mass m and radius r is pivoted smoothly with a block of mass M. A force F_1 acts on m and another force F_2 acts on M. Assume that ground is smooth. Find the:



(i) linear acceleration of the bodies.

(ii) angular acceleration of the disc.

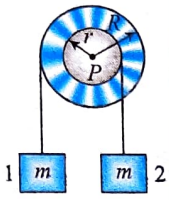
7. A light rigid rod is connected rigidly with two identical particles each of mass m as shown. The free end of the rod is smoothly pivoted at O. The rod is released from rest from its horizontal position at $t = 0$. Find:



(i) angular acceleration of the rod, at $t = 0$.

(ii) reaction offered by the pivot, at $t = 0$.

8. Two blocks each of mass m are hanging from a stepped pulley by an inextensible light string which passes over a stepped pulley. The mass of the pulley is m and its radius of gyration is k. If the string does not slide on the pulley, find the accelerations of the blocks.



ANSWERS

1. (a) $\frac{3g}{2l}$ (b) $\frac{3g}{2}$; 0 **2.** $\frac{mg}{4}$ **3.** $\frac{(I+Mr^2)\omega^2}{2Mg\sin\theta}$

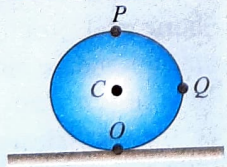
4. 20 N **5.** $\frac{F(3m_1+2m_2)}{m_1(m_1+m_2)}$ **6.** (i) $\frac{F_1-F_2}{m+M}$ (ii) $\frac{2F_1}{mr}$

7. (i) $\frac{3g}{5l}$ (ii) $\frac{mg}{5}$ **8.** $a_1=\frac{gR(R-r)}{k^2+(R^2+r^2)}$, $a_2=\frac{gr(R-r)}{k^2+(R^2+r^2)}$

Solved Examples

EXAMPLE 3.1

Find the radius of curvature of the path traced by a peripheral point of a perfectly rolling disc of radius R, at its highest position P.



Sol. The radius of curvature of the path traced by P is,

$$R_P=\frac{v_P^2}{(a_P)_n} \qquad ...(i)$$

where $(a_P)_n=|\vec{a}_{PC_n}+\vec{a}_{C_n}|$

Since, $a_{PC_n}=\frac{v_{PC}^2}{r_{PC}}=\frac{|\vec{v}_P-\vec{v}_C|^2}{r_{PC}}$

Substituting $\vec{v}_P=2v\hat{i}$, $\vec{v}_C=v\hat{i}$ for pure rolling and $r_{PC}=R$,

we have $a_{PC_n}=\frac{v^2}{R}$

Since, the point 'C' moves in a straight line, the radius of curvature of a straight line is infinite.

Then, we have $a_{C_n}=\frac{v_C^2}{R_C}=0$ (where $R_C=\infty$)

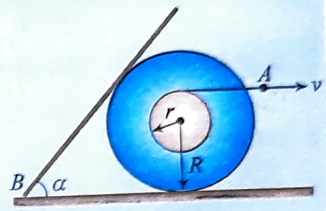
Therefore, $a_{P_n}=|\vec{a}_{PC_n}+\vec{a}_{C_n}|=\frac{v^2}{R}+0=\frac{v^2}{R}$

Substituting $v_P=2v$ and $a_{P_n}=v^2/R$ in Eq. (i),

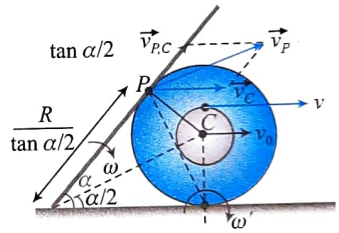
$$R_P=\frac{v_P^2}{(a_P)_n}=\frac{(2v)^2}{v^2/R}=4R$$

EXAMPLE 3.2

A bobbin rolls without slipping over a horizontal surface so that the velocity v of the end of the thread (point A) is directed along the horizontal. A board hinged at B leans against the bobbin as shown in figure. The inner and outer radii of the bobbin are r and R respectively. Determine the angular velocity ω of the board as a function of angle α.



Sol. Let P be the point of contact of the board and the bobbin at a certain instant. As the bobbin is in pure rolling, the velocity of point P, $\vec{v}_P=\vec{v}_{\text{rotation}}+\vec{v}_{\text{translation}}=\vec{v}_{P,C}+\vec{v}_C$



If the angular velocity of the board at this instant is ω, the linear velocity of the point of the board in contact with the bobbin is $\frac{\omega R}{\tan\frac{\alpha}{2}}$ normal to the board. If the board is to remain in contact with the bobbin, their normal velocities should be same, i.e.,

$$\omega\frac{R}{\tan\frac{\alpha}{2}}=v_0\sin\alpha \qquad ...(i)$$

As I is instantaneous centre of rotation,

$$\frac{v_0}{R}=\frac{v}{R+r}=\text{angular velocity of bobbin } \omega'$$

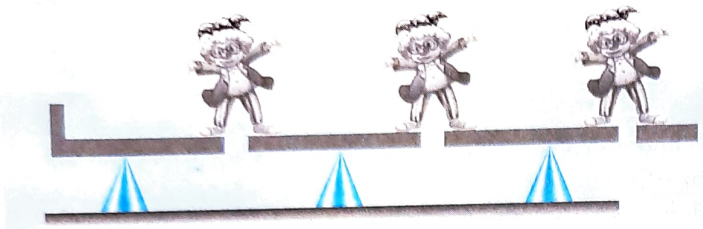
$$\therefore \quad v_0=\frac{vR}{R+r} \qquad ...(ii)$$

Now Eq. (i) becomes $\omega R=\frac{vR}{R+r}\sin\alpha\tan\frac{\alpha}{2}$

or $$\omega=\frac{2v\sin^2\frac{\alpha}{2}}{(R+r)\cos\frac{\alpha}{2}}$$

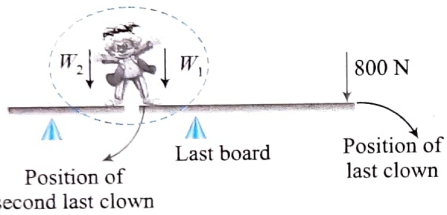
EXAMPLE 3.3

In a circus show are used large numbers of light boards, each of which can rotate around a fixed fulcrum. Fulcrum of reach board divides the length of the board in ration 2:1. At the one end of the left-most board is placed a small block of mass 30 kg and a team of clowns stand keeping their feet at the ends of adjacent boards as shown in the figure. The mass of each clown is 80 kg. What maximum number of clowns can keep balance in this way?



Sol. **Method 1:** Consider the last board.

The clown on the last board will keep his total weight on the extreme end.



Balancing torque about fulcrum of last board.

$$W_1 \times \frac{2l}{3} = 800 \times \frac{l}{3}$$

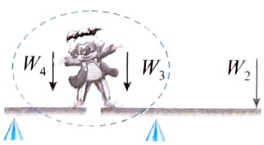
$$W_1 = 400\ \text{N}$$

Hence $W_2 = 800 - 400 = 400\ \text{N}$

Now balancing the board next to last (second last board)
Again, balancing torque about fulcrum of second last board

$$W_3 \times \frac{2}{3}l = W_2 \times \frac{l}{3}$$

$$W_3 = \frac{W_2}{2} = 200\ \text{N}$$



Hence $W_4 = 800 - 200 = 600\ \text{N}$

Now considering next to second last board.
Balancing torque about fulcrum

$$W_5 \times \frac{2l}{3} = W_4 \times \frac{l}{3} \Rightarrow W_5 = \frac{W_4}{2} = 300\ \text{N}$$

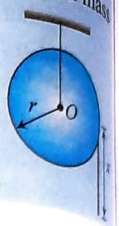
But on 1st board on extreme end 300 N is applied.
Hence total number of boards are three.

Method 2: On first board at extreme end 300 N is placed and on last board on extreme right end the clown will apply his total weight (800 N).

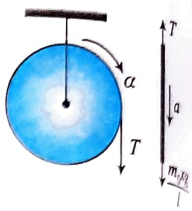
For balancing any board, the ratio of force applied on left and right side on the board should be 1:2.
If $W = 300$ N, $W_2 = 600$ N
Hence number of boards should be three.

EXAMPLE 3.4

A uniform cylinder of radius r and mass m can rotate freely about a fixed horizontal axis. A thin cord of length l and mass m_0 is wound on the cylinder in a single layer. Find the angular acceleration of the cylinder as a function of the length x of the hanging part of the cord. The wound part of the cord is supposed to have its centre of gravity on the cylinder axis as shown in figure.



Sol. Figure shows the free-body diagram of the two bodies, viz., cylinder and overhanging portion of the cord.



If α is the angular acceleration of the cylinder, then from $\tau = I\alpha$, we have

$$Tr = \left[\frac{mr^2}{2} + \frac{m_0}{l}(l-x)r^2\right]\alpha$$

$$T = \left[\frac{m}{2} + m_0 - \frac{m_0 x}{l}\right] r\alpha \quad \ldots(i)$$

If a be the acceleration of the hanging part, then from $F = ma$

$$\frac{m_0 xg}{l} - T = \frac{m_0 x}{l} a = \frac{m_0 x}{l}(\alpha r) \quad \ldots(ii)$$

[Linear acceleration of the rim of the cylinder is same as that of the cord]

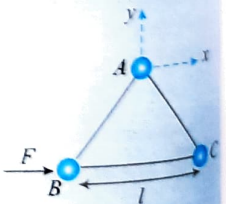
Adding Eqs. (i) and (ii), we get

$$\frac{m_0 x}{l} g = \alpha\left[\frac{m}{2} + m_0\right] r$$

$$\Rightarrow \quad \alpha = \frac{m_0 xg}{rl\left[\frac{m}{2} + m_0\right]}$$

EXAMPLE 3.5

Three particles A, B and C each of mass m are connected to each other by three massless rigid rods to form a rigid, equilateral triangular body of side l. This body is placed on a horizontal frictionless table (x-y plane) and is hinged at point A so that it can move without friction about the vertical axis through A (see figure). The body is set into rotational motion on the table about this axis with a constant angular velocity α.



(a) Find the magnitude of the horizontal force exerted by the hinge on the body.

(b) At time T, when side BC is parallel to x-axis, force F is applied on B along BC (as in the figure). Obtain the x-component and the y-component of the force exerted by the hinge on the body, immediately after time T.

Sol.

(a) The centre of mass of the system is at the centroid of the triangular assembly. The CM moves along a circular path

with constant angular velocity. Therefore, there must be a horizontal centripetal force directed towards the axis at the hinge.

From Fig. (a), we find

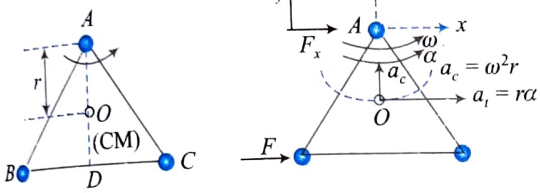
$$AD = l\sin 60° = \frac{l\sqrt{3}}{2} \Rightarrow AO = \frac{2}{3}AD = \frac{l^2}{\sqrt{3}} = r$$

The centripetal acceleration, $a_c = \omega^2 r = \omega^2\left(\frac{l}{\sqrt{3}}\right)$

Tangential acceleration, $a_t = \alpha r = \alpha\left(\frac{l}{\sqrt{3}}\right)$

Horizontal force $= ma_c = m\omega^2 l/\sqrt{3}$

(b) Let F_x and F_y be the forces applied by the hinges along x-axis and y-axis, respectively. The system is in non-centroidal rotation. The three equations of motion are



$$\Sigma F_x = F_x + F = (3m)a_t = 3m\left(\frac{l}{\sqrt{3}}\alpha\right) \quad ...(i)$$

$$\Sigma F_y = F_y = 3m\left(\frac{l}{\sqrt{3}}\right)\omega^2 \quad ...(ii)$$

$$\Sigma\tau = F\times\left(\frac{\sqrt{3}}{2}l\right) = 2ml^2\alpha \quad ...(iii)$$

From Eq. (iii), $\alpha = \frac{\sqrt{3}F}{4ml}$

From Eq. (i), $F_x + F = 3m\left(\frac{l}{\sqrt{3}}\right)\times\frac{\sqrt{3}F}{4ml} = \frac{3F}{4}$

$\Rightarrow F_x = -\frac{F}{4}$

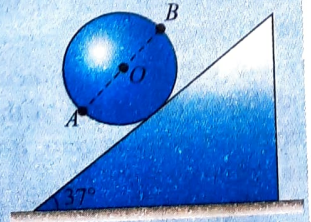
From Eq. (ii), $F_y = \sqrt{3}\,ml\omega^2$

EXAMPLE 3.6

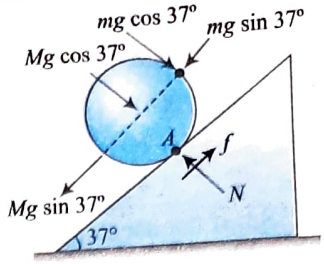
A uniform solid sphere of mass 1 kg and radius 10 cm is kept stationary on a rough inclined plane by fixing a highly dense particle at B. Inclination of plane is 37° with horizontal and AB is the diameter of the sphere which is parallel to the plane, as shown in figure. Calculate

(a) mass of the particle fixed at B

(b) minimum required coefficient of friction between sphere and plane to keep sphere in equilibrium.



Sol. Let us assume the mass of the particle to be m. Let us draw the free body diagram of the system of sphere and particle. Since system is in static equilibrium, torque of forces acting on the system should be zero. Taking torque about point A of all the forces acting on the system,



$$(Mg\sin 37°) + (mg\sin 37°)R = (mg\cos 37°)R$$

$\Rightarrow \quad m = 3$ kg

Considering forces normal the plane,

$$N = Mg\cos 37° + mg\cos 37° = 32 \text{ N}$$

The friction force between sphere and plane is static nature. Now considering forces along the plane,

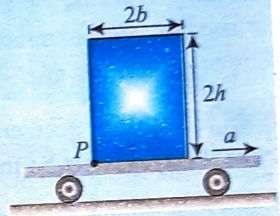
$$f = Mg\sin 37° + mg\sin 37° = 24 \text{ N}$$

But $f \le \mu N$ where μ is coefficient of friction which gives

$$\mu \ge \frac{f}{N} \text{ or } \mu_{min} = 0.75$$

EXAMPLE 3.7

A block of mass m height $2h$ and width $2b$ rests on a flat car which moves horizontally with constant acceleration 'a' as shown in figure. Determine:

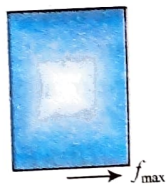


(a) the value of the acceleration at which slipping of the block on the car starts, if the coefficient of friction is μ.

(b) the value of the acceleration at which block topples about A, assuming sufficient friction to prevent slipping and

(c) the shortest distance in which it can be stopped from a speed of 20 m s^{-1} with constant deceleration so that the block is not disturbed. The following data are given $b = 0.6$ m, $h = 0.9$ m, $\mu = 0.5$ and $g = 10$ m s^{-2}

Sol.



(a) Maximum frictional force between block and car will be limiting force.

$\Rightarrow \quad f_{max} = \mu mg$

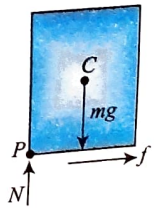
where m = mass of block

The maximum possible acceleration of block can be

$$a_{max} = \frac{f_{max}}{m} = \frac{\mu mg}{m} = \mu g = (0.5)(10) = 5 \text{ m s}^{-2}$$

If acceleration of car, exceeds this value slipping will start between the block and the car.

(b) In the case when the block is about to topple about P, normal reaction will pass through P. The friction will be static in nature.



$$f = ma \quad ...(i)$$

$$N = mg \quad ...(ii)$$

The block topples when torque of f about CM exceeds the torque of N.

In case of the limiting value

$$\tau_f = \tau_N$$

or $(f)(h) = (N)(b)$ or $(ma)(h) = (mg)(b)$

or $a = \left(\frac{b}{h}\right)g = \left(\frac{0.6}{0.9}\right)(10) = \frac{60}{9}\,\text{ms}^{-2}$

So, if the acceleration of car exceeds this value, the block will topple about A.

(c) The maximum acceleration/retardation at which the block is not disturbed, is the smaller of the two values, obtained above i.e., 5 m s^{-2}

Hence the maximum retardation can be 5 m s^{-2}

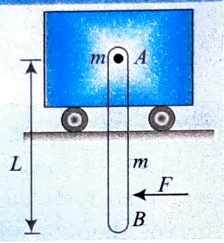
Now using $v^2 = u^2 - 2as$

Initial velocity $u = 0$

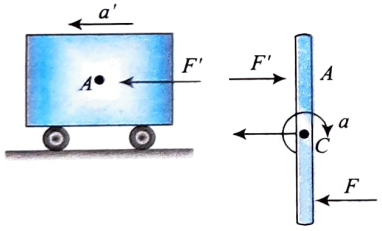
and $a = 5\,\text{m/s}^2$, $s = \frac{u^2}{2a} = \frac{(20)^2}{(2)(5)}$ or $s = 40\,\text{m}$

EXAMPLE 3.8

A uniform slender bar AB of mass m is suspended as shown from a small cart of the same mass m. Neglecting the effect of friction, determine the accelerations of points A and B immediately after a horizontal force F has been applied at B.



Sol. Let the acceleration of bar and cart be a and a', respectively; also α be the angular acceleration of the bar.



Writing Newton's equation of motion

For the bar: $F - F' = ma$ ...(i)

For the cart: $F' = ma'$ ...(ii)

Torque equation for rod about center of mass

For the rotation of the rod

$$(F + F')\frac{L}{2} = \frac{ML^2}{12}\alpha$$

$$F + F' = \frac{ML\alpha}{6} \quad \text{...(iii)}$$

The acceleration of point A on the rod will be same as the acceleration of the cart

$$\vec{a}_A = \vec{a}_{A,C} + \vec{a}_C$$

$$-a' = \alpha\frac{L}{2} - a \text{ or } a' = a - \frac{L}{2}\alpha \quad \text{... (iv)}$$

On solving Eqs. (i), (ii), (iii), and (iv), we get

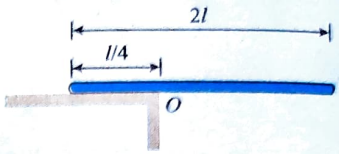
$$a = \frac{7F}{5m} \text{ and } \alpha = \frac{18F}{5mL}$$

$$|a'| = |a_A| = a - \frac{L}{2}\alpha = -\frac{2F}{5m}$$

$$|a_B| = a + \frac{L}{2}\alpha = \frac{16F}{5m}$$

EXAMPLE 3.9

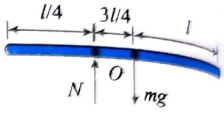
One fourth length of a uniform rod of length $2l$ and mass m is placed on a horizontal table and the rod is held horizontal. The rod is released from rest. Find the normal reaction on the rod as soon as the rod is released.



Sol. The forces acting on the rod are shown in FBD.

The torque applied by gravity about the edge O,

$$\tau = mg \times \frac{3}{4}l$$



$$\Rightarrow I_0\alpha = \frac{3}{4}mgl \quad \text{...(i)}$$

where $I_0 = I_g + m\left(\frac{3l}{4}\right)^2$

$$I_0 = \frac{ml^2}{12} + \frac{9}{16}ml^2 = \frac{ml^2}{4}\left(\frac{1}{3} + \frac{9}{4}\right) = \frac{31ml^2}{48} \quad \text{...(ii)}$$

Equations (i) and (ii) yield,

$$\alpha = \frac{3mgl/4}{31ml^2/48} = \frac{36g}{31l} \quad \text{...(iii)}$$

Newton's second law of motion:

$$mg - N = ma \quad \text{...(iv)}$$

But, $a = \left(\frac{3l}{4}\right)\alpha = \frac{36g}{31}$ ...(v)

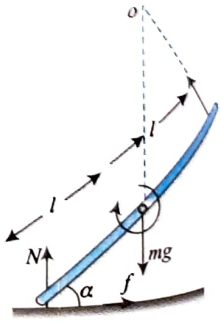
Using Eqs. (iv) and (v) we obtain

$$N = mg - m\left(\frac{27g}{31}\right) \Rightarrow N = \frac{4mg}{27}$$

EXAMPLE 3.10

Determine the minimum coefficient of friction μ_{min} between a thin homogeneous rod and a floor at which a person can slowly lift the rod from the floor without slippage to the vertical position, applying to it a force perpendicular to it.

Sol. We consider the equilibrium of the rod when it is inclined at an angle α with the horizontal. The forces acting on the rod are shown in figure.



We consider torque about the point of intersection of the force of gravity, mg; and the force F applied by the person perpendicular to the rod. Since the torque of these forces about this point is zero, it eliminates the unknown force F.

$$\sum \tau = Nl\cos\alpha - fl\left(\frac{1}{\sin\alpha} + \sin\alpha\right) = 0 \quad \text{...(i)}$$

Moment arm for N is $l\cos\alpha$, while for friction is,

$$\frac{l}{\sin\alpha} + l\sin\alpha$$

From Eq. (i), we get

$$f = N\frac{\cos\alpha\sin\alpha}{1+\sin^2\alpha} = N\frac{\cos\alpha\sin\alpha}{2\sin^2\alpha+\cos^2\alpha}$$

$$= N\frac{1}{2\tan\alpha\cot\alpha}$$

As the force of friction cannot exceed μN, we have

$$\mu \geq \frac{1}{2\tan\alpha+\cot\alpha}$$

μ is minimum when $2\tan\alpha+\cot\alpha$ is maximum,

$$\frac{d}{d\alpha}(2\tan\alpha+\cot\alpha) = 0 \Rightarrow 2\sec^2\alpha-\text{cosec}^2\alpha = 0$$

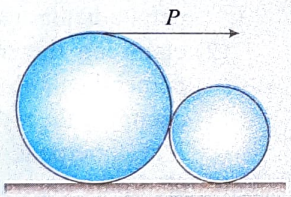
$$\tan\alpha = \frac{1}{\sqrt{2}}$$

Thus, the required coefficient of friction is

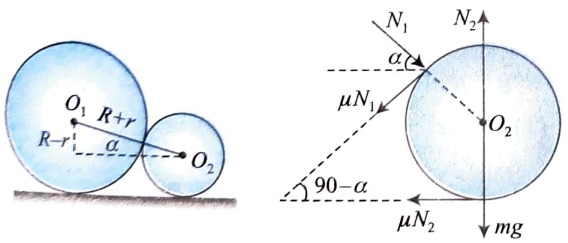
$$\mu_{min} = \frac{1}{2\sqrt{2}}$$

EXAMPLE 3.11

Consider two heavy right circular rollers of radii R and r, respectively, and rest on a rough horizontal plane as shown in figure. The larger roller has a string wound around it to which a horizontal force P can be applied as shown. Assume the coefficient of friction μ has the same value for all surfaces of contact, determine the necessary condition under which the larger roller can be pulled over the smaller one. Assume the smaller cylinder should neither roll nor slide.



Sol. Figure shows the forces acting on the smaller cylinder. N_1 is reaction between the two cylinders and N_2 the normal reaction between the smaller cylinder and the horizontal surface.



(a) (b)

The conditions of equilibrium are

$$\sum F_x = N_1\cos\alpha-\mu N_2-\mu N_1\cos(90-\alpha) = 0 \quad \ldots(i)$$

Taking torque about O_2, centre of smaller sphere,

$$\sum\tau = \mu N_1 r-\mu N_2 r = 0$$

$$N_1 = N_2 \quad \ldots(ii)$$

On substituting Eq. (ii) in Eq. (i), we obtain

$$\mu N_1+\mu N_1\sin\alpha = N_1\cos\alpha$$

$$\mu(1+\sin\alpha) = \cos\alpha \quad \ldots(iii)$$

From Fig. (b), $\sin\alpha = \sqrt{\frac{R-r}{R+r}}$, $\cos\alpha = \frac{2\sqrt{Rr}}{R+r}$

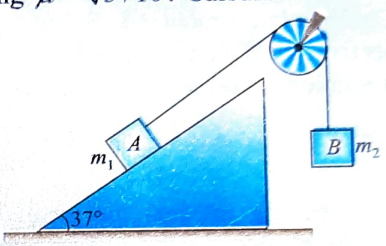
Now Eq. (iii) becomes,

$$\mu\left(1+\frac{R-r}{R+r}\right) = \frac{2\sqrt{Rr}}{R+r} = \mu = \sqrt{\frac{r}{R}}$$

Hence the required condition is $\mu \geq \sqrt{\frac{r}{R}}$

EXAMPLE 3.12

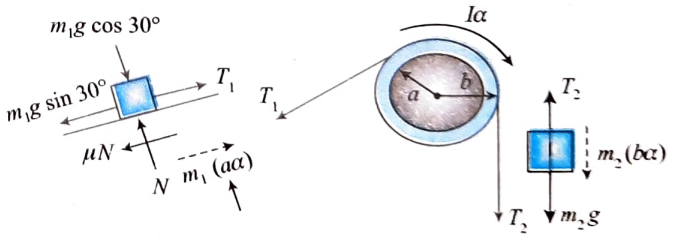
In the system shown in figure blocks A and B have mass m_1 = 2 kg and m_2 = 26/7 kg respectively. Pulley having moment of inertia $I = 0.11\ \text{kg m}^{-2}$ can rotate without friction about a fixed axis. Inner and outer radii of pulley are a = 10 cm and b = 15 cm respectively. B is hanging with the thread wrapped around the pulley, while A lies on a rough inclined plane. Coefficient of friction being $\mu = \sqrt{3}/10$. Calculate:



(a) tension in each thread, and
(b) acceleration of each block. ($g = 10\,\text{ms}^{-2}$).

Sol. When the system is released, weight m_2g tries to rotate the pulley in clockwise direction while down the plane component $m_1g\sin 30°$ of weight of block A tries to rotate it anticlockwise. But moment produced by m_2g is greater; therefore, the pulley has tendency to rotate clockwise. Let its angular acceleration be 'α'. Then acceleration of blocks A and B will be equal to $a\alpha$ (up the plane) and $b\alpha$ (vertically downwards) respectively.

Let tension in threads connected with blocks A and B be T_1 and T_2, respectively. Considering free body diagrams,



For forces on block A,

$$N = m_1g\cos 30° \quad \ldots(i)$$

$$T_1-m_1g\sin 30°-\mu N = m_1(a\alpha) \quad \ldots(ii)$$

For forces on block B, $m_2g-T_2 = m_2(b\alpha)$ $\ldots(iii)$

Taking moments of forces acting on pulley, about axis of rotation,

$$T_2b-T_1a = I\alpha$$

From above equations, $N = 10\sqrt{3}\ \text{N}$

$$T_1 = 17\ \text{N},\ T_2 = 26\ \text{N} \text{ and } \alpha = 20\ \text{rad s}^{-2}$$

Acceleration of block, $A = a\alpha = 2\ \text{m s}^{-2}$ (up the plane)

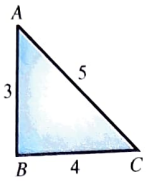
Acceleration of block, $B = b\alpha = 3\ \text{m s}^{-2}$ (downward)

Exercises

Single Correct Answer Type

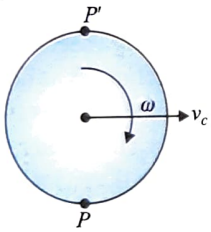
1. The wheels of an airplane are set into rotation just before landing so that the wheels do not slip on the ground. If the airplane is travelling in the east direction, what should be the direction of angular velocity vector of the wheels?
(1) East (2) West
(3) South (4) North

2. ABC is a triangular plate of uniform thickness. The sides are in the ratio shown in the figure. I_{AB}, I_{BC}, I_{CA} are the moments of inertia of the plated about AB, BC and CA respectively. Which one of the following relation is correct?



(1) I_{CA} is maximum (2) $I_{AB} > I_{BC}$
(3) $I_{BC} > I_{AB}$ (4) $I_{AB} + I_{BC} = I_{CA}$

3. A sphere is moving towards the positive x-axis with a velocity v_c and rotates clockwise with angular speed ω as shown in figure, such that $v_c > \omega R$. The instantaneous axis of rotation will be

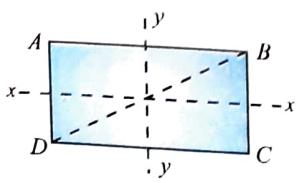


(1) on point P (2) on point P'
(3) inside the sphere (4) outside the sphere

4. Let I_A and I_B be moments of inertia of a body about two axes A and B respectively. The axis A passes through the centre of mass of the body but B does not.
(1) $I_A < I_B$
(2) If $I_A < I_B$ the axes are parallel
(3) If the axes are parallel, $I_A < I_B$
(4) If the axes are not parallel, then $I_A \geq I_B$.

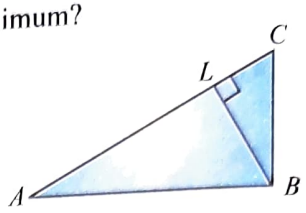
5. For the same total mass, which of the following will have the largest moment of inertia about an axis passing through the centre of gravity and perpendicular to the plane of body?
(1) A ring of radius l
(2) A disc of radius l
(3) A square lamina of side $2l$
(4) Four rods forming square of side $2l$

6. In rectangle $ABCD$, $AB = 2l$ and $BC = l$. Axes xx and yy pass through the centre of the rectangle. The moment of inertia is least about



(1) DB (2) BC
(3) xx (4) yy

7. About which axis moment of inertia in the given triangular lamina is maximum?



(1) AB (2) BC
(3) AC (4) BL

8. Three point masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of side a. What is the moment of inertia of the system about an axis along the altitude of the triangle passing through m_1?
(1) $(m_1 + m_2)\frac{a^2}{4}$ (2) $(m_2 + m_3)\frac{a^2}{4}$
(3) $(m_1 + m_3)\frac{a^2}{4}$ (4) $(m_1 + m_2 + m_3)\frac{a^2}{4}$

9. The density of a rod continuously increases from A to B. It is easier to set it into rotation by
(1) clamping the rod at A and applying a force F at B perpendicular to the rod
(2) clamping the rod at B and applying a force F at A perpendicular to the rod
(3) clamping the rod at mid point of AB and applying a force F at A, perpendicular to the rod
(4) clamping the rod at mid-point of AB and applying force F at B, perpendicular to the rod

10. Two rings of same radius and mass are placed such that their centres are at a common point and their planes are perpendicular to each other. The moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings is (mass of the ring = m, radius = r)
(1) $\frac{1}{2}mr^2$ (2) mr^2
(3) $\frac{3}{2}mr^2$ (4) $2mr^2$

11. The moment of inertia of a solid sphere about an axis passing through the centre of gravity is $2/5\ MR^2$; then its radius of gyration about a parallel axis at a distance $2R$ from first axis is
(1) $5R$ (2) $\sqrt{\frac{22}{5}}R$
(3) $\frac{5}{2}R$ (4) $\sqrt{\frac{12}{5}}R$

12. From a given sample of uniform wire, two circular loops P and Q are made, P of radius r and Q of radius nr. If the M.I. of Q about its axis is four times that of P about its axis (assuming the wire to be of diameter much smaller than either radius), the value of n is
(1) $(4)^{\frac{2}{3}}$ (2) $(4)^{\frac{1}{3}}$
(3) $(4)^{\frac{1}{2}}$ (4) $(4)^{\frac{1}{4}}$

13. Two circular discs A and B are of equal masses and thicknesses but made of metal with densities d_A and d_B ($d_A > d_B$). If their moments of inertia about an axis passing through their centres and perpendicular to circular faces are I_A and I_B, then

(1) $I_A = I_B$ (2) $I_A > I_B$

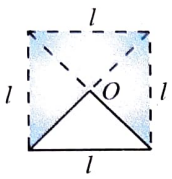
(3) $I_A < I_B$ (4) $I_A \geq I_B$

14. Four identical rods are joined end to end to form a square. The mass of each rod is M. The moment of inertia of the square about the median line is

(1) $\frac{Ml^2}{3}$ (2) $\frac{Ml^2}{4}$

(3) $\frac{Ml^2}{6}$ (4) none of these

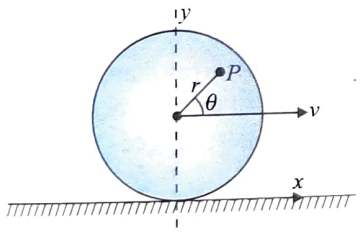
15. An isosceles triangular piece is cut from a square plate of side l. The piece is one-fourth of the square and mass of the remaining plate is M. The moment of inertia of the plate about an axis passing through O and perpendicular to its plane is



(1) $\frac{Ml^2}{6}$ (2) $\frac{Ml^2}{12}$

(3) $\frac{Ml^2}{24}$ (4) $\frac{Ml^2}{3}$

16. A disc of radius R rolls without slipping at speed v along positive x-axis. Velocity of point P at the instant shown in figure is



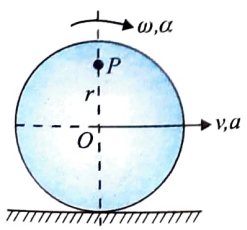
(1) $\vec{v}_p = \left(v + \frac{vr\sin\theta}{R}\right)\hat{i} + \frac{vr\cos\theta}{R}\hat{j}$

(2) $\vec{v}_p = \left(v + \frac{vr\sin\theta}{R}\right)\hat{i} - \frac{vr\cos\theta}{R}\hat{j}$

(3) $\vec{v}_p = \frac{vr\sin\theta}{R}\hat{i} + \frac{vr\cos\theta}{R}\hat{j}$

(4) $\vec{v}_p = \frac{vr\sin\theta}{R}\hat{i} - \frac{vr\cos\theta}{R}\hat{j}$

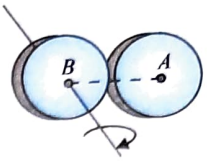
17. A disc of radius R rolls on a horizontal ground with linear acceleration a and angular acceleration α as shown in figure. The magnitude of acceleration of point P as shown in the figure at an instant when its linear velocity is v and angular velocity is ω will be



(1) $\sqrt{(a + r\alpha)^2 + (r\omega^2)^2}$ (2) $\frac{ar}{R}$

(3) $\sqrt{r^2\alpha^2 + r^2\omega^4}$ (4) $r\alpha$

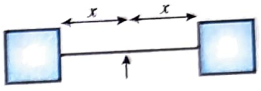
18. Two thin discs, each of mass M and radius r metre, are attached as shown in figure, to form a rigid body. The rotational inertia of this body about an axis perpendicular to the plane of disc B passing through its centre is



(1) $2Mr^2$ (2) $3Mr^2$

(3) $4Mr^2$ (4) $5Mr^2$

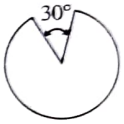
19. Two identical masses are connected to a horizontal thin massless rod as shown in the figure. When their distance from the pivot is x, a torque produces an angular acceleration α_1. If the masses are now repositioned so that they are at distance $2x$ each from the pivot, the same torque will produce an angular acceleration α_2 such that



(1) $\alpha_2 = 4\alpha_1$ (2) $\alpha_2 = \alpha_1$

(3) $\alpha_2 = \frac{\alpha_1}{2}$ (4) $\alpha_2 = \frac{\alpha_1}{4}$

20. From a complete ring of mass M and radius R, a 30° sector is removed. The moment of inertia of the incomplete ring about an axis passing through the centre of the ring and perpendicular to the plane of the ring is



(1) $\frac{9}{12}MR^2$ (2) $\frac{11}{12}MR^2$

(3) $\frac{11.3}{12}MR^2$ (4) MR^2

21. A uniform disc of mass M and radius R is mounted on an axle supported in frictionless bearings. A light cord is wrapped around the rim of the disc and a steady downward pull T is exerted on the cord. The angular acceleration of the disc is

(1) $\frac{T}{MR}$ (2) $\frac{MR}{T}$

(3) $\frac{2T}{MR}$ (4) $\frac{MR}{2T}$

22. In problem 21, the tangential acceleration of a point on the rim is

(1) $\frac{T}{M}$ (2) $\frac{MR}{T}$

(3) $\frac{2T}{M}$ (4) $\frac{MR}{2T}$

23. If we hang a body of mass m with the cord in problem 21, the tangential acceleration of the disc will be

(1) $\frac{mg}{M+m}$ (2) $\frac{mg}{M+2m}$

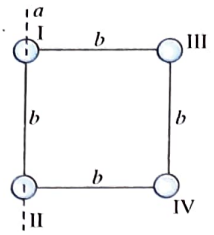
(3) $\frac{2mg}{M+2m}$ (4) $\frac{M+2m}{2mg}$

24. The tension in the cord in problem 21 is

(1) $\frac{mg}{M+m}$ (2) $\frac{Mmg}{M+2m}$

(3) $\frac{2mg}{Mmg}$ (4) none of these

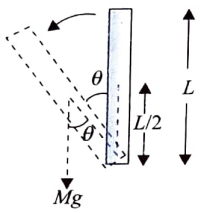
25. Four spheres, each of mass m and radius r, are placed with their centres on the four corners of a square of side b. The moment of inertia of the system about any side of the square is



(1) $\frac{8}{5}mr^2 + mb^2$ (2) $\frac{8}{5}mr^2 + 2mb^2$

(3) $\frac{8}{5}mr^2 + 4mb^2$ (4) none of these

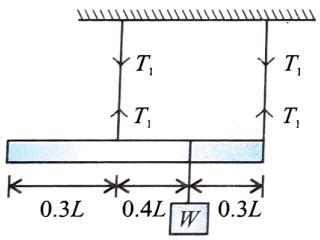
26. A uniform rod of mass M and length L is pivoted at one end such that it can rotate in a vertical plane. There is negligible friction at the pivot. The free end of the rod is held vertically above the pivot and then released. The angular acceleration of the rod when it makes an angle θ with the vertical is



(1) $g \sin\theta$ (2) $\frac{g}{L}\sin\theta$

(3) $\frac{3g}{2L}\sin\theta$ (4) $6gL\sin\theta$

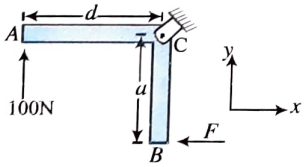
27. In figure, the bar is uniform and weighing 500 N. How large must W be if T_1 and T_2 are to be equal?



(1) 500 N (2) 300 N

(3) 750 N (4) 1500 N

28. Find force F required to keep the system in equilibrium. The dimensions of the system are $d = 0.3$ m and $a = 0.2$ m. Assume the rods to be massless.

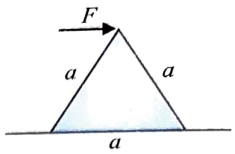


(1) 150 $(\hat{i})$ (2) 150 $(-\hat{k})$

(3) 150 $(-\hat{i})$

(4) It cannot be in equilibrium

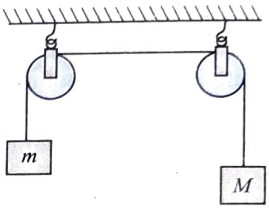
29. An equilateral prism of mass m rests on a rough horizontal surface with coefficient of friction μ. A horizontal force F is applied on the prism as shown in figure. If the coefficient of friction is sufficiently high so that the prism does not slide before toppling, the minimum force required to topple the prism is



(1) $\frac{mg}{\sqrt{3}}$ (2) $\frac{mg}{4}$

(3) $\frac{\mu mg}{\sqrt{3}}$ (4) $\frac{\mu mg}{4}$

30. Each pulley in figure has radius r and moment of inertia I. The acceleration of the block is



(1) $\frac{(M-m)g}{\left(M+m+\frac{2I}{r^2}\right)}$ (2) $\frac{(M-m)g}{\left(M+m-\frac{2I}{r^2}\right)}$

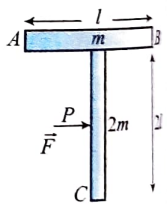
(3) $\frac{(M-m)g}{\left(M+m+\frac{I}{r^2}\right)}$ (4) $\frac{(M-m)g}{\left(M+m-\frac{I}{r^2}\right)}$

31. The line of action of the resultant of two like parallel forces shifts by one-fourth of the distance between the forces when the two forces are interchanged. The ratio of the two forces is

(1) 1 : 2 (2) 2 : 3

(3) 3 : 4 (4) 3 : 5

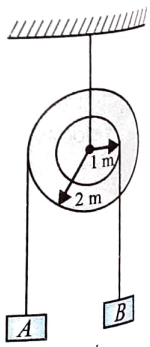
32. A T shaped object with dimensions shown in the figure is lying on a smooth floor. A force is applied at the point P parallel to AB such that the object has only the translational motion without rotation. Find the location of P with respect to C.



(1) (3/4)l (2) l

(3) (4/3)l (4) (3/2)l

33. In the pulley system shown, if radii of the bigger and smaller pulley are 2 m and 1 m, respectively, and the acceleration of block A is 5 ms^{-2} in the downward direction, the acceleration of block B will be



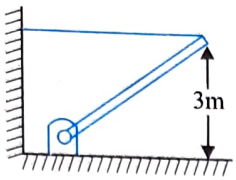
(1) 0 ms^{-2}

(2) 5 ms^{-2}

(3) 10 ms^{-2}

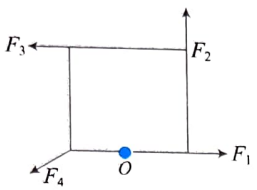
(4) $\frac{5}{2}$ ms^{-2}

34. A uniform rod of mass 15 kg is held stationary with the help of a light string as shown in figure. The tension in the string is



(1) 150 N
(2) 225 N
(3) 100 N
(4) none of the above

35. Four forces of the same magnitude act on a square as shown in figure. The square can rotate about point O; mid point of one of the edges. The force which can produce greatest torque is

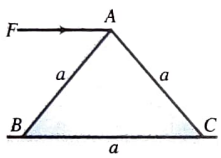


(1) F_1 (2) F_2
(3) F_3 (4) F_4

36. Given a uniform disc of mass M and radius R. A small disc of radius $R/2$ is cut from this disc in such a way that the distance between the centres of the two discs is $R/2$. Find the moment of inertia of the remaining disc about a diameter of the original disc perpendicular to the line connecting the centres of the two discs

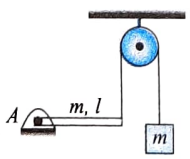
(1) $3MR^2/32$ (2) $5MR^2/16$
(3) $11MR^2/64$ (4) none of these

37. A horizontal force F is applied at the top of an equilateral triangular block having mass m. The minimum coefficient of friction required to topple the block before translation will be



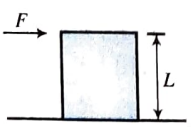
(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{1}{3}$
(3) $\frac{1}{\sqrt{3}}$ (4) $\frac{1}{2}$

38. Uniform rod AB is hinged at end A in horizontal position as shown in the figure. The other end is connected to a block through a massless string as shown. The pulley is smooth and massless. Masses of block and rod is same and is equal to m. Then acceleration of block just after release from this position is



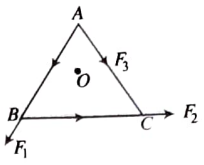
(1) 6g/13 (2) g/4
(3) 3g/8 (4) None

39. A cubical block of side L rests on a rough horizontal surface with coefficient of friction μ. A horizontal force F is applied on the block as shown in the figure. If the coefficient of friction is sufficiently high so that the block does not slide before toppling, the minimum force required to topple the block is



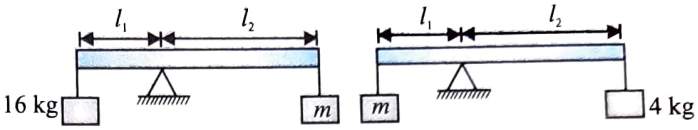
(1) infinitesimal (2) $\frac{mg}{4}$
(3) $\frac{mg}{2}$ (4) $mg(1-\mu)$

40. O is the centre of an equilateral triangle ABC. F_1, F_2 and F_3 are the three forces acting along the sides AB, BC and AC respectively. What should be the value of F_3 so that the total torque about O is zero?



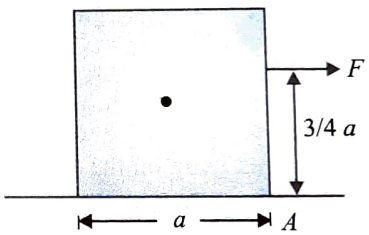
(1) $2(F_1+F_2)$ (2) $\frac{F_1+F_2}{2}$
(3) F_1-F_2 (4) F_1+F_2

41. In an experiment with a beam balance, an unknown mass m is balanced by two known masses of 16 kg and 4 kg as shown in figure.



The value of the unknown mass m is
(1) 10 kg (2) 6 kg
(3) 8 kg (4) 12 kg

42. A cube of side a and mass m is to be tilted at point A by applying a force F as shown in figure. The minimum force required is

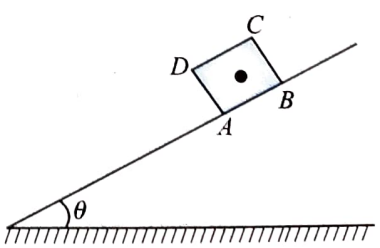


(1) mg (2) $\frac{2}{3}mg$
(3) $\frac{3}{2}mg$ (4) $\frac{3}{4}mg$

43. A ladder of length l and mass m is placed against a smooth vertical wall, but the ground is not smooth. Coefficient of friction between the ground and the ladder is μ. The angle θ at which the ladder will stay in equilibrium is

(1) $\theta = \tan^{-1}(\mu)$ (2) $\theta = \tan^{-1}(2\mu)$
(3) $\theta = \tan^{-1}\left(\frac{\mu}{2}\right)$ (4) none of these

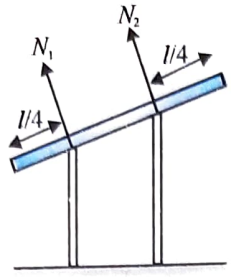
44. A cube of side a is placed on an inclined plane of inclination θ. What is the maximum value of θ for which the cube will not topple?



(1) 15° (2) 30°
(3) 45° (4) 60°

45. A uniform rod of length l is placed symmetrically on two walls as shown in figure. The rod is in equilibrium. If N_1

and N_2 are the normal forces exerted by the walls on the rod, then

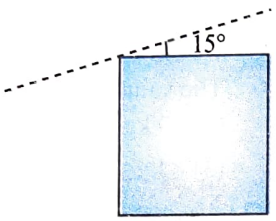


(1) $N_1 > N_2$ (2) $N_1 < N_2$

(3) $N_1 = N_2$

(4) N_1 and N_2 would be in the vertical directions

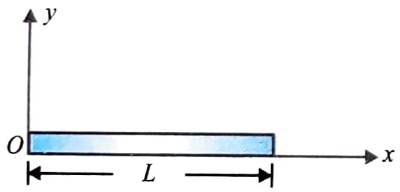
46. A square plate of mass M and edge L is shown in the figure. The moment of inertia of the plate about the axis in the plane of plate and passing through one of its vertex making an angle 15° horizontal is



(1) $\frac{ML^2}{12}$ (2) $\frac{11ML^2}{24}$

(3) $\frac{7ML^2}{12}$ (4) none of these

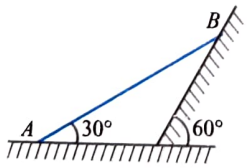
47. The figure shows a uniform rod lying along the x-axis. The locus of all the points lying on the x–y plane, about which the moment of inertia of the rod is same as that about O, is



(1) an ellipse (2) a circle

(3) a parabola (4) a straight line

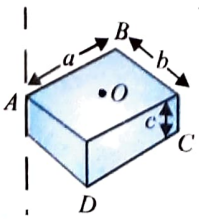
48. In the figure shown, the instantaneous speed of end A of the rod is v to the left. The angular velocity of the rod of length L must be



(1) $\frac{v}{2L}$ (2) $\frac{v}{L}$

(3) $\frac{v\sqrt{3}}{2L}$ (4) none of these

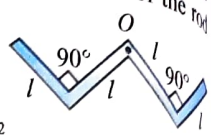
49. Figure shows a uniform solid block of mass M and edge lengths a, b and c. Its MI about an axis through one edge and perpendicular (as shown) to the large face of the block is



(1) $\frac{M}{3}(a^2 + b^2)$ (2) $\frac{M}{4}(a^2 + b^2)$

(3) $\frac{7M}{12}(a^2 + b^2)$ (4) $\frac{M}{12}(a^2 + b^2)$

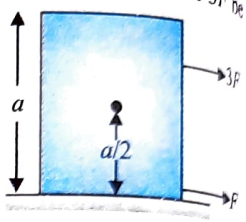
50. A thin rod of length $4l$ and mass $4m$ is bent at the points as shown in figure. What is the moment of inertia of the rod about the axis passing through point O and perpendicular to the plane of the paper?



(1) $\frac{Ml^2}{3}$ (2) $\frac{10Ml^2}{3}$

(3) $\frac{Ml^2}{12}$ (4) $\frac{Ml^2}{24}$

51. A rectangular block of mass M and height a is resting on a smooth level surface. A force F is applied to one corner as shown in figure. At what point should a parallel force $3F$ be applied in order that the block shall undergo pure translational motion? Assume normal contact force between the block and surface passes through the centre of gravity of the block.



(1) $\frac{a}{3}$ vertically above centre of gravity

(2) $\frac{a}{6}$ vertically above centre of gravity

(3) No such point exists

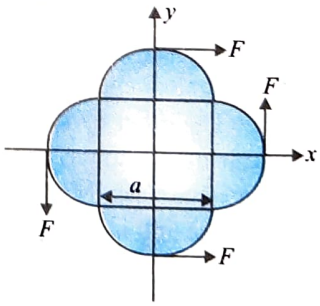
(4) it is not possible

52. If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass, and I_2 is the moment of inertia (about central axis) of the ring formed by bending the rod, then

(1) $I_1 : I_2 = 1 : 1$ (2) $I_1 : I_2 = \pi^2 : 3$

(3) $I_1 : I_2 = \pi : 4$ (4) $I_1 : I_2 = 3 : 5$

53. A planar object made up of a uniform square plate and four semicircular discs of the same thickness and material is being acted upon by four forces of equal magnitude as shown in figure. The coordinates of point of application of forces is given by



(1) $(0, a)$ (2) $(0, -a)$

(3) $(a, 0)$ (4) $(-a, 0)$

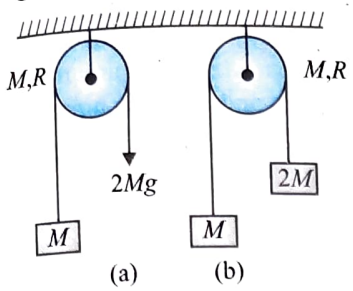
54. A uniform disc of radius R lies in the x-y plane, with its centre at origin. Its moment of inertia about z-axis is equal to its moment of inertia about line $y = x + c$. The value of c will be

(1) $-\frac{R}{2}$ (2) $\pm\frac{R}{\sqrt{2}}$

(3) $+\frac{R}{4}$ (4) $-R$

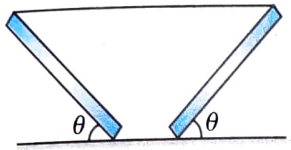
55. A cord is wrapped on a pulley (disk) of mass M and radius R as shown in figure. To one end of the cord, a block of mass

M is connected as shown and to other end in (a) a force of $2Mg$ and in (b) a block of mass $2M$. Let angular acceleration of the disk in A and B is α_A and α_B, respectively, then (cord is not slipping on the pulley).



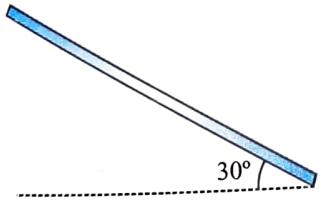
(1) $\alpha_A = \alpha_B$
(2) $\alpha_A > \alpha_B$
(3) $\alpha_A < \alpha_B$
(4) None of these

56. Two uniform boards, tied together with the help of a string, are balanced on a surface as shown in figure.



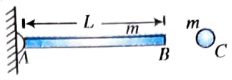
The coefficient of static friction between boards and surface is 0.5. The minimum value of θ, for which this type of arrangement is possible is
(1) 30°
(2) 45°
(3) 37°
(4) it is not possible to have this type of balanced arrangement

57. A slender rod of mass m and length L is pivoted about a horizontal axis through one end and released from rest at an angle of 30° above the horizontal. The force exerted by the pivot on the rod at the instant when the rod passes through a horizontal position is



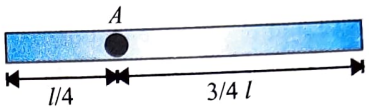
(1) $\sqrt{\frac{10}{4}}\,mg$ along horizontal
(2) mg along vertical
(3) $\frac{\sqrt{10}}{4}\,mg$ along a line making an angle of $\tan^{-1}\left(\frac{1}{3}\right)$ with the horizontal
(4) $\frac{\sqrt{10}}{4}\,mg$ along a line making an angle of $\tan^{-1}(3)$ with the horizontal

58. A uniform bar AB of mass m and a ball of the same mass are released from rest from the same horizontal position. The bar is hinged at end A. There is gravity downwards. What is the distance of the point from point B that has the same acceleration as the ball, immediately after release?



(1) $\frac{2L}{3}$
(2) $\frac{L}{3}$
(3) $\frac{L}{2}$
(4) $\frac{3L}{4}$

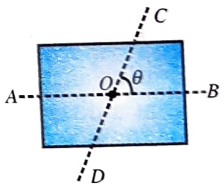
59. A uniform rod of mass m and length l is fixed from point A, which is at a distance $l/4$ from one end as shown in the figure. The rod is free to rotate in a vertical plane. The rod is released from the horizontal position.



What is the reaction at the hinge, when kinetic energy of the rod is maximum?
(1) $\frac{4}{7}\,mg$
(2) $\frac{5}{7}\,mg$
(3) $\frac{13}{7}\,mg$
(4) $\frac{11}{7}\,mg$

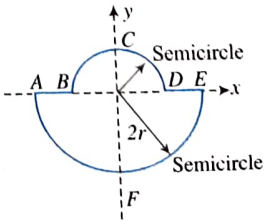
60. A rod of length L is held vertically on a smooth horizontal surface. The top end of the rod is given a gentle push. At a certain instant of time, when the rod makes an angle 37° with horizontal the velocity of COM of the rod is 2 ms^{-1}. The velocity of the end of the rod in contact with the surface at that instant is
(1) 2 $m\,s^{-1}$
(2) 1 ms^{-1}
(3) 4 $m\,s^{-1}$
(4) 1.5 $m\,s^{-1}$

61. Let I be the moment of inertia of a uniform square plate about an axis AB that passes through its centre and is parallel to two of its sides. CD is a line in the plane of the plate and it passes through the centre of the plate, making an angle θ with AB. The moment of inertia of the plate about the axis CD is equal to



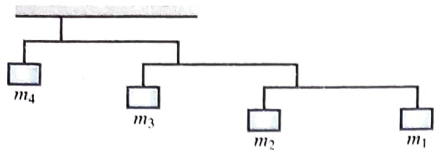
(1) I
(2) $I\sin^2\theta$
(3) $I\cos^2\theta$
(4) $I\cos^2\left(\frac{\theta}{2}\right)$

62. A uniform thin rod is bent in the form of closed loop $ABCDEFA$ as shown in the figure. The ratio of moment of inertia of the loop about x-axis to that about y-axis is



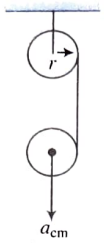
(1) > 1
(2) < 1
(3) = 1
(4) = 1/2

63. Figure shows an arrangement of masses hanging from a ceiling. In equilibrium, each rod is horizontal, has negligible mass and extends three times as far to the right of the wire supporting it as to the left. If mass m_4 is 48 kg then mass m_1 is equal to



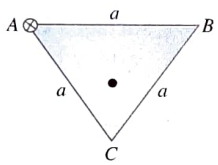
(1) 1 kg (2) 2 kg
(3) 3 kg (4) 4 kg

64. Two identical uniform discs of mass m and radius r are arranged as shown in the figure. If α is the angular acceleration of the lower disc and a_{cm} is acceleration of centre of mass of the lower disc, then relation among a_{cm}, α and r is



(1) $a_{cm} = \dfrac{\alpha}{r}$ (2) $a_{cm} = 2\,\alpha r$
(3) $a_{cm} = \alpha r$ (4) none of these

65. A uniform triangular plate ABC of mass m and moment of inertia I (about an axis passing through A and perpendicular to plane of the plate) can rotate freely in the vertical plane about point 'A' as shown in figure. The plate is released from the position shown in the figure. Line AB is horizontal. The acceleration of centre of mass just after the release of plate is

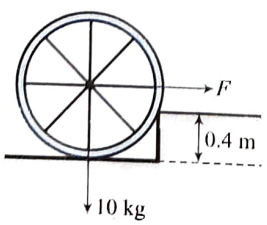


(1) $\dfrac{mga^2}{\sqrt{3}\,I}$ (2) $\dfrac{mga^2}{4I}$
(3) $\dfrac{mga^2}{2\sqrt{3}\,I}$ (4) $\dfrac{mga^2}{3I}$

66. Three children are sitting on a see-saw in such a way that it is balanced. A 20 kg and a 30 kg boy are on opposite sides at a distance of 2m from the pivot. It the third boy jumps off, thereby destroying balance, then the initial angular acceleration of the board is (Neglect weight of board)
(1) 0.01 rad s^{-2} (2) 1.0 rad s^{-2}
(3) 10 rad s^{-2} (4) 100 rad s^{-2}

67. A wheel of radius R has an axle of radius $R/5$. A force F is applied tangentially to the wheel. To keep the system in a state of "rotational" rest, a force F' is applied tangentially to the axle. The value of F' is
(1) F (2) $3F$
(3) $5F$ (4) $7F$

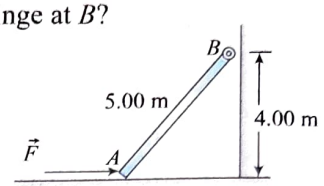
68. Calculate the force F that is applied horizontally at the axle of the wheel which is necessary to raise the wheel over the obstacle of height 0.4 m. Radius of wheel is 1m and mass = 10 kg . F is



(1) 100 N (2) 66 N
(3) 167 N (4) 133.3 N

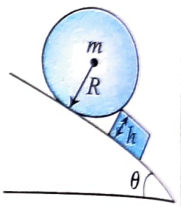
69. A string is wrapped around a cylinder of mass M and radius R. The string is pulled vertically upwards to prevent the centre of mass from falling as the cylinder upwinds the string. The tension in the string is
(1) $2Mg/3$ (2) $Mg/2$
(3) $Mg/3$ (4) $Mg/6$

70. End A of the bar AB in figure rests on a frictionless horizontal surface and end B is hinged. A horizontal force $\vec{F}$ of magnitude 120 N is exerted on end A. You can ignore the weight of the bar. What is the net force exerted by the bar on the hinge at B?



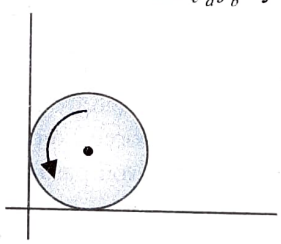
(1) 200 N (2) 140 N
(3) 100 N (4) None of these

71. Find the minimum height of the obstacle so that the sphere can stay in equilibrium.

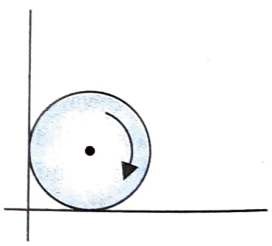


(1) $\dfrac{R}{1+\cos\theta}$
(2) $\dfrac{R}{1+\sin\theta}$
(3) $R(1-\sin\theta)$
(4) $R(1-\cos\theta)$

72. A sphere is placed rotating with its centre initially at rest in a corner as shown in Figs. (a) and (b). Coefficient of friction between all surfaces and the sphere is 1/3. Find the ratio of the friction forces f_a/f_b by ground in situations (a) and (b).



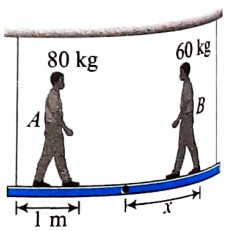
(a)



(b)

(1) 1 (2) $\dfrac{9}{10}$
(3) $\dfrac{10}{9}$ (4) None of these

73. Two painters are working from a wooden board 5 m long suspended from the top of a building by two ropes attached to the ends of the plank. Either rope can withstand a maximum tension of 1040 N. Painter A of mass 80 kg is working at a distance of 1 m from one end. Painter B of mass 60 kg is working at a distance of x m from the centre of mass of the board on the other side. Take mass of the board as 20 kg and $g = 10\ m\,s^{-2}$. The range of x so that both the painters can work safely is



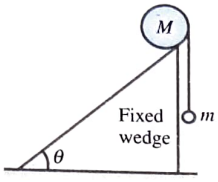
(1) $\dfrac{1}{3} < x < \dfrac{11}{6}$ (2) $0 < x < \dfrac{11}{6}$
(3) $0 < x < 2$ (4) $\dfrac{1}{3} < x < 2$

74. In figure, a massive rod AB is held in horizontal position by two massless strings. If the string at B breaks and if the horizontal acceleration of centre of mass, vertical acceleration and angular acceleration of rod about the centre of mass are a_x, a_y and α, respectively, then

(1) $2\sqrt{3}a_y = \sqrt{3}\alpha l + 2a_x$ (2) $\sqrt{3}a_y = \sqrt{3}\alpha l + a_x$

(3) $a_y = \sqrt{3}\alpha l + 2a_x$ (4) $2a_y = \alpha l + 2\sqrt{3}a_x$

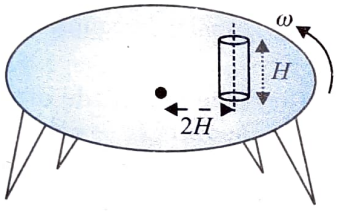
75. A uniform cylinder of mass M lies on a fixed plane inclined at an angle θ with horizontal. A light string is tied to the cylinder at the right most point, and a mass m hangs from the string, as shown. Assume that the coefficient of friction between the cylinder and the incline plane is sufficiently large to prevent slipping. For the cylinder the remain static, the value of m is



(1) $\frac{M\sin\theta}{1-\sin\theta}$ (2) $\frac{M\cos\theta}{1+\sin\theta}$

(3) $\frac{M\sin\theta}{1+\sin\theta}$ (4) $\frac{M\cos\theta}{1-\sin\theta}$

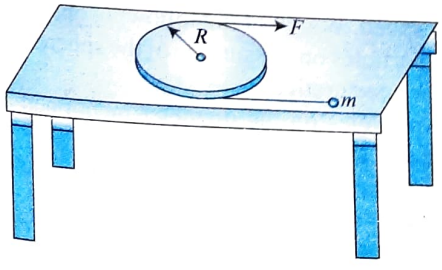
76. A cylinder of height H and diameter $H/4$ is kept on a frictional turntable as shown in figure. The axis of the cylinder is perpendicular to the surface of the table and the distance of axis of the cylinder is $2H$ from the centre of the table. The angular speed of the turntable at which the cylinder will start toppling (assume that friction is sufficient to prevent slipping) is



(1) $\sqrt{\frac{g}{2}\left(\frac{1}{2}-H\right)}$ (2) $\sqrt{g\left(\frac{1}{2}-H\right)}$

(3) $\sqrt{\frac{g}{4H}}$ (4) $\sqrt{\frac{g}{8H}}$

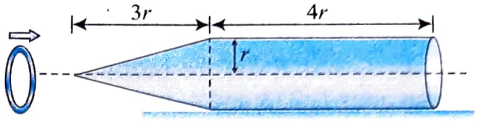
77. A disc of mass m and radius R placed on a smooth horizontal table as shown in figure. A light ideal string passes through the disc such that the one end of the string is attached to a small body of mass m and the other end is being pulled with a force F. The circumference of the disc is sufficiently rough so that the string does not slip over it. Find acceleration of the small body.



(1) $\frac{F}{4m}$ (2) $\frac{F}{m}$

(3) $\frac{F}{2m}$ (4) $\frac{F}{3m}$

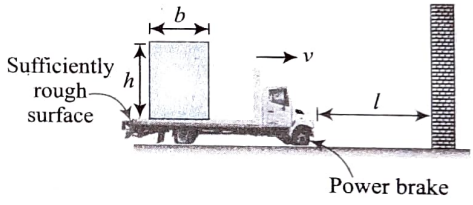
78. A solid uniform cylindrical rod of radius r has a conical nose is placed on smooth horizontal surface as shown in figure. The length of the cylindrical and conical nose are $4r$ and $3r$ respectively. Mass of the conical part is M. A ring of radius $r/2$ is to be tightly fitted on the nose of the body. The maximum permissible mass of the ring so that the body does not topple is



(1) $\frac{21}{5}M$ (2) $\frac{29}{6}M$

(3) $\frac{11}{3}M$ (4) $\frac{16}{3}M$

79. Find minimum value of l so that truck can avoid the dead end, without toppling the block kept on it.



(1) $\frac{2hv^2}{bg}$ (2) $\frac{bv^2}{2hg}$

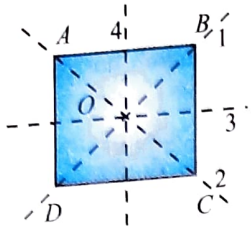
(3) $\frac{2bv^2}{hg}$ (4) $\frac{hv^2}{2bg}$

Multiple Correct Answers Type

1. The radius of gyration of a body depends upon
   (1) mass of the body
   (2) nature of distribution of mass
   (3) axis of rotation
   (4) none of these
2. A rigid body is in pure rotation, that is, undergoing fixed axis rotation. Then which of the following statement(s) are true?
   (1) You can find two points in the body in a plane perpendicular to the axis of rotation having the same velocity.
   (2) You can find two points in the body in a plane perpendicular to the axis of rotation having the same acceleration.
   (3) Speed of all the particles lying on the curved surface of a cylinder whose axis coincides with the axis of rotation is the same.

(4) Angular speed of the body is the same as seen from any point in the body.

3. The moments of inertia of a thin square plate $ABCD$ of uniform thickness about an axis passing through the centre O and perpendicular to the plate are



(1) $I_1 + I_2$ (2) $I_3 + I_4$

(3) $I_1 + I_3$ (4) $I_1 + I_2 + I_3 + I_4$

(where I_1, I_2, I_3 and I_4 are, respectively, the moments of inertia about axes 1, 2, 3 and 4, where axes are in the plane of the plate)

4. A bucket of water of mass 21 kg is suspended by a rope wrapped around a solid cylinder 0.2 m in diameter. The mass of the solid cylinder is 21 kg. The bucket is released from rest. Which of the following statements are correct?

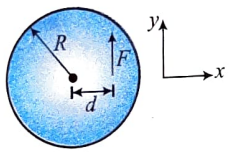
(1) The tension in the rope is 70 N

(2) The acceleration of the bucket is (20/3) m/s^2

(3) The acceleration of the bucket is independent of the mass of the bucket

(4) None of these

5. A uniform thin flat isolated disc is floating in space. It has radius R and mass m. A force F is applied to it at a distance $d = (R/2)$ from the centre in the y-direction. Treat this problem as two-dimensional. Just after the force is applied



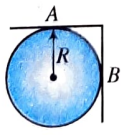
(1) acceleration of the centre of the disc is F/m

(2) angular acceleration of the disk is F/mR

(3) acceleration of leftmost point on the disc is zero

(4) point which is instantaneously unaccelerated is the rightmost point

6. A rod bent at right angle along its centre line, is placed on a rough horizontal fixed cylinder of radius R as shown in figure. Mass of rod is $2m$ and rod is in equilibrium. Assume that friction force on rod at A and B are equal in magnitude.



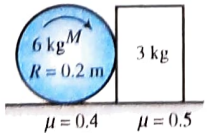
(1) Normal force applied by cylinder on rod at A is $3mg/2$

(2) Normal force applied by cylinder on rod at B must be zero

(3) Friction force acting on rod at B is upward

(4) Normal force applied by cylinder on rod at A is mg

7. A clockwise torque of 6 Nm is applied to the circular cylinder as shown in the figure. There is no friction between the cylinder and the block.



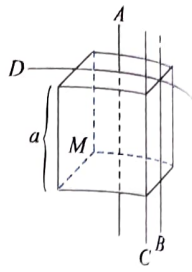
(1) The cylinder will be slipping but the system does not move forward

(2) The system cannot move forward for any torque applied to the cylinder

(3) The acceleration of the system will be 1m/s^2 forward

(4) The angular acceleration of the cylinder is 10 rad s^{-2}

8. Illustrated below is a uniform cubical block of mass M and side a. Mark the correct statement(s).



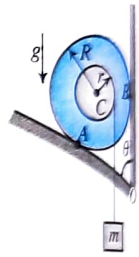
(1) The moment of inertia about axis A, passing through the centre of mass is $I_A = \frac{1}{6}Ma^2$

(2) The moment of inertia about axis B, which bisects one of the cube faces is $I_B = \frac{5}{12}Ma^2$

(3) The moment of inertia about axis C, along one of the cube edges is $I_C = \frac{2}{3}Ma^2$

(4) The moment of inertia about axis D, which bisects one of the horizontal cube faces is $\frac{7}{12}$

9. A massless spool of inner radius r, outer radius R is placed against vertical wall and tilted split floor as shown. A light inextensible thread is tightly wound around the spool through which a mass m is hanging. There exists no friction at point A, while the coefficient of friction between spool and point B is μ. The angle between two surfaces is θ.



(1) the magnitude of force on the spool at B in order to maintain equilibrium is $mg\sqrt{\left(\frac{r}{R}\right)^2 + \left(1-\frac{r}{R}\right)^2 \frac{1}{\tan^2\theta}}$

(2) the magnitude of force on the spool at B in order to maintain equilibrium is $mg\left(1-\frac{r}{R}\right)\frac{1}{\tan\theta}$

(3) the minimum value of μ for the system to remain in equilibrium is $\frac{\cot\theta}{(R/r)-1}$

(4) the minimum value of μ for the system to remain in equilibrium is $\frac{\tan\theta}{(R/r)-1}$

10. For a body having pure rotation motion. The axis of rotation

(1) may pass through the centre of mass

(2) must pass through the centre of mass

(3) may pass through a particle of the body

(4) must pass through a particle of the body

11. A solid sphere is rotating about a diameter with uniform angular speed. Which of the following options is/are correct

(1) The particles on the surface of the sphere do not have any linear acceleration

(2) The particles on the diameter mentioned above do not have any linear acceleration

(3) Different particles on the surface have different angular speeds

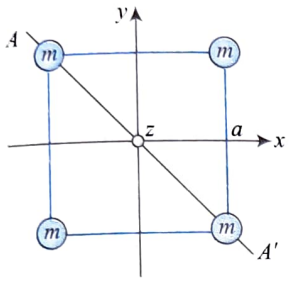
(4) All the particles on the surface have the different linear speed

12. Two identical semicircular discs of mass 'm' each and radius 'R' are placed in the XY (horizontal) plane and the YZ (vertical) plane, respectively. They are so placed that they have their common diameter along the Y-axis. Then, the moment of inertia of the system

(1) $I_x = \frac{1}{2}mR^2$ (2) $I_y = \frac{1}{2}mR^2$

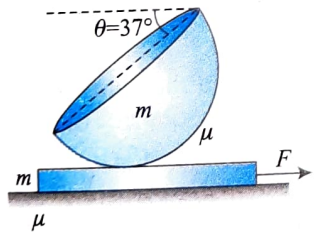
(3) $I_z = \frac{3}{4}mR^2$ (4) $I_x = I_z$

13. Four point mass, each of mass m are connected at a corner of a square of side 'a', by massless rods as shown in the figure, x and y axis are in the plane of the system and z axis is perpendicular to the plane and passing through the centre of the square.



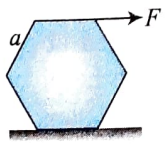
(1) Moment of inertia of the system about y-axis is $I_y = ma^2$
(2) Moment of inertia of the system about the diagonal axis AA' is $I_{AA'} = ma^2$
(3) Moment of inertia of the system about z-axis is $I_z = ma^2$
(4) Moment of inertia of the system about x-axis is $I_x = ma^2$

14. A force F is applied on the plank such that the hollow hemispherical shell is in equilibrium as shown in figure. The coefficient of friction μ is same between the with respect to plank hemispherical shell and the plank as its between the plank and the ground. Friction is just sufficient to prevent the slipping (Take $g = 10$ m/s^2 and $m = 5$ kg). Then



(1) the minimum magnitude of coefficient of friction $\mu = 1/2$
(2) the minimum magnitude of coefficient of friction $\mu = 1/4$
(3) the magnitude of applied force is 100 N.
(4) the acceleration of plank is 10 m/s^2.

15. A hexagonal body lies on a horizontal rough surface. A force F acts on the side of a hexagonal body as shown in figure. Choose the correct option(s).



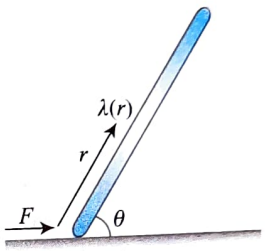
(1) For the toppling, the minimum value of coefficient of friction is $\frac{1}{2\sqrt{3}}$.

(2) For the toppling, the minimum value of coefficient of friction is $\frac{2}{\sqrt{3}}$.

(3) If $\mu = 2\mu_{min}$, and applied force $F = \frac{mg}{\sqrt{3}}$ then angular acceleration of the body is $\frac{6}{17}\frac{g}{a}$.

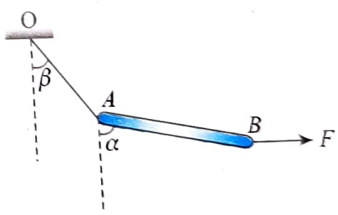
(4) If $\mu = 2\mu_{min}$, and applied force $F = \frac{mg}{\sqrt{3}}$, then angular acceleration of the body is $\frac{6}{5}\frac{g}{a}$.

16. A rod, placed in vertical plane, is being pushed along a smooth horizontal surface by a horizontal force $F = \frac{2mg}{3}$. Mass per unit length of the rod is given $\lambda(r) = \left(\frac{m}{l^2}r\right)$, where r is distance from the action point of applied force F and l is the length of the rod. Then,



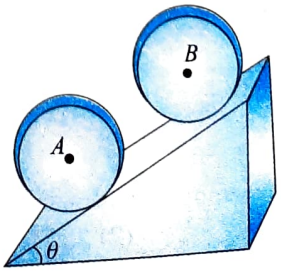
(1) angle θ for which rod only translates is 37°
(2) acceleration of the rod for which rod only translates is $\frac{4}{3}g$
(3) normal force for which rod only translates is $\frac{m}{2}g$
(4) normal force for which rod only translates is mg

17. A stick AB of mass M is tied at one end to a light string OA. A horizontal force $F = Mg$ is applied at end B of the stick and its remains in equilibrium in position shown. Then,



(1) α is $\tan^{-1}$ (2)
(2) the value of β is 45°
(3) tension in string OA is $\sqrt{2}Mg$
(4) the value of α is $\tan^{-1}\left(\frac{1}{2}\right)$

18. Two cylinders A and B have been placed in contact on an incline. They remain in equilibrium. The dimensions of the two cylinders are same. Choose the correct statement.



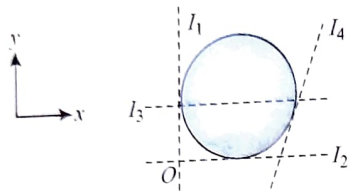
(1) The magnitude of friction in all surfaces should be equal
(2) The value of friction between the cylinder is

$$\left(\frac{M_A + M_B}{2}\right) g \sin\theta$$

(3) For two cylinders to be in contact mass of cylinder B should be greater than the mass of cylinder A
(4) The normal reaction between the cylinders should be

$$\left(\frac{M_A + M_B}{2}\right) g \sin\theta$$

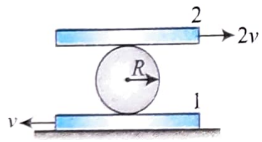
19. A disc has radius R, mass m and has moment of inertia I_1, I_2, I_3 and I_4 about axis as shown in figure. Moment of inertia of disc about z-axis passing through O will be



(1) $I_1 + I_2$ (2) $\frac{5}{2}mR^2$

(3) $3mR^2$ (4) $I_1 + I_4$

20. A cylinder is rolling without sliding over two horizontal planks (surfaces) 1 and 2. If the velocities of the surfaces A and B are $-v\hat{i}$ and $2v\hat{i}$ respectively, find the

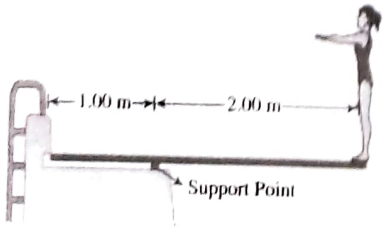


(1) The instantaneous axis of rotation is located at a distance of $4R/3$ below the top of the disc
(2) The instantaneous axis of rotation is located at a distance of $3R/2$ below the top of the disc
(3) Angular velocity of the cylinder is $\frac{3v}{2R}$
(4) Angular velocity of the cylinder is $\frac{5v}{3R}$

Linked Comprehension Type

For Problems 1–2

A diving board 3.00 m long is supported at a point 1.00 m from the end and a diver weighing 500 N stands at the free end. The diving board is of uniform cross section and weighs 280 N. Find

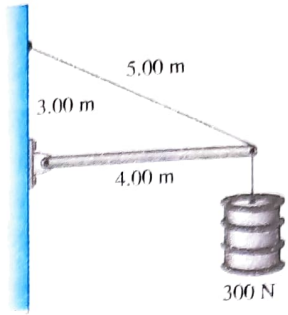


1. The forces at the support point
(1) 780 N (2) 220 N
(3) 1920 N (4) 1140 N

2. The force at the end that is held down.
(1) 780 N (2) 220 N
(3) 1920 N (4) 1140 N

For Problems 3–4

The horizontal beam in figure weighs 150 N, and its centre of gravity is at its centre. Find

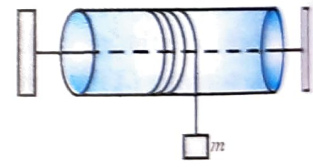


3. The tension in the cable
(1) 75 N (2) 500 N
(3) 300 N (4) 625 N

4. The horizontal and vertical components of the force exerted on the beam at the wall
(1) Horizontal component is 500 N towards left and vertical component 75 N downwards
(2) Horizontal component is 500 N towards right and vertical component 75 N upwards
(3) Horizontal component is 625 N towards left and vertical component 150 N upwards
(4) Horizontal component is 625 N towards right and vertical component 150 N downwards

For Problems 5–6

A cord is wound round the circumference of a solid cylinder of radius R and mass M. The axis of the cylinder is horizontal. A weight mg is attached to the end of the cord and falls from rest. After falling through a distance h.



5. The angular velocity of the cylinder will be

(1) $\frac{2mg}{M+2m}$ (2) $\sqrt{\frac{2gh}{R^2}}$

(3) $\sqrt{\frac{4mgh}{(M+2m)R^2}}$ (4) $\sqrt{\frac{4mgh}{M+2m}}$

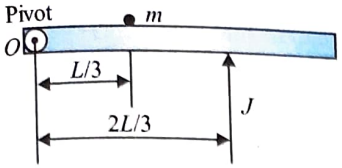
6. If the mass starts from rest and falls a distance h, then its speed at that instant is

(1) proportional to R (2) proportional to $\frac{1}{R}$

(3) proportional to $\frac{1}{R^2}$ (4) independent of R

For Problems 7–9

A uniform rod of length L and mass M is lying on a frictionless horizontal plane and is pivoted at one of its ends as shown in figure. There is no friction at the pivot. An inelastic ball of mass m is fixed with the rod at a distance $L/3$ from O. A horizontal impulse J is given to the rod at a distance $2L/3$ from O in a direction perpendicular to the rod. Assume that the ball remains in contact with the rod after the collision and impulse J acts for a small time interval Δt.



Now answer the following questions:

7. Find the resulting instantaneous angular velocity of the rod after the impulse.

(1) $\frac{3J}{(m+3M)L}$ (2) $\frac{6J}{(m+3M)L}$

(3) $\frac{3J}{(3m+M)L}$ (4) $\frac{6J}{(3m+M)L}$

8. Find the impulse acted on the ball during the time interval Δt.

(1) $\frac{2MJ}{(3m+M)}$ (2) $\frac{2MJ}{(m+3M)}$

(3) $\frac{2mJ}{(3m+M)}$ (4) $\frac{2mJ}{(m+3M)}$

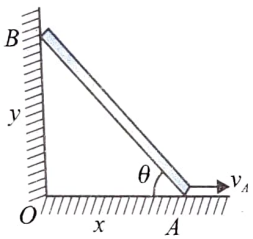
9. Find the magnitude of the impulse applied by the pivot during the time interval Δt.

(1) $\frac{mJ}{(m+3M)}$ (2) $\frac{mJ}{(3m+M)}$

(3) $\frac{MJ}{(m+3M)}$ (4) $\frac{MJ}{(3m+M)}$

For Problems 10–12

End A of a rod AB is being pulled on the floor with a constant velocity v_0 as shown. Taking the length of the rod as L at an instant when the rod makes an angle 37° with the horizontal, calculate



10. The velocity of end B

(1) $\frac{3}{5}v_0$ (2) $\frac{4}{5}v_0$

(3) $\frac{5}{3}v_0$ (4) $\frac{5}{4}v_0$

11. The angular velocity of the end

(1) $\frac{5v_0}{3l}$ (2) $\frac{3v_0}{5l}$

(3) $\frac{5v_0}{4l}$ (4) $\frac{4v_0}{5l}$

12. The velocity of the CM of the rod

(1) $\frac{5}{7}v_0$ at $\tan^{-1}\frac{4}{3}$ below horizontal

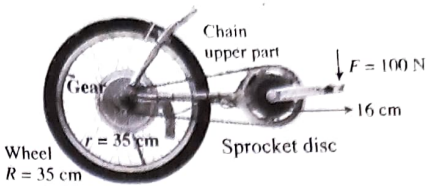
(2) $\frac{5}{7}v_0$ at $\tan^{-1}\frac{3}{4}$ below horizontal

(3) $\frac{5}{6}v_0$ at $\tan^{-1}\frac{3}{4}$ below horizontal

(4) $\frac{5}{6}v_0$ at $\tan^{-1}\frac{4}{3}$ below horizontal

For Problems 13–17

A bicycle has pedal rods of length 16 cm connected to a sprocketed disc of radius 10 cm. The bicycle wheels are 70 cm in diameter and the chain runs over a gear of radius 4 cm. The speed of the cycle is constant and the cyclist applies 100 N force that is always perpendicular to the pedal rod, as shown in the figure. Assume tension in the lower part of chain is negligible. The cyclist is peddling at a constant rate of two revolutions per second. Assume that the force applied by other foot is zero when one foot is exerting 100 N force. Neglect friction within cycle parts and the rolling friction.



13. The tension in the upper portion of the chain is equal to

(1) 100 N (2) 120 N

(3) 160 N (4) 240 N

14. Net torque on the rear wheel of the bicycle is equal to

(1) zero (2) 16 Nm

(3) 6.4 Nm (4) 4.8 Nm

15. The power delivered by the cyclist is equal to

(1) 280 W (2) 100 W

(3) 64π W (4) 32 W

16. The speed of the bicycle is

(1) 6.4π m s^{-1} (2) 3.5π m s^{-1}

(3) 2.8π m s^{-1} (4) 5.6π m s^{-1}

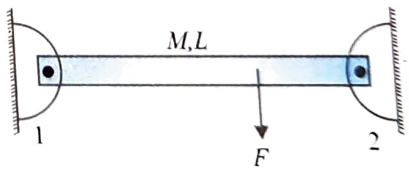
17. The net force of the friction on the rear wheel due to the road is

(1) 100 N (2) 62 N

(3) 32.6 N (4) 18.3 N

For Problems 18–20

A uniform rod of mass $M = 2$ kg and length L is suspended by two smooth hinges 1 and 2 as shown in figure. A force $F = 4$ N is applied downward at a distance $L/4$ from hinge 2. Due to the application of force F, hinge 2 breaks. At this instant, applied force F is also removed. The rod starts to rotate downward about hinge 1. ($g = 10$ m/s^2)



18. The reaction at hinge 1, before hinge 2 breaks, is

(1) 24 N (2) 12 N

(3) 11 N (4) 10 N

19. The reaction at hinge 1, just after breaking of hinge 2, is

(1) 20 N (2) 10 N

(3) 5 N (4) 0

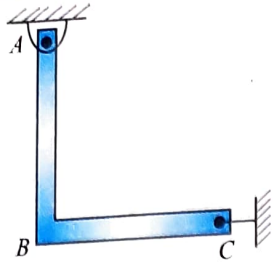
20. The acceleration of the end point of the rod of small mass dm at the end point of the rod, when the rod becomes vertical is

(1) 30 m/s^2 (2) 20 m/s^2

(3) 10 m/s^2 (4) 0

For Problems 21–23

An L shaped uniform rod of mass $2M$ and length $2L$ ($AB = BC = L$) is held as shown in figure with a string fixed between C and wall so that AB is vertical and BC is horizontal. There is no friction between the hinge and the rod at A.



21. Find the tension in the string.

(1) $\frac{Mg}{3}$ (2) $\frac{Mg}{4}$

(3) Mg (4) $\frac{Mg}{2}$

22. What will be the reaction between hinge and rod at point A?

(1) $\frac{\sqrt{65}Mg}{4}$ (2) $2\,Mg$

(3) $\frac{\sqrt{17}Mg}{4}$ (4) $\frac{\sqrt{17}Mg}{2}$

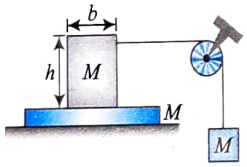
23. If the string is burnt, find the angle between AB and the vertical at equilibrium position.

(1) $\tan^{-1}\left(\frac{1}{3}\right)$ (2) $\tan^{-1}\left(\frac{1}{4}\right)$

(3) $\tan^{-1}(3)$ (4) $\tan^{-1}\left(\frac{1}{2}\right)$

For Problems 24–25

A block of mass $M = 4$ kg of height h' and breath b is placed on a rough plank of same mass M. A light inextensible string is connected to the upper end of the block and passed through a light smooth pulley as shown in figure. A mass $m = 1$ kg is hung to the other end of the string.



24. What should be the minimum value of coefficient of friction between the block and the plank so that, there is no slipping between the block and the wedge?

(1) $\frac{3}{5}$ (2) $\frac{5}{9}$

(3) $\frac{4}{5}$ (4) $\frac{1}{9}$

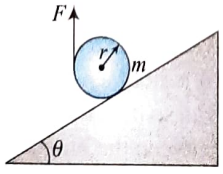
25. Find the minimum value of b/h so that the block does not topple over the plank, friction is absent between the plank and the ground.

(1) $\frac{2}{5}$ (2) $\frac{4}{9}$

(3) $\frac{4}{5}$ (4) $\frac{5}{9}$

For Problems 26–28

A solid cylinder of mass m is kept in balance on a fixed incline of angle $\alpha = 37°$ with the help of a thread fastened to its jacket. The cylinder does not slip.



26. What force F is required to keep the cylinder in balance when the thread is held vertically?

(1) $\frac{mg}{2}$ (2) $\frac{3mg}{4}$

(3) $\frac{3mg}{8}$ (4) None of these

27. The value of minimum coefficient of static friction between cylinder and incline in the case mentioned in previous case.

(1) $\frac{3}{5}$ (2) $\frac{2}{3}$

(3) $\frac{3}{4}$ (4) $\frac{4}{5}$

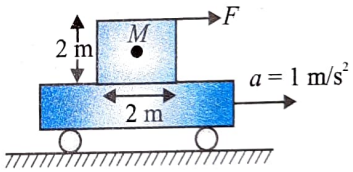
28. In what direction should the thread be pulled to minimise the force required to hold the cylinder? What is the magnitude of this force?

(1) $\frac{3mg}{10}$ parallel to incline (2) Vertical $\frac{3mg}{8}$

(3) Horizontal $\frac{mg}{5}$ (4) None of these

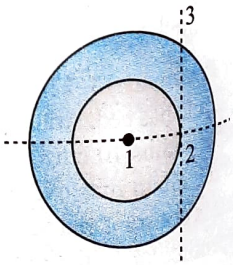
Matrix Match Type

1. A cubical block having square cross section of side 2 m and of mass $M = 10$ kg is resting over a platform moving at constant acceleration $a = 1$ m/s^2. Coefficient of friction between the block and the platform is $\mu = 0.1$. A force F acts at the top of the cube as shown in figure. Now match Column I with Column II.



Column I	Column II
i. $F = 0$	a. Block neither topples nor slips over the platform
ii. $F = 45$ N	b. Block topples but does not slip over the platform
iii. $F = 15$ N	c. Block slips but does not topple over the platform
iv. $F = 25$ N	d. Block slips as well as topples on the platform

2. From a uniform disc of mass M and radius R, a concentric disc of radius $R/2$ is cut out. For the remaining annular disc I_1 is the moment of inertia about axis '1', I_2 about '2', I_3 about '3' and I_4 about '4'. Axes '1' and '2' are perpendicular to the disc and '3' and '4' are in the plane of the disc. Axes '2', '3' and '4' intersect at a common point. Match the following

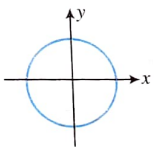


Column I	Column II
i. I_1 is equal to	a. $\frac{21}{32}MR^2$
ii. I_2 is equal to	b. $I_1/2$
iii. $I_3 + I_4$ is equal to	c. $\frac{15}{32}MR^2$
iv. $I_2 - I_3$ is equal to	d. none of these

3. In Column I, information about the force(s) acting on a body is mentioned, while in Column II information about the motion of a body is given. Match the entries in Column II with the entries in Column I.

Column I	Column II
i. A single force through centre of mass	a. Rotational motion
ii. Equal and opposite forces separated by non-zero distance	b. Translational motion
iii. Equal and opposite forces acting at the same point	c. No motion
iv. A single force not through centre of mass	d. Centre of mass performs curvilinear motion

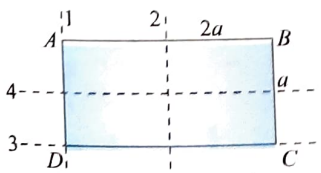
4. A body of Mass M and radius R is as shown in figure.



The body can be a ring, a disc, a hollow sphere or a solid sphere. I_x, I_y and I_z are the moments of inertia about x, y and z axis respectively. Origin coincides with centre of mass of the body.

Column I	Column II
i. $I_x = I_y = I_z$	a. possible for ring or disc
ii. $I_z = I_y + I_z$	b. possible for hollow sphere or solid sphere
iii. $I_y = I_z + I_x$	c. possible for all of the bodies mentioned
iv. $I_z = I_x + I_y$	d. not possible for any of the body mentioned

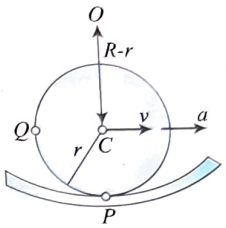
5. A rectangular slab ABCD have dimensions $a \times 2a$ as shown in figure. Match the following two columns.



Column I	Column II
i. Radius of gyration about axis-1	a. $\frac{a}{\sqrt{12}}$
ii. Radius of gyration about axis-2	b. $\frac{2a}{\sqrt{3}}$
iii. Radius of gyration about axis-3	c. $\frac{a}{\sqrt{3}}$
iv. Radius of gyration about axis-4	d. None

(1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – c, iii – c, iv – a
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b

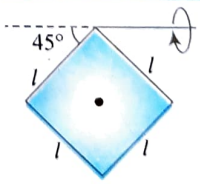
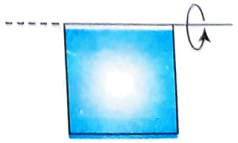
6. A body of radius r rolls without sliding on a curved surface of radius of curvature R as shown in the figure. Match the Columns:

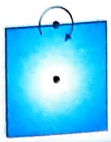


Column I	Column II
i. ω_{CO}	a. $\frac{v}{r}$
ii. ω_{CP}	b. $\frac{v}{R-r}$
iii. α_{PC}	c. $\frac{a}{R-r}$
iv. α_{CO}	d. $\frac{a}{r}$

(1) i – b, ii – a, iii – d, iv – c
(2) i – a, ii – b, iii – c, iv – d
(3) i – a, ii – b, iii – d, iv – c
(4) i – b, ii – a, iii – c, iv – d

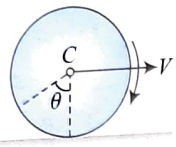
7. Square frame is made of four uniform rods. Mass of each rod is m and length l.

Column I (Axis of rotation)	Column II (Moment of inertia about axis of rotation)
i. 	a. $\frac{2}{3}ml^2$
ii. 	b. $\frac{5}{3}ml^2$

iii.	c. $\frac{7}{3}ml^2$
iv. 	d. $\frac{8}{3}ml^2$

(1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – a, iii – d, iv – c
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b

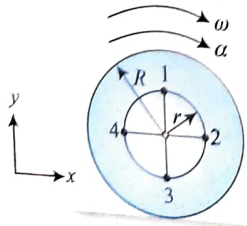
8. A disc rolls on ground without slipping. Velocity of centre of mass is v. There is a point P on circumference of disc at angle θ. Suppose v_p is the speed of this point. Then, match the following:



Column I	Column II
i. If $\theta = 60°$	a. $v_p = \sqrt{2}v$
ii. If $\theta = 90°$	b. $v_p = v$
iii. If $\theta = 120°$	c. $v_p = 2v$
iv. If $\theta = 18°$	d. $v_p = \sqrt{3}v$

(1) i – b, ii – a, iii – c, iv – d
(2) i – b, ii – a, iii – d, iv – c
(3) i – b, ii – d, iii – a, iv – c
(4) i – a, ii – b, iii – d, iv – c

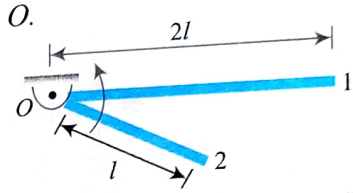
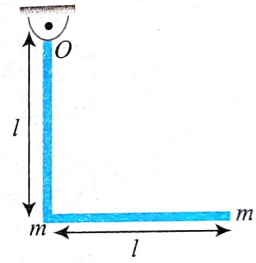
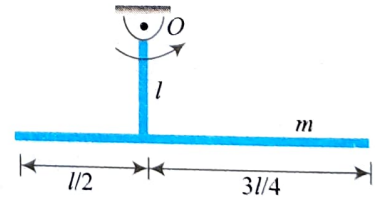
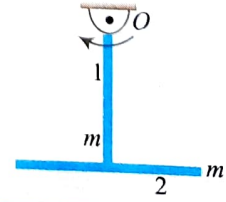
9. A disc of radius R is rolling without slipping with an angular acceleration α, on a horizontal plane. Four points are marked at the end of horizontal and vertical diameter of a circle of radius r ($<R$) on the disc. If horizontal and vertical direction are chosen as x and y axis as shown in the figure, then acceleration of points 1, 2, 3 and 4 are $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{a}_4$ respectively, at the moment when angular velocity of the disc is ω. Match the following



Column I	Column II
i. $\vec{a}_1$	a. $(R\alpha - r\alpha)\hat{i} + (r\omega^2)\hat{j}$
ii. $\vec{a}_2$	b. $(R\alpha + r\alpha)\hat{i} - (r\omega^2)\hat{j}$
iii. $\vec{a}_3$	c. $(R\alpha - r\omega^2)\hat{i} - (r\alpha)\hat{j}$
iv. $\vec{a}_4$	d. $(R\alpha + r\omega^2)\hat{i} + (r\alpha)\hat{j}$
	e. None of these

(1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – c, iii – a, iv – d
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b

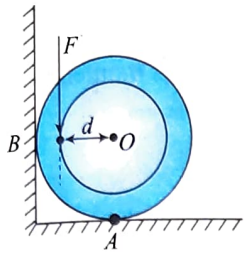
10. For the situations given in column-I match the corresponding values of moment of inertia about an axis passing through O.

Column I (Axis of rotation)	Column II (Moment of inertia about axis of rotation)
i. A rod of mass m length $3l$ is bent at O. 	a. $\frac{7}{3}ml^2$
ii. L-shaped composite rod comprising two identical rods each of mass m and length l. 	b. $\frac{17}{12}ml^2$
iii. A rod of mass m length l is rotating about an axis passing through O. 	c. ml^2
iv. Two identical rods each of mass m and length l are welded as shown. 	d. $\frac{55}{48}ml^2$

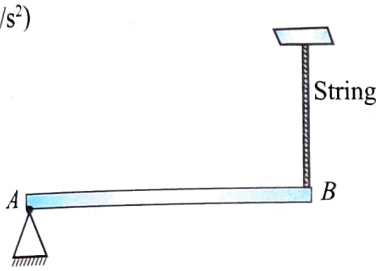
(1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – a, iii – d, iv – c
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b

Numerical Value Type

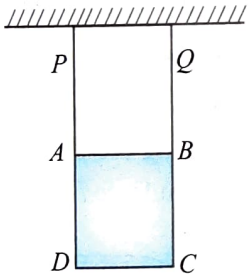
1. A solid cylinder with $r = 0.1$ m and mass $M = 2$ kg is placed such that it is in contact with the vertical and a horizontal surface as shown in figure. The coefficient of static friction is $\mu = (1/3)$ for both the surfaces. Find the distance d (in cm) from the centre of the cylinder at which a force $F = 40$ N should be applied vertically so that the cylinder just starts rotating in anticlockwise direction.



2. A uniform rod AB of mass 2 kg is hinged at one end A. The rod is kept in the horizontal position by a massless string tied to point B. Find the reaction of the hinge (in N) on end A of the rod at the instant when string is cut. ($g = 10$ m/s^2)

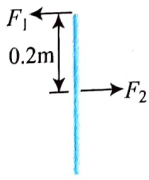


3. A square plate $ABCD$ of mass m and side l is suspended with the help of two ideal strings P and Q as shown. Determine the acceleration (in m/s^2) of corner A of the square just at the moment the string Q is cut. ($g = 10$ m/s^2)

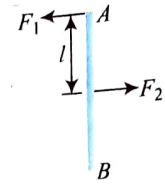


4. A uniform cylinder rests on a cart as shown. The coefficient of static friction between the cylinder and the cart is 0.5. If the cylinder is 4 cm in diameter and 10 cm in height, then what is the minimum acceleration (in ms^{-2}) of the cart needed to cause the cylinder to tip over?

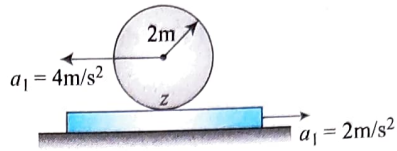
5. A thin uniform rod AB of mass 1 kg moves translationally with acceleration $a = 2$ m/s^2 due to two antiparallel forces F_1 and F_2. The distance between the points at which these forces are applied is equal to $l = 0.2$ m. If force $F_1 = 8$ N, then find the length of the rod in meter.



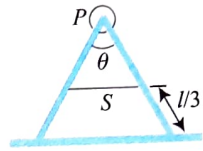
6. A thin uniform rod AB of mass $m = 1$ kg moves translationally with acceleration $a = 2$ m/s^2 due to two antiparallel forces F_1 and F_2. The distance between the points at which these forces are applied is equal to $l = 20$ cm. Besides, it is known that $F_2 = 5$ N. Find the length of the rod in m.



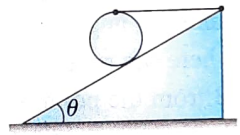
7. In figure, a sphere of radius 2 m rolls on a plank. The accelerations of the sphere and the plank are indicated. Find the value of α in rad/s^2.



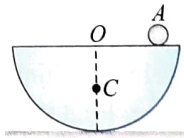
8. Two rods each of mass $m = 4$ kg and length $l = 1$ m are in equilibrium being smoothly hinged at P. Find the tension in the connecting string as shown in figure.



9. A cylinder is in rotational equilibrium on an inclined plane, inclined at angle $\theta = 53°$. Find the minimum μ between the wooden log and the inclined plane.
(Given: $\sin 53° = \frac{4}{5}$ and $\cos 53° = \frac{3}{5}$)

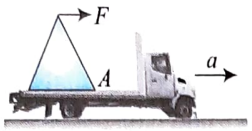


10. A uniform hemisphere with radius $R = 10$ m is placed on a horizontal smooth surface. Its centre of mass C is below the centre O. An object 'A' is placed on the flat surface of the hemisphere at point O. The mass of the object is 1/8th of mass of the hemisphere. The coefficient of friction between the object and the surface of the hemisphere is $\mu = 0.2$. Now the object 'A' is displaced slowly along the flat surface of the hemisphere towards its outer rim. Find the maximum distance (in m) of the object from the centre O without slipping on the hemisphere.

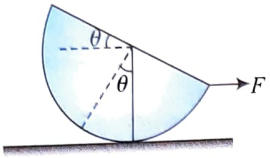


11. Figure show a uniform block of mass $m = \sqrt{\frac{3}{13}}$ kg having square base and equilateral triangular cross section placed at the edge of the floor of a truck which is under constant acceleration $a = \frac{g\sqrt{3}}{4}$ m/s^2 to the right. A horizontal force on the apex of the triangle F is just enough to just tilt the block. Find the force (in newton) on the block at point A

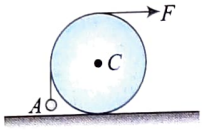
where the block is in contact with the step edge when F is applied. (Take $g = 10\ m/s^2$)



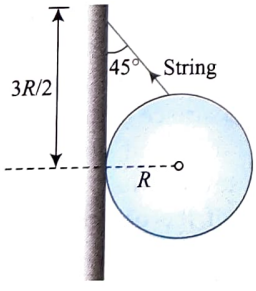
12. A semicircular disc of radius R and mass M is pulled by a horizontal force F so that it moves with uniform velocity. The coefficient of friction between disc and ground is $\mu = \frac{4}{3\pi}$. Find the value of θ in degree.



13. A cylinder C rests on a horizontal surface. A small particle of mass $m = 5$ kg is held in equilibrium connected to an overhanging string as shown. The other end of the massless string is being pulled horizontally by a force F as shown. Find F (in N).

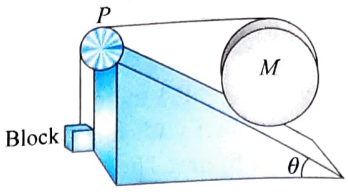


14. A sphere of radius R is supported by a rope attached to the wall. The rope makes an angle $\theta = 45°$ with respect to the wall. The point where the rope is attached to the wall is at a distance of $3R/2$ from the point where the sphere touches the wall. Find the minimum coefficient of friction (μ) between the wall and the sphere for this equilibrium to be possible.

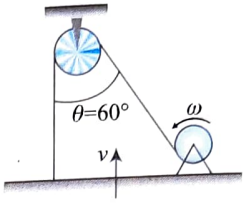


15. A wheel of radius $R = 1$ m is rolling without sliding uniformly on a horizontal surface. Find the radius of curvature of the path (in m), of a point on its circumference when it is at highest point in its path.

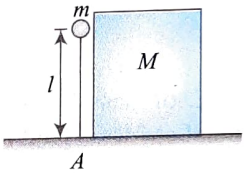
16. In the arrangement shown in figure the cylinder of mass $M = 9$ kg is at rest on an incline. The plane is inclined at angle 37° with horizontal. The string between the cylinder and the pulley (P) is horizontal. Find the minimum the mass of the block (in kg) to keep the system in equilibrium.



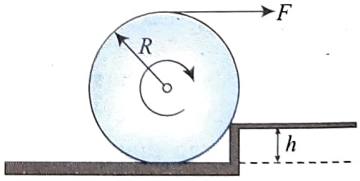
17. In the instant shown in the figure, the board is moving up vertically with constant velocity (v). The drum winds up at a constant rate ω=15 rad/s. If the radius of the drum is $R = 10$ cm and the board always remains horizontal. Find its velocity of board at this moment(in m/s).



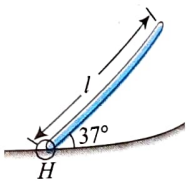
18. A weightless rod of length l with a small load of mass m at the end is hinged at point A as shown and occupies a strictly vertical position, touching a block of mass M. A light jerk sets the system in motion. For what mass ratio M/m will the rod form an angle $\theta = \pi/6$ with the horizontal at the moment of the separation from the block?



19. A cylinder of mass 15 kg and radius R is to be raised onto a horizontal step of height $h = \frac{R}{5}$ as shown in figure. A rope is wrapped around the cylinder and pulled horizontally with force F. Assuming the cylinder does not slip on the step, find the minimum force F (in N) necessary to raise the cylinder.



20. A uniform rod of mass $m = 2\sqrt{10}$ kg and length l can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find angular acceleration a of the rod just after it is released from initial position making an angle of 37° with horizontal from rest? Find force (in N) exerted by the hinge just after the rod is released from rest.



Archives

JEE MAIN

Single Correct Answer Type

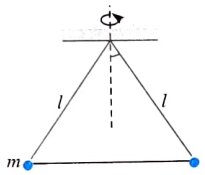
1. A mass m hangs with the help of a string wrapped around a pulley on a frictionless bearing. The pulley has mass m and radius R. Assuming pulley to be a perfect uniform circular disc, the acceleration of the mass m, if the string does not slip on the pulley, is

(1) $3g/2$ (2) g
(3) $2g/3$ (4) $g/3$ **(AIEEE 2011)**

2. A pulley of radius 2 m is rotated about its axis by a force $F = (20t - 5t^2)$ newton (where t is measured in seconds) applied tangentially. If the moment of inertia of the pulley about its axis of rotation is 10 kg m^2, the number of rotations made by the pulley before its direction of motion is reversed, is

(1) less than 3
(2) more than 3 but less than 6
(3) more than 6 but less than 9
(4) more than 9 **(AIEEE 2011)**

3. A mass m is supported by a massless string wound around a uniform hollow cylinder of mass m and radius R. If the string does not slip on the cylinder, with what acceleration will the mass fall on release?



(1) $\frac{5g}{6}$ (2) g

(3) $\frac{2g}{3}$ (4) $\frac{g}{2}$ **(JEE Main 2014)**

4. From a solid sphere of mass M and radius R, a cube of maximum possible volume is cut. The moment of inertia of cube about an axis passing through its centre and perpendicular to one of its faces is

(1) $\frac{MR^2}{32\sqrt{2}\pi}$ (2) $\frac{MR^2}{16\sqrt{2}\pi}$

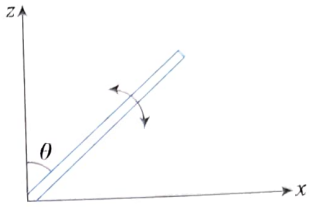
(3) $\frac{4MR^2}{9\sqrt{3}\pi}$ (4) $\frac{4MR^2}{3\sqrt{3}\pi}$ **(JEE Main 2015)**

5. The moment of inertia of a uniform cylinder of length l and radius R about its perpendicular bisector is I. What is the ratio l/R such that the moment of inertia is minimum?

(1) 1 (2) $\frac{3}{\sqrt{2}}$

(3) $\sqrt{\frac{3}{2}}$ (4) $\frac{\sqrt{3}}{2}$ **(JEE Main 2017)**

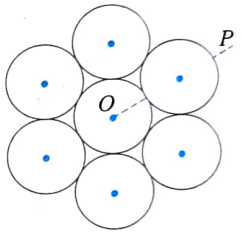
6. A slender uniform rod of mass M and length l is pivoted at one end so that it can rotate in a vertical plane (see figure). There is negligible friction at the pivot. The free end is held vertically above the pivot and then released. The angular acceleration of the rod when it makes an angle θ with the vertical is



(1) $\frac{3g}{2l}\cos\theta$ (2) $\frac{2g}{3l}\cos\theta$

(3) $\frac{3g}{2l}\sin\theta$ (4) $\frac{2g}{3l}\sin\theta$ **(JEE Main 2017)**

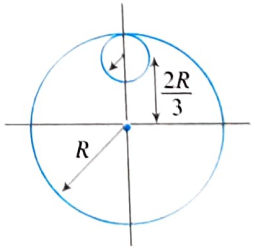
7. Seven identical circular planar disks, each of mass M and radius R are welded symmetrically as shown. The moment of inertia of the arrangement about the axis normal to the plane and passing through the point P is



(1) $\frac{181}{2}MR^2$ (2) $\frac{19}{2}MR^2$

(3) $\frac{55}{2}MR^2$ (4) $\frac{73}{2}MR^2$ **(JEE Main 2018)**

8. From a uniform circular disc of radius R and mass $9M$, a small disc of radius $R/3$ is removed as shown in the figure. The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through centre of disc is



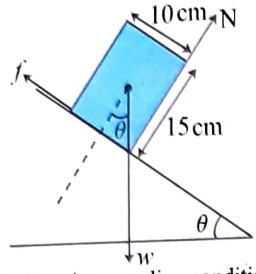
(1) $\frac{37}{9}MR^2$ (2) $4\,MR^2$

(3) $\frac{40}{9}MR^2$ (4) $10\,MR^2$ **(JEE Main 2018)**

JEE ADVANCED

Single Correct Answer Type

1. A block of base 10 cm × 10 cm and height 15 cm is kept on an inclined plane. The coefficient of friction between them is 3. The inclination θ of this inclined plane from the horizontal plane is gradually increased from 0°. Then

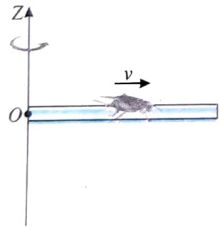


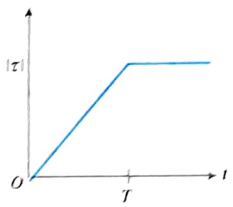
FBD at just toppling condition

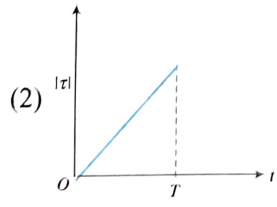
(1) at $\theta = 30°$, the block will start sliding down the plane
(2) the block will remain at rest on the plane up to certain θ and then it will topple
(3) at $\theta = 60°$, the block will start sliding down the plane and continue to do so at higher angles
(4) at $\theta = 60°$, the block will start sliding down the plane and on further increasing θ, it will topple at certain θ

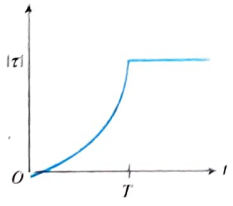
(IIT-JEE 2009)

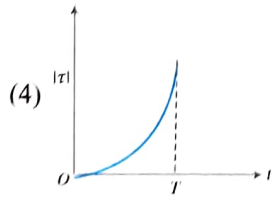
2. A thin uniform rod, pivoted at O, is rotating in the horizontal plane with constant angular speed ω, as shown in the figure. At time $t = 0$, a small insect starts from O and moves with constant speed v with respect to the rod towards the other end. It reaches the end of the rod at $t = T$ and stops. The angular speed of the system remains ω throughout. The magnitude of the torque $(\vec{\tau})$ on the system about O, as a function of time is best represented by which plot?



(1) 

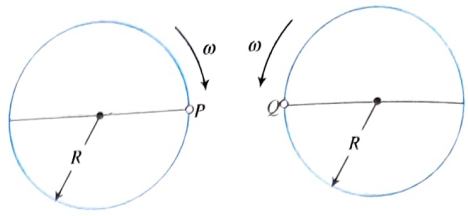
(2) 

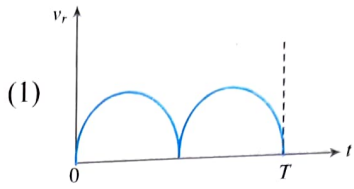
(3) 

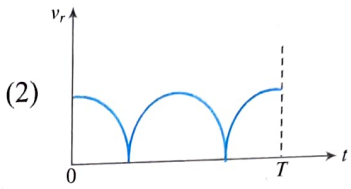
(4) 

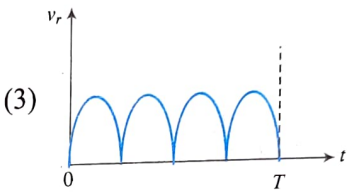
(IIT-JEE 2012)

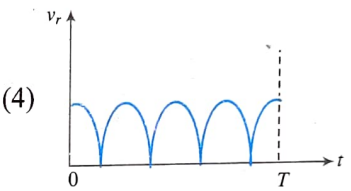
3. Two identical discs of same radius R are rotating about their axes in opposite directions with the same constant angular speed ω. The discs are in the same horizontal plane. At time $t = 0$, the points P and Q are facing each other as shown in the figure. The relative speed between the two points P and Q is v_r. In one time period (T) of rotation of the discs, v_r as a function of time is best represented by



(1) 

(2) 

(3) 

(4) 

(IIT-JEE 2012)

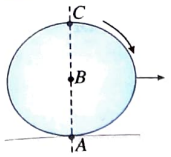
4. A uniform wooden stick of mass 1.6 kg and length l rests in an inclined manner on a smooth, vertical wall of height h ($< l$) such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle of 30° with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio h/l and the frictional force f at the bottom of the stick are ($g = 10\ \text{ms}^{-2}$)

(1) $\frac{h}{l} = \frac{\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}\text{N}$
(2) $\frac{h}{l} = \frac{3}{16}, f = \frac{16\sqrt{3}}{3}\text{N}$
(3) $\frac{h}{l} = \frac{3\sqrt{3}}{16}, f = \frac{8\sqrt{3}}{3}\text{N}$
(4) $\frac{h}{l} = \frac{3\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}\text{N}$

(JEE Advanced 2016)

Multiple Correct Answers Type

1. A sphere is rolling without slipping on a fixed horizontal plane surface. In the figure, A is the point of contact, B is the centre of the sphere and C is its topmost point. Then,

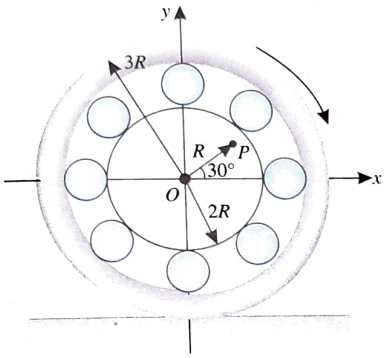


(1) $\vec{V}_C - \vec{V}_A = 2(\vec{V}_B - \vec{V}_C)$ (2) $\vec{V}_C - \vec{V}_A = \vec{V}_B - \vec{V}_A$

(3) $|\vec{V}_C - \vec{V}_A| = 2|\vec{V}_B - \vec{V}_C|$ (4) $|\vec{V}_C - \vec{V}_A| = 4|\vec{V}_B|$

(IIT-JEE 2009)

2. The figure shows a system consisting of (i) a ring of outer radius $3R$ rolling clockwise without slipping on a horizontal surface with angular speed ω and (ii) an inner disc of radius $2R$ rotating anti-clockwise with angular speed $\omega/2$. The ring and disc are separated by frictionless ball bearings. The system is in the x–z plane. The point P on the inner disc is at distance R from the origin, where OP makes an angle of $30°$ with the horizontal. Then with respect to the horizontal surface,



(1) The point O has a linear velocity $3R\omega\hat{i}$

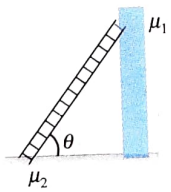
(2) The point P has a linear velocity $\frac{11}{4}R\omega\hat{i} + \frac{\sqrt{3}}{4}R\omega\hat{k}$

(3) The point P has a linear velocity $\frac{13}{4}R\omega\hat{i} + \frac{\sqrt{3}}{4}R\omega\hat{k}$

(4) The point P has a linear velocity $\left(3 - \frac{\sqrt{3}}{4}\right)R\omega\hat{i} + \frac{1}{4}R\omega\hat{k}$

(IIT-JEE 2012)

3. In the figure, a ladder of mass m is shown leaning against a wall. It is in static equilibrium making an angle θ with the horizontal floor. The coefficient of friction between the wall and the ladder is μ_1 and that between the floor and the ladder is μ_2. The normal reaction of the wall on the ladder is N_1 and that of the floor is N_2. If the ladder is about to slip, then



(1) $\mu_1 = 0\ \mu_2 \neq 0$ and $N_2 \tan\theta = mg/2$

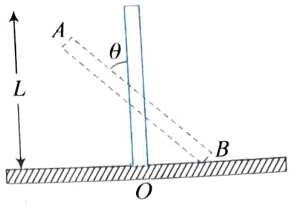
(2) $\mu_1 \neq 0\ \mu_2 = 0$ and $N_1 \tan\theta = mg/2$

(3) $\mu_1 \neq 0\ \mu_2 \neq 0$ and $N_2 = \dfrac{mg}{1 + \mu_1\mu_2}$

(4) $\mu_1 = 0\ \mu_2 \neq 0$ and $N_1 \tan\theta = mg/2$

(JEE Advanced 2014)

4. A rigid uniform bar AB of length L is slipping from its vertical position on a frictionless floor (as shown in the figure). At some instant of time, the angle made by the bar with the vertical is θ. Which of the following statements about its motion is/are correct?



(1) When the bar makes an angle θ with the vertical, the displacement of its midpoint from the initial position is proportional to $(1 - \cos\theta)$

(2) The midpoint of the bar will fall vertically downward

(3) Instantaneous torque about the point in contact with the floor is proportional to $\sin\theta$

(4) The trajectory of the point A is a parabola

(JEE Advanced 2017)

5. Consider a body of mass 1.0 kg at rest at the origin at time $t = 0$. A force $\vec{F} = (\alpha t\hat{i} + \beta\hat{j})$ is applied on the body, where $\alpha = 1.0\ \text{Ns}^{-1}$ and $\beta = 1.0$ N. The torque acting on the body about the origin at time $t = 1.0$ s is $\vec{\tau}$. Which of the following statements is (are) true?

(1) $|\vec{\tau}| = \frac{1}{3}$ Nm

(2) The torque $\vec{\tau}$ is in the direction of the unit vector $+\hat{k}$

(3) The velocity of the body at $t = 1$ s is $\vec{v} = \frac{1}{2}\left(\hat{i} + 2\hat{j}\right)\text{ms}^{-1}$

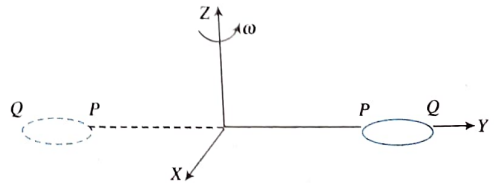
(4) The magnitude of displacement of the body at $t = 1$ s is $\frac{1}{6}$ m

(JEE Advanced 2018)

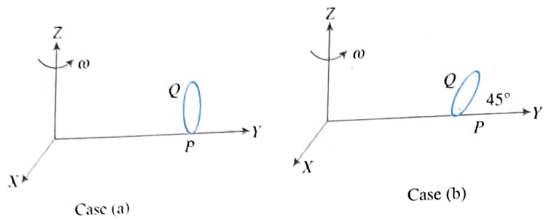
Linked Comprehension Type

For Problems 1–2

The general motion of a rigid body can be considered to be a combination of (i) a motion.... of its centre of mass about an axis, and (ii) its motion about an instantaneous axis passing through the centre of mass. These axes need not be stationary. Consider, for example, a thin uniform disc welded (rigidly fixed) horizontally at its rim to a massless stick, as shown in the figure. When the disc–stick system is rotated about the origin on a horizontal frictionless plane with angular speed ω, the motion at any instant can be taken as a combination of (i) a rotation of the centre of mass of the disc about the z-axis, and (ii) a rotation of the disc through an instantaneous vertical axis passing through its centre of mass (as is seen from the changed orientation of points P and Q). Both these motions have the same angular speed ω in this case.



Now consider two similar systems as shown in the figure: case (a) the disc with its face vertical and parallel to x–z plane; case (b) the disc with its face making an angle of 45° with x–y plane and its horizontal diameter parallel to x-axis. In both the cases, the disc is welded at point P, and the system are rotated with constant angular speed ω about the z-axis.

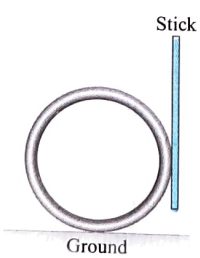


(IIT-JEE 2012)

1. Which of the following statements regarding the angular speed about the instantaneous axis (passing through the centre of mass) is correct?
   (1) It is $\sqrt{2}\,\omega$ for both the cases
   (2) It is ω for case (a); and $\omega/\sqrt{2}$ for case (b)
   (3) It is ω force case (a); and $\sqrt{2}\,\omega$ for case (b)
   (4) It is ω for both the cases
2. Which of the following statements about the instantaneous axis (passing through the centre of mass) is correct?
   (1) It is vertical for both the cases (a) and (b)
   (2) It is vertical for case (a); and is at 45° to the x–z plane and lies in the plane of the disc for case (b)
   (3) It is horizontal for case (a); and is at 45° to the x–z plane and is normal to the plane of the disc for case (b)
   (4) It is vertical for case (a); and is at 45° to the x–z plane and is normal to the plane of the disc for case (b)

Numerical Value Type

1. A boy is pushing a ring of mass 2 kg and radius 0.5 m with a stick as shown in the figure. The stick applies a force of 2 N on the ring and rolls it without slipping with an acceleration of 0.3 m/s². The coefficient of friction between the ground and the ring is large enough that rolling always occurs and the coefficient of friction between the stick and the ring is $(P/10)$. The value of P is

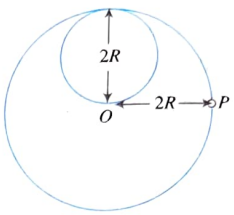


(IIT-JEE 2011)

2. Four solid spheres each of diameter $\sqrt{5}$ cm and mass 0.5 kg are placed with their centers at the corners of a square of side 4 cm. The moment of inertia of the system about the diagonal of the square is $N \times 10^{-4}$ kg-m², then N is

(IIT-JEE 2011)

3. A lamina is made by removing a small disc of diameter $2R$ from a bigger disc of uniform mass density and radius $2R$, as shown in the figure. The moment of inertia of this lamina about axes passing through O and P is I_O and I_P respectively. Both these axes are perpendicular to the plane of the lamina. The ratio I_P/I_O to the nearest integer is

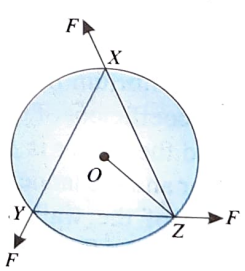


(IIT-JEE 2012)

4. A uniform circular disc of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad/s about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disc in such a manner that they are touching each other along the axis of the disc and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disc and the system rotates about the original axis. The new angular velocity (in rad s⁻¹) of the system is

(JEE Advanced 2013)

5. A uniform circular disc of mass 1.5 kg and radius 0.5 m is initially at rest on a horizontal frictionless surface. Three forces of equal magnitude $F = 0.5$ N are applied simultaneously along the three sides of an equilateral triangle XYZ with its vertices on the perimeter of the disc (see figure). One second after applying the forces, the angular speed of the disc in rad s⁻¹ is



(JEE Advanced 2014)

Answers Key

EXERCISES

Single Correct Answer Type

1. (4)	2. (2)	3. (4)	4. (3)	5. (4)
6. (3)	7. (2)	8. (2)	9. (2)	10. (3)
11. (2)	12. (2)	13. (3)	14. (4)	15. (1)
16. (2)	17. (1)	18. (4)	19. (4)	20. (2)
21. (3)	22. (3)	23. (3)	24. (2)	25. (2)
26. (3)	27. (4)	28. (3)	29. (1)	30. (1)
31. (4)	32. (3)	33. (4)	34. (3)	35. (3)
36. (3)	37. (3)	38. (3)	39. (3)	40. (4)
41. (3)	42. (2)	43. (4)	44. (3)	45. (3)

46. (2) 47. (2) 48. (2) 49. (1) 50. (2)
51. (2) 52. (2) 53. (2) 54. (2) 55. (2)
56. (2) 57. (3) 58. (2) 59. (3) 60. (4)
61. (1) 62. (2) 63. (1) 64. (2) 65. (3)
66. (2) 67. (3) 68. (4) 69. (3) 70. (1)
71. (4) 72. (2) 73. (3) 74. (4) 75. (1)
76. (4) 77. (1) 78. (2) 79. (4)

Multiple Correct Answers Type

1. (2),(3) 2. (3),(4) 3. (1),(2),(3)
4. (1),(2),(3) 5. (1),(2),(3) 6. (1),(3)
7. (3),(4) 8. (1),(2),(3) 9. (1),(4)
10. (1),(3) 11. (2),(4) 12. (2),(3)
13. (1),(2),(4) 14. (1),(3) 15. (1),(3)
16. (1),(2),(3) 17. (1),(2),(3) 18. (1),(2),(3)
19. (1),(2),(4) 20. (1),(3)

Linked Comprehension Type

1. (3) 2. (4) 3. (4) 4. (2) 5. (3)
6. (4) 7. (2) 8. (4) 9. (1) 10. (2)
11. (1) 12. (4) 13. (3) 14. (1) 15. (3)
16. (2) 17. (4) 18. (3) 19. (3) 20. (1)
21. (4) 22. (4) 23. (1) 24. (4) 25. (2)
26. (3) 27. (3) 28. (1)

Matrix Match Type

1. i. → a.; ii. → d.; iii.→ a.; iv. → c.
2. i. → c.; ii. → a.; iii. → a.; iv. → b.
3. i. → b.; ii. → a.; iii. → c.; iv. → a., b., d.
4. i. → b.; ii. → d.; iii. → d.; iv. → a.

5. (2) 6. (1) 7. (3) 8. (2) 9. (2)
10. (4)

Numerical Value Type

1. (6) 2. (5) 3. (6) 4. (4) 5. (2.00)
6. (1.0) 7. (3.00) 8. (30.00) 9. (0.50) 10. (6.00)
11. (5.00) 12. (30) 13. (25.00) 14. (0.50) 15. (4.00)
16. (30.00) 17. (1.00) 18. (4) 19. (50.00) 20. (40.00)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3) 2. (2) 3. (4) 4. (3) 5. (3)
6. (3) 7. (1) 8. (2)

JEE Advanced

Single Correct Answer Type

1. (2) 2. (2) 3. (1) 4. (4)

Multiple Correct Answers Type

1. (2),(3) 2. (1),(2) 3. (3),(4)
4. (1),(3),(4) 5. (1),(3)

Linked Comprehension Type

1. (4) 2. (1)

Numerical Value Type

1. (4) 2. (9) 3. (3) 4. (8) 5. (2)

4 Rigid Body Dynamics: Part 2

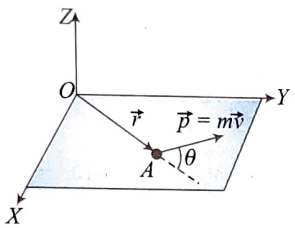
INTRODUCTION

Even though the force and torque are different quantities, let us try to discover some similarity between them. As we know, force can change the linear momentum. Following this logic, let us enquire if a torque can change something.

ANGULAR MOMENTUM

Just as the idea of the linear momentum helps us to analyze translational motion, a rotational analog **angular momentum** helps us to analyze the objects undergoing rotational motion.

Consider a particle of mass m located at the vector position $\vec{r}$ and moving with linear momentum $\vec{p}$ as shown in figure.



In describing translational motion, we found that the net force on the particle equals the time rate of change of its linear momentum,

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad ...(i)$$

Let us take the cross product of each side of Eq. (i) with $\vec{r}$, which gives the net torque on the particle on the left side of the equation:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Now let's add to the right side, which is zero because $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{v}$ and $\vec{p}$ are parallel.

Therefore, $\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$

We recognize the right side of this equation as the derivative of $\vec{r} \times \vec{p}$.

Therefore, $\vec{\tau} = \frac{d(\vec{r} \times \vec{p})}{dt}$ ...(ii)

which looks very similar in form to Eq. (i). Because torque plays the same role in rotational motion that force plays in translational motion, this result suggests that the combination $\vec{r} \times \vec{p}$ should play the same role in rotational motion that $\vec{p}$ plays in translational motion. We call this combination as the **angular momentum** of the particle.

The instantaneous angular momentum $\vec{L}$ of a particle relative to an axis through the origin O is defined by the cross product of the particle's instantaneous position vector $\vec{r}$ and its instantaneous linear momentum $\vec{p}$:

$$\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \qquad ...(iii)$$

As torque $(\vec{\tau} = \vec{r} \times \vec{F})$ is defined as the 'moment of force', angular momentum is also referred sometimes as 'moment of linear momentum'.

We can now write Eq. (ii) as,

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad ...(iv)$$

which is the rotational analog of Newton's second law, $\vec{F} = d\vec{p}/dt$. Torque causes the angular momentum $\vec{L}$ to change just as force causes linear momentum $\vec{p}$ to change.

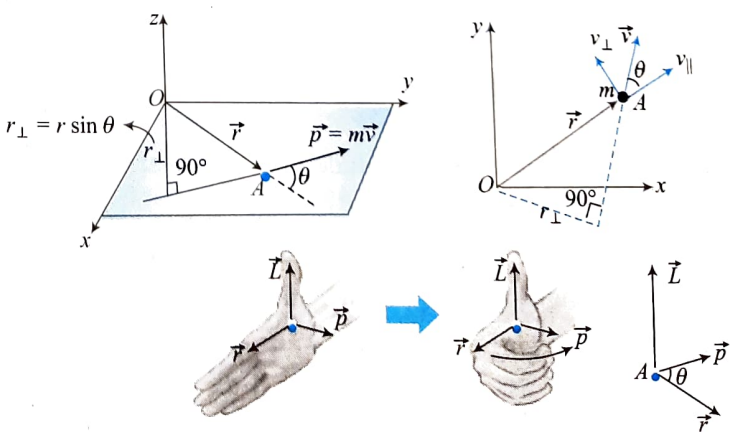
Note: Equation (iv) is valid only if $\vec{\tau}$ and $\vec{L}$ are measured about the same axis. Furthermore, the expression is valid for any axis fixed in an inertial frame.

The SI unit of angular momentum is kg m^2/s. Note that both the magnitude and the direction of $\vec{L}$ depend on the choice of axis. Regarding angular momentum, it is worth nothing that it is an axial vector (i.e., always perpendicular to the plane of motion), following the right-hand rule, we see that the direction of $\vec{L}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$. In figure, $\vec{r}$ and $\vec{p}$ are in the x-y plane, so $\vec{L}$ points in the z direction. Because $\vec{p} = m\vec{v}$ and θ is the angle between $\vec{r}$ and $\vec{p}$, we can write the magnitude of $\vec{L}$ as

$$L = mv(r\sin\theta) = mvr_\perp \qquad ...(v)$$

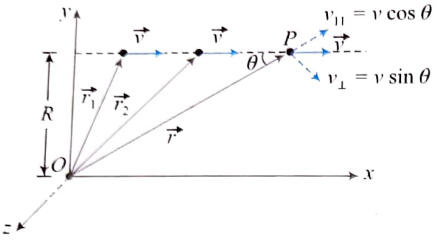
Also, $L = m(v\sin\theta)r = mv_\perp r$ ...(vi)

where $r_\perp$ is the perpendicular distance from O called lever arm of angular momentum and the direction of $\vec{p}$ and $v_\perp$ is the component of $\vec{v}$ perpendicular to $\vec{r}$.



CAN A PARTICLE MOVING IN A STRAIGHT LINE HAVE ANGULAR MOMENTUM?

Let a particle of mass m moving in a straight line with constant velocity as shown in figure. Generally, we think that when a particle moves in a straight line, how can it possess an "angular momentum" without undergoing angular motion or circular motion etc.? In other words, how can we use angular quantity like angular momentum when the particle moves in a straight line? Let us explain the idea behind this.



We can provide a logical interpretation of this by saying that when a particle moves in a straight line or any curve, its position vector rotates about a fixed point. Hence, we can say that the particle is turning about that point.

We can see that in the given figure, the particle P turns about the fixed-point O with an angular velocity given by

$$\omega_{PO} = \frac{v_{PO_\perp}}{r} = \frac{v_\perp}{r}$$

It means, the turning of the particle about O is decided by the component of its velocity perpendicular to $\vec{r}$, that is $v_\perp$. On the other hand, the particle approaches to (or) departs from the point O with a velocity $v_\parallel$.

Hence, the angular momentum of the particle

$$|\vec{L}| = mv_\perp r = m(v\sin\theta)r = mv(r\sin\theta) = mvR$$

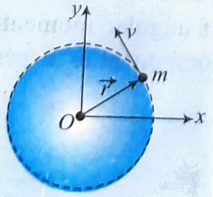
$\therefore$ $|\vec{L}|$ = constant as m, v and R all are constants.

But the direction of $\vec{r}\times\vec{v}$ also remains the same. Therefore, angular momentum of the particle about the origin remains constant with due course of time. In vector form, we can write

$$\vec{L} = mvR(\hat{k})$$

ILLUSTRATION 4.1

A particle moves in the x-y plane in a circular path of radius r as shown in figure. Find the magnitude and direction of its angular momentum relative to an axis through O when its velocity is $\vec{v}$.



Sol. Let's evaluate the magnitude of the angular momentum of a particle by

$$L = mvr\sin\theta$$

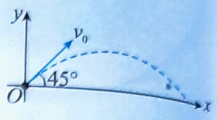
where θ is the angle between $\vec{r}$ and $\vec{p}$. Here in this case it is 90°.

Hence, $L = mvr\sin 90° = mvr$

This value of L is constant because all three factors on the right are constant. The direction of $\vec{L}$ also is constant, even though the direction of $\vec{p} = m\vec{v}$ keeps changing. To verify this statement, apply the right-hand rule to find the direction of $\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v}$. Your thumb points out of the page, so that is the direction of $\vec{L}$. Hence, we can write the vector expression $\vec{L} = (mvr)\hat{k}$. If the particle were to move clockwise, $\vec{L}$ would point downward and into the page and $\vec{L} = -(mvr)\hat{k}$. A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path.

ILLUSTRATION 4.2

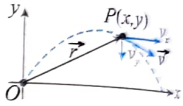
A particle is projected at time $t = 0$ from a point O with a speed v_0 at an angle of 45° to the horizontal. Find the magnitude and the direction of the angular momentum of the particle about the point O at time $t = v_0/g$.



Sol. In this case the angular momentum of the particle will not remain constant as there is always torque of gravity is acting. Let us take the origin at O, x-axis along the horizontal and y-axis along the vertically upward direction as shown in figure.

For horizontal motion during the time 0 to t,

$$v_x = v_0\cos 45° = \frac{v_0}{\sqrt{2}}$$



and $x = v_x t = \frac{v_0}{\sqrt{2}}\frac{v_0}{g} = \frac{v_0^2}{\sqrt{2}g}$

For vertical motion, using $v_y = v_{oy} - gt$, we get

$$v_y = v_0\sin 45° - g\left(\frac{v_0}{g}\right) = \frac{(1-\sqrt{2})}{\sqrt{2}}v_0$$

and $y = (v_0\sin 45°)t - \frac{1}{2}gt^2$

$$\Rightarrow\quad y = \frac{v_0}{\sqrt{2}}\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g}(\sqrt{2}-1)$$

The angular momentum of the particle at time t about the origin is

$$L = \vec{r}\times\vec{p} = m(\vec{r}\times\vec{v})$$

$$= (x\hat{i} + y\hat{j})\times(v_x\hat{i} + v_y\hat{j}) = m(xv_y\hat{k} - yv_x\hat{k})$$

$$= m\left[\left(\frac{v_0^2}{\sqrt{2}g}\right)\frac{v_0}{\sqrt{2}}(1-\sqrt{2}) - \frac{v_0^2}{2g}(\sqrt{2}-1)\frac{v_0}{\sqrt{2}}\right](\hat{k})$$

$$= \frac{mv_0^2}{2\sqrt{2}g}(-\hat{k})$$

Thus, the angular momentum of the particle is $mv_0^2/(2\sqrt{2}g)$ in the negative z-direction, i.e., perpendicular to the plane of motion, going into the plane.

ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

The total angular momentum of a system of particles about some axis is defined as the vector sum of the angular momenta of the individual particles about that axis:

$$\vec{L}_{tot} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_n = \sum_i \vec{L}_i$$

where the vector sum is over all n particles in the system.

Differentiating this equation with respect to time gives

$$\frac{d\vec{L}_{tot}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_i$$

The torques acting on the particles of the system are those associated with internal forces between particles and those associated with external forces. The net torque associated with all internal forces, however, is zero. In the summation, therefore, the net internal torque is zero. We conclude that the total angular momentum of a system can vary with time only if a net external torque is acting on the system. Therefore,

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}_{tot}}{dt}$$

The net external torque on a system equals the time rate of change of angular momentum of the system

ILLUSTRATION 4.3

A monkey of mass M is hanging through a light rope passes over a light, frictionless pulley. Other end of the rope is fastened to a bunch of bananas of mass M. The monkey climbs the rope in an attempt to reach the bananas. Will the monkey reach the bananas?

Sol. Assuming the rope is massless, the tension is the same on both sides of the pulley. Net torque

$$\sum \tau = TR - TR = 0$$

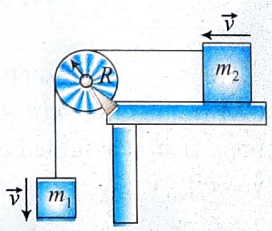
We know $\sum \tau = \frac{dL}{dt}$, and since $\sum \tau = 0, L =$ constant.

Since the total angular momentum of the system is initially zero, the total angular momentum remains zero, so the monkey and bananas move upward with the same speed at any instant.

The monkey will not reach the bananas. The motions of the monkey and bananas are identical, so the bananas remain out of the monkey's reach—until they get tangled in the pulley.

ILLUSTRATION 4.4

Two blocks '1'and '2' of masses m_1 and m_2 respectively are connected by a light cord that passes over a pulley as shown in figure. The radius of the pulley is R, and its moment of inertia about its axis of rotation is I. The block '2' slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.



Sol. Let us take the block, pulley, and sphere as a system for angular momentum, subject to the external torque due to the gravitational force on the block '1'. Let us calculate the angular momentum about an axis that coincides with the axle of the pulley. The angular momentum of the system includes that of two objects moving translationally (the block '1' and the block '2') and one object undergoing pure rotation (the pulley).

At any instant of time, the block '1' and the block '2' have a common speed v, so the angular momentum of the block '1' about the pulley axle

$$L_1 = m_1 vR \quad ...(i)$$

We have taken counter clockwise rotation as positive

Similarly, for block '2':

$$L_2 = m_2 vR \quad ...(ii)$$

At the same instant, all points on the rim of the pulley also move with speed v, so the angular velocity of the pulley is $\omega = \frac{v}{R}$.

The pulley is rotating about a fixed axis, hence the angular momentum of the pulley is

$$L_{Pulley} = I\omega = I\frac{v}{R} \quad ...(iii)$$

Now let's address the total external torque acting on the system about the pulley axle. The force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on the block '2'is balanced by the gravitational force $m_2\vec{g}$, so these forces do not contribute to the torque. The gravitational force $m_1\vec{g}$ acting on the block '1' produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. This result is the total external torque about the pulley axle, that is,

$$\sum \tau_{ext} = m_1 gR \quad ...(iv)$$

Write an expression for the total angular momentum of the system:

$$L_{total} = m_1 vR + m_2 vR + I\frac{v}{R} = \left(m_1 + m_2 + \frac{I}{R^2}\right)vR \quad ...(v)$$

Now using, $\sum \tau_{ext} = \frac{dL}{dt}$

Hence, $m_1 gR = \frac{d}{dt}\left(m_1 + m_2 + \frac{I}{R^2}\right)vR$

$$m_1 g = \left(m_1 + m_2 + \frac{I}{R^2}\right)\frac{dv}{dt} \quad ...(vi)$$

Recognizing that $dv/dt = a$, solve Eq. (vi) for a:

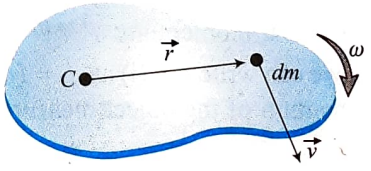
$$a = \frac{m_1 g}{m_1 + m_2 + \frac{I}{R^2}}$$

Important Point:

When we evaluated the net torque about the axle, we did not include the forces that the cord exerts on the objects because these forces are internal to the system under consideration. Instead, we analyzed the system as a whole. Only *external* torques contribute to the change in the system's angular momentum.

ANGULAR MOMENTUM OF A ROTATING RIGID OBJECT ABOUT AN AXIS PASSING THROUGH CENTRE OF MASS

For the sake of simplicity let us consider a plate and try to find its angular momentum about a centroidal axis. Here, we will discuss the cases when the axis is perpendicular to the plane of the plate.



To find the angular momentum, first we consider an elementary mass "dm" which moves with a velocity $\vec{v}$ relative to the centre of mass C. Since the body spins with angular speed ω, we have

$$v = r\omega$$

Then, the angular momentum of the mass "dm" about C is

$$d\vec{L}_C = dm\vec{r} \times \vec{v}$$

Since $\vec{v} \perp \vec{r}$ and $v = r\omega$, we have

$$d\vec{L}_C = dmr\{r\omega(-\hat{k})\} = (r^2 dm)\vec{\omega}$$

Integrating $d\vec{L}_C$, the total angular momentum is $\vec{L}_C = \int d\vec{L}_C$.

Then, $\vec{L}_C = \vec{\omega}\int r^2 dm$

Substituting $\int r^2 dm = I_C$, we have $\vec{L}_C = I_C\vec{\omega}$

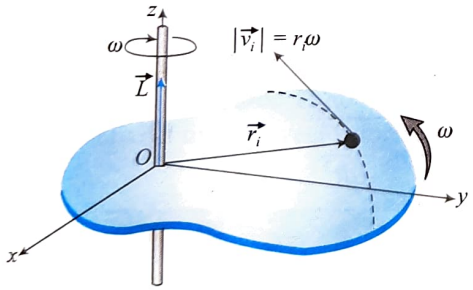
where I_C is called moment of inertia about centroidal axis.

L_C is called spin angular momentum. It is independent of coordinate system. It is an intrinsic property of a rotating body.

Angular Momentum of a Rotating Rigid Object About a Fixed Axis of Rotation

Consider a rigid object rotating about a fixed axis that coincides with the z-axis of a coordinate system as shown in figure. Let's determine the angular momentum of this object. Each particle of the object rotates in the x-y plane about the z-axis with an angular speed ω. The magnitude of the angular momentum of a particle of mass m_i about the z-axis is $m_i v_i r_i$. Because $v_i = r_i\omega$, we can express the magnitude of the angular momentum of this particle as

$$\vec{L}_i = m_i r_i^2 \vec{\omega}$$



The vector $\vec{L}_i$ for this particle is directed along the z-axis, as is the vector $\vec{\omega}$.

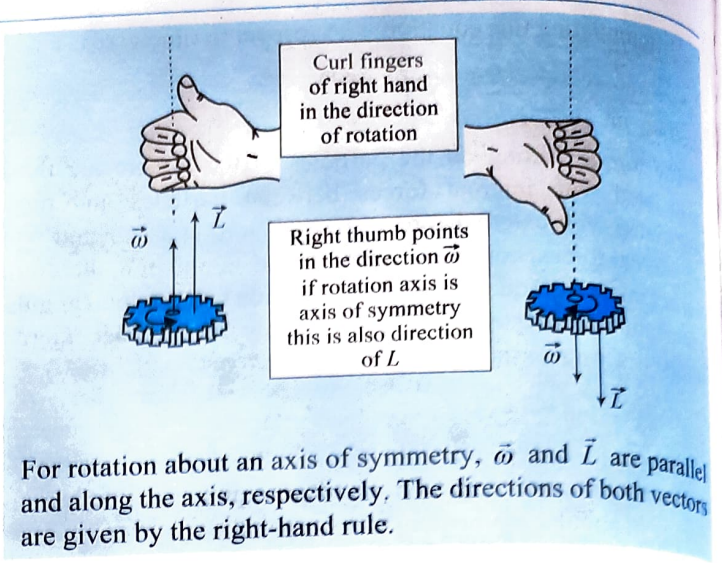
We can now find the angular momentum (which in this situation has only a z component) of the whole object by taking the sum of $\vec{L}_i$ over all particles:

$$\vec{L}_z = \sum_i \vec{L}_i = \sum_i m_i v_i^2 \vec{\omega} = \left(\sum_i m_i r_i^2\right)\vec{\omega}$$

$$\vec{L}_z = I\vec{\omega} \qquad \ldots(i)$$

where we have recognized $\Sigma\, m_i\, r_i^2$ as the moment of inertia I of the object about the z-axis. Notice that Eq. (i) is mathematically similar in form for linear momentum: $\vec{p} = m\vec{v}$

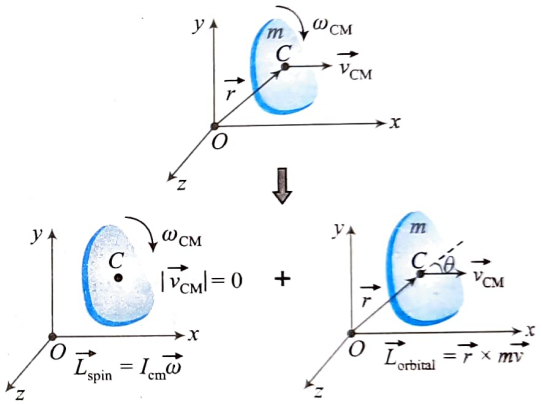
In vector form, we can write Eq. (i) as $\vec{L} = I\vec{\omega}$, where $\vec{L}$ is the total angular momentum of the object measured with respect to the axis of rotation.



For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis, respectively. The directions of both vectors are given by the right-hand rule.

Angular Momentum of a Rigid Body Due to Translation and Rotation Both

Consider a body of mass m is rotating with angular velocity ω about CM axis and translating with a linear velocity $\vec{v}$ as shown in figure.



The angular momentum of the body is given by

$$\vec{L} = \vec{L}_{spin} + \vec{L}_{orbital}$$

$$\vec{L} = \vec{L}_{spin} + m(\vec{r} \times \vec{v}_{cm}) = I_{cm}\vec{\omega} + m(\vec{r} \times \vec{v}_{cm})$$

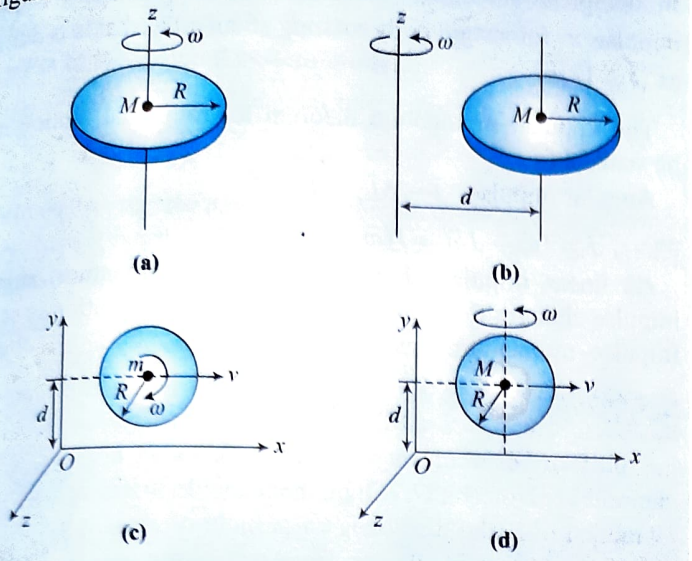
Here $\vec{L}_{spin}$ represents the angular momentum of the body about CM frame, and $m(\vec{r} \times \vec{v})$ equals the angular momentum of the body if it is assumed to be concentrated at CM translating with velocity $\vec{v}_{cm}$.

The first term "$I_{cm}\vec{\omega}$" appears because of spin of the rigid body. Hence, we call it "spin angular momentum" denoted by $\vec{L}_{spin}$. We can see that $\vec{L}_{spin}$ does not depend on the reference frame. This can also be termed as "internal angular momentum" because it arises due to the relative motion (velocity) between particles of the rigid body.

The second term "$\vec{r} \times m\vec{v}$" is named as "orbital angular momentum" denoted as $\vec{L}_{orbital}$ which depends on the position of CM relative to the reference frame.

ILLUSTRATION 4.5

Find the angular momentum of a disc about the axis shown in figure in the following situations.



Sol. **Case (a):** In this case the disc is rotating about fixed axis which is parallel to the z-axis and passing through its centre and perpendicular to the plane of the disc.

$$\text{Hence, } \vec{L} = I_z\omega(\hat{k}) = \left(\frac{MR^2}{2}\right)\omega(\hat{k})$$

Direction can be determined using right hand rule.

Case (b): In this case also the disc rotates about fixed axis which is not passing through the disc. We can write angular momentum for this situation.

$$\vec{L} = I_z\omega(\hat{k})$$

Using parallel axis theorem, we can find the value of

$$I_z = \frac{MR^2}{2} + Md^2$$

$$\text{Hence, } \vec{L} = \left(\frac{MR^2}{2} + Md^2\right)\omega(\hat{k})$$

Case (c): In this case the disc is rotating about an axis parallel to the z-axis and centre of mass of disc is also translating.

Hence, angular momentum

$$\vec{L} = \vec{L}_{spin} + \vec{L}_{orbital} = I_{cm}\vec{\omega} + M(\vec{r}\times\vec{v}_{cm})$$

$$\Rightarrow \vec{L} = \frac{MR^2}{2}\omega(-\hat{k}) + Mvd(-\hat{k}) = \left(\frac{MR^2}{2}\omega + Mvd\right)(-\hat{k})$$

Case (d): In this case also the disc is rotating about an axis passing through it centre and parallel to y-axis and centre of mass of disc is translating with velocity v parallel to x-axis.

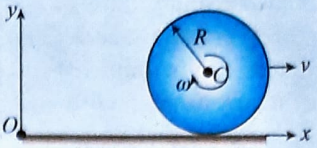
$$\vec{L} = I_{CM}\omega(\hat{j}) + Mvd(-\hat{k})$$

$$\text{Here, } I_{CM} = \frac{MR^2}{4}$$

$$\text{Hence, } \vec{L} = \frac{MR^2}{4}\omega(\hat{j}) + Mvd(-\hat{k})$$

ILLUSTRATION 4.6

A cylinder of mass m and radius R is moving on smooth horizontal surface as shown in figure.



(i) Find its angular momentum about point O on ground.
(ii) Find its angular momentum about point O on if cylinder is rolling without sliding.
(iii) What happens to this angular momentum if the cylinder is rotating in opposite direction but moving translationally in same direction?

Sol. In the given situation, cylinder has both translational and rotational motion. We find its angular momentum about any point by using the equation:

$$\vec{L} = \vec{L}_{spin} + \vec{L}_{orbital}$$

$$\vec{L} = \vec{L}_{spin} + m(\vec{r}\times\vec{v}_{cm}) = I_{cm}\vec{\omega} + m(\vec{r}\times\vec{v}_{cm})$$

(i) The magnitude of angular momentum of the cylinder about its axis of rotation passing through its centre of mass (called spin angular momentum) is given as

$$L_{spin} = I_c\omega = \left(\frac{1}{2}mR^2\right)\omega$$

For direction we can use right hand rule, the cylinder is rotating in clockwise sense, so writing spin angular momentum in vector form,

$$\vec{L}_{spin} = \left(\frac{1}{2}mR^2\right)\omega(-\hat{k})$$

the magnitude of angular momentum due to its translational motion

$$L_{orbital} = mvr_{\perp} = mvR$$

In vector form,

$$\vec{L}_{orbital} = mvR(-\hat{k})$$

Total angular momentum of cylinder about point O is given as

$$\vec{L}_{total} = \left(\frac{1}{2}mR^2\right)\omega(-\hat{k}) + mvR(-\hat{k}) = \left(\frac{1}{2}mR^2\omega + mvR\right)(-\hat{k})$$

$$\Rightarrow \vec{L}_{total} = \left(\frac{1}{2}mR^2\omega + mvR\right)(-\hat{k}) \quad \ldots(i)$$

(ii) If cylinder is rolling, $v = \omega R$ $\ldots(ii)$

Then from (i) and (ii), we get

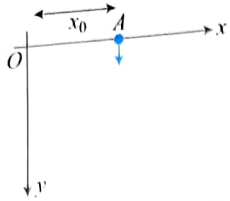
$$\vec{L}_{total} = \left(\frac{3}{2}mR^2\omega\right)(-\hat{k})$$

(iii) If cylinder rotates in anticlockwise direction and moving in same direction, its direction of spin angular momentum will change, and other factors will remain same as case (i). Total angular momentum of cylinder about point O is given as

$$\vec{L}_{total} = \frac{1}{2}mR^2\omega(\hat{k}) + mvR(-\hat{k})$$

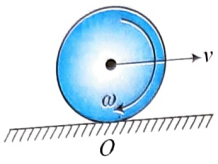
CONCEPT APPLICATION EXERCISE 4.1

1. A particle of mass m is released from rest at point A in figure, falling parallel to the (vertical) y-axis. Find the angular momentum of the particle at any time t with respect to the same origin O.



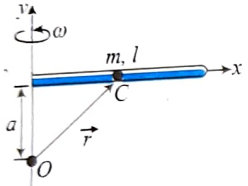
2. A particle of mass m is projected with a speed u at an angle θ to the horizontal at time $t = 0$. Find its angular momentum about the point of projection O at time t, vectorially. Assume the horizontal and vertical lines through O as X and Y axes, respectively.

3. A circular disc of mass m and radius R is set into motion on a horizontal floor with a linear speed v in the forward direction and an angular speed $\omega = v/R$ in the clockwise direction as shown in figure. Find the magnitude of the total angular momentum of the disc about the bottommost point O of the disc.

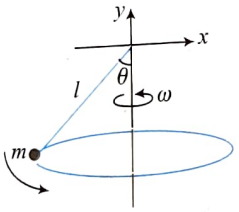


4. Calculate the angular momentum about point "O" on the axis of rotation in following cases.

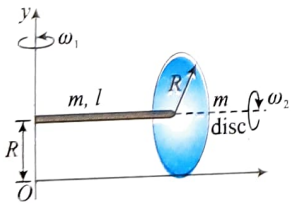
(i)



(ii)



5. Calculate the angular momentum in following situation. A disc is connected with a rod of mass m and length l. The disc is free to rotate about its own axis. If whole system is rotating with $\omega_1\hat{j}$ and disc also rotate about its own axis with $\omega_2\hat{i}$ then calculate $\vec{L}$ about origin.



6. A sphere rolls without slipping on a rough surface with centre of mass has constant speed v_0. If mass of the sphere is M and its radius be R, then find the angular momentum of the sphere about the point of contact.

ANSWERS

1. $mgx_0t(\hat{k})$ 2. $\frac{mu^2g\cos\theta}{2}(-\hat{k})$ 3. $\frac{3}{2}mvR$

4. (i) $\frac{ml^2}{3}\omega\hat{j} - \frac{mla\omega}{2}\hat{i}$ (ii) $-2m\omega l^2\cos\theta\sin\theta\,\hat{i} + ml^2\sin^2\theta\omega\,\hat{j}$

5. $\frac{4ml^2}{3}\omega_1\hat{j} - \left(\frac{mR^2}{2}\omega_2 - \frac{3}{2}m\omega_1 lR\right)\hat{i} + \frac{mR^2}{4}\omega_1\hat{j}$

6. $\frac{7}{5}Mv_0R(-\hat{k})$

ANGULAR IMPULSE: IMPULSE–MOMENTUM THEOREM

In complete analogue with the linear impulse, the angular impulse $\vec{J}$ for a rigid body rotating about a fixed axis is defined as $\vec{J} = \int\vec{\tau}\,dt$.

The impulse–momentum theorem for rotational motion may be stated as

Angular impulse, $\vec{J} = \Delta\vec{L}_f = \vec{L}_f - \vec{L}_i$

$\Rightarrow \quad \vec{J} = I\vec{\omega}_f - I\vec{\omega}_i = I(\vec{\omega}_f - \vec{\omega}_i)$

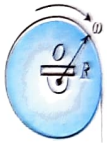
As linear impulse changes the linear momentum, angular impulse changes the angular momentum. This equation is called Impulse–momentum equation.

ILLUSTRATION 4.7

A uniform disc of mass M and radius R is smoothly pivoted at O. A light inextensible string wrapped over the disc hangs a particle of mass m. If the system is released from rest, assuming that the string does not slide on the disc, find the angular speed of the disc as the function of time using impulse–momentum equation.

Sol. The net torque about O is

$$\vec{\tau} = -mgR\hat{k} \qquad ...(i)$$



The angular momentum of the disc particle system about O is

$$L = I_0\omega + mvR, \text{ where } v = R\omega$$

$$L = \frac{MR^2}{2}\omega + mR^2\omega = \left(\frac{MR^2}{2} + mR^2\right)\omega$$

$$\vec{L} = \left(\frac{M+2m}{2}\right)R^2\omega\,(-\hat{k}) \qquad ...(ii)$$

Impulse–momentum equation is

$$\Delta\vec{L} = \int\vec{\tau}\,dt \qquad ...(iii)$$

Using Eqs. (i), (ii) and (iii), we have

$$\left(\frac{M+2m}{2}\right)R^2\omega = mgR\int_0^t dt$$

or $$\omega = \frac{2mgt}{(M+2m)R}$$

LAW OF CONSERVATION OF ANGULAR MOMENTUM

We know that the total linear momentum of a system of particles remains constant if the system is isolated, that is, if the net external force acting on the system is zero. We have an analogous conservation law in rotational motion:

For rotational motion, we define torque.

$$\vec{\tau} = \frac{d\vec{L}}{dt}\quad\left[\equiv\frac{d\vec{p}}{dt} = \vec{F}\right]$$

So, if the net torque on a particle (or system) is zero $\frac{d\vec{L}}{dt} = 0$, i.e., $\vec{L}$ = constant.

"The total angular momentum of a system is constant in both magnitude and direction if the net external torque acting on the system is zero, that is, if the system is isolated."

This statement is often called the principle of **conservation of angular momentum** and is the basis of the **angular momentum version of the isolated system model**.

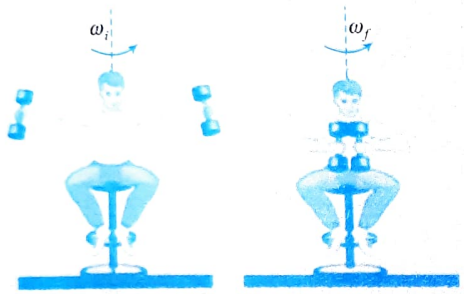
Hence, when the total angular momentum is conserved,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots = \text{constant}$$

This shows that the angular momentum of an individual particle may change, but their sum remains constant in the absence of an external torque.

ILLUSTRATION 4.8

A man sits on a freely rotating stool holding two dumbbells, each of mass 2.0 kg. When his arms are extended horizontally [Fig.(a)], the dumbbells are 1.0 m from the axis of rotation and the man rotates with an angular speed of 0.50 rad/s. The moment of inertia of the man plus stool is 5.0 kgm^2 and is assumed to be constant. The man pulls the dumbbells inward horizontally to a position 0.50 m from the rotation axis (Fig. (b)). Find the new angular speed of the man.



Sol. A man is seated on a stool that can rotate freely about a vertical axis with his arms stretched horizontally with equal weights in each hand. As no net external torque acts on the system that consist of the 'student, stool, and dumbbells'. It means, the angular momentum of that system about the rotation axis must remain constant. The total angular momentum of the system is conserved then,

$$\vec{L}_i = \vec{L}_f \quad \text{or} \quad I_i\omega_i = I_f\omega_f$$

Moment of inertia of the system,

$$I_{\text{total}} = I_{\text{student}} + I_{\text{weights}} = 5.0 + 2(mr^2)$$

Initially, $r_{\text{initial}} = 1.0$ m

Thus, $I_i = 5.0 + 2\times(2.0)(1.0)^2 = 9.0$ kg m^2

Finally, $r_{\text{final}} = 0.50$ m

Thus, $I_f = 5.0 + 2(2.0)(0.50)^2 = 6.0$ kg m^2

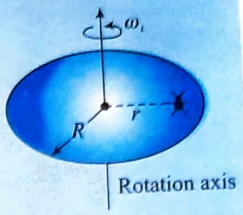
We now using conservation of angular momentum.

$$I_f\omega_f = I_i\omega_i$$

or $$\omega_f = \left(\frac{I_i}{I_f}\right)\omega_i = \left(\frac{9.0}{6.0}\right)(0.50) = 0.75 \text{ rad/s}$$

ILLUSTRATION 4.9

In figure, a cockroach with mass m rides on a disk of mass $6.00m$ and radius R. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.50$ rad/s. The cockroach is initially at radius $r = 0.800R$, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?



Sol. The cockroach's crawl changes the mass distribution (and thus the rotational inertia) of the cockroach-disk system. The angular momentum of the system does not change because there is no external torque to change it. (The forces and torques due to the cockroach's crawl are internal to the system.)

We want to find the final angular speed. Our key is to equate the final angular momentum L_f to the initial angular momentum L_i, because both involve angular speed. They also involve rotational inertia I. So, let's start by finding the rotational inertia of the system of cockroach and disk before and after the crawl.

The rotational inertia of a disk rotating about its central axis is given by $I_d = \frac{1}{2}MR^2$. Substituting $6.00m$ for the mass M, our disk here has rotational inertia,

$$I_d = 3.00mR^2 \qquad \text{...(i)}$$

We know that the rotational inertia of the cockroach (a particle) is equal to mr^2. Substituting the cockroach's initial radius ($r = 0.800R$) and final radius ($r = R$), we find that its initial rotational inertia about the rotation axis is,

$$I_{ci} = 0.64mR^2 \qquad \text{...(ii)}$$

and its final rotational inertia about the rotation axis is

$$I_{cf} = mR^2 \qquad \text{...(iii)}$$

So, the cockroach-disk system initially has the rotational inertia,

$$I_i = I_d + I_{ci} = 3.64mR^2 \qquad \text{...(iv)}$$

and finally has the rotational inertia,

$$I_f = I_d + I_{cf} = 4.00mR^2 \qquad \text{...(v)}$$

Next, we use equation ($L = I\omega$) to write the fact that the system's final angular momentum L_f is equal to the system's initial angular momentum L_i:

$$I_f\omega_f = I_i\omega_i \quad \text{or} \quad 4.00mR^2\omega_f = 3.64mR^2(1.50 \text{ rad/s})$$

After canceling the unknowns m and R, we come to,

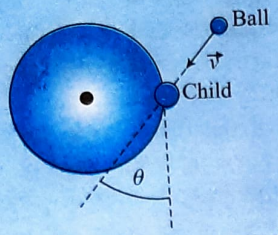
$$\omega_f = 1.37 \text{ rad/s}$$

Note that ω decreased because part of the mass moved outward, thus increasing that system's rotational inertia.

ILLUSTRATION 4.10

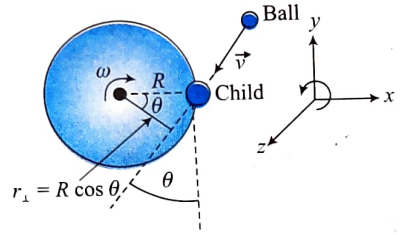
A child of mass 39 kg stands on the edge of a stationary merry-go-round of radius 2.5 m. The moment of inertia of the merry-go-round about its rotation axis is 150 kg m^2. The child catches a ball of mass 1.0 kg thrown by a friend. Just before the ball

is caught, it has a horizontal velocity of magnitude 50 m/s, at angle $\theta = 37°$ with a line tangent to the outer edge of the merry-go-round, as shown. What is the angular speed of the merry-go-round just after the ball is caught?



Top view of child on merry-go-round

Sol. There is no net external torque acts on the system that consist of the 'child, merry-go-round and ball'. It means, the angular momentum of that system about the rotation axis must remain constant.



We define the magnitude of angular momentum of a particle,

$$L = mvr_\perp$$

where $r_\perp$ is the perpendicular distance between the axis of rotation to line of action of $\vec{P}$.

Let m and v_0 be the mass and initial speed of the ball and R the radius of the merry-go-round. In this case,

$$r_\perp = R\cos 37° = 2.5 \times \frac{4}{5} = 2 \text{ m} \quad \text{...(i)}$$

The initial angular momentum is

$$L_i = mv_0 r_\perp = 1 \times 50 \times 2 = 100 \text{ kg m}^2/\text{s}$$

in clockwise direction. In vector form, we can write

$$\vec{L}_i = 100(-\hat{k}) \text{kg m}^2/\text{s}$$

After catching the ball, the system 'child, merry-go-round and ball' rotate with common angular velocity in clockwise direction to keep angular momentum conserve. Let this angular velocity be ω, then final angular momentum

$$L_f = I_{\text{system}}\omega \quad \text{...(ii)}$$

Now let us calculate moment of inertia of the system,

$$I_{\text{system}} = I_{\text{merry go round}} + I_{\text{child}} + I_{\text{ball}}$$

or, $I_{\text{system}} = 150 + (39)R^2 + (1.0)R^2 = 150 + 40R^2 = 400 \text{ kg m}^2$

Substituting the value of I_{system} in Eq. (i), we get

$$L_f = 400\omega$$

In vector form we can write,

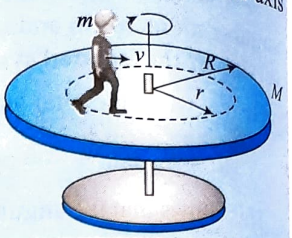
$$\vec{L}_f = 400\omega(-\hat{k}) \text{kg m}^2/\text{s} \quad \text{...(iii)}$$

As angular momentum is conserved then, $\vec{L}_i = \vec{L}_f$

or, $100(-\hat{k}) = 400\omega(-\hat{k})$, hence $\omega = 0.25$ rad/s.

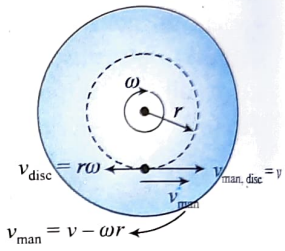
ILLUSTRATION 4.11

A man of mass m stands on a horizontal platform in the shape of a disc of mass M and radius R, pivoted on a vertical axis through its centre about which it can freely rotate. The man starts to move around the centre of the disc in a circle of radius r with a velocity v relative to the disc. Calculate the angular velocity of the disc.



Sol. If we take 'disc + man' as a system, there is no torque acting about the axis of rotation of the disc, so the angular momentum of the system remains constant. Initially it is zero. When the man starts rotating about the axes of the disc, to maintain zero angular momentum the disc start rotating in opposite sense of rotation of man. Suppose ω is the angular velocity of the disc in clockwise direction.

Let us take counter clockwise rotation as positive. The velocity of the man with respect to the ground will be:



$$\vec{v}_{\text{man, disc}} = \vec{v}_{\text{man}} - \vec{v}_{\text{disc}}$$

$$\Rightarrow \vec{v}_{\text{man}} = \vec{v}_{\text{man, disc}} + \vec{v}_{\text{disc}}$$

$$\Rightarrow |\vec{v}_{\text{man}}| = v - \omega r$$

Thus, angular momentum of the man $\vec{L}_{\text{man}} = m(v - \omega r)r\hat{k}$

and $\vec{L}_{\text{disc}} = I_{\text{disc}}\vec{\omega} = -\frac{MR^2}{2}\omega\hat{k}$

(taking counter clockwise rotation as positive)

By conservation of angular momentum, we have

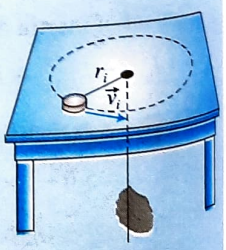
$$\vec{L}_{\text{initial}} = \vec{L}_{\text{final}} \Rightarrow 0 = m(v - \omega r)r - \frac{MR^2}{2}\omega$$

After solving, we get

$$\omega = \frac{mvr}{\left(mr^2 + \frac{MR^2}{2}\right)}$$

ILLUSTRATION 4.12

A puck of mass $m = 50.0$ g is attached to a taut cord passing through a small hole in a frictionless, horizontal surface (figure). The puck is initially orbiting with speed $v_i = 1.50$ m/s in a circle of radius $r_i = 0.300$ m. The cord is then slowly pulled from below, decreasing the radius of the circle to $r = 0.100$ m.



(a) What is the puck's speed at the smaller radius?

(b) Find the tension in the cord at the smaller radius.

Sol.

(a) The puck's linear momentum is always changing. Its mechanical energy changes as work is done on it. But its angular momentum stays constant because although an

external force (the tension of the rope) acts on the puck, no external torques act. Therefore, L = constant, and at any time, $mvr = mv_i r_i$ giving us,

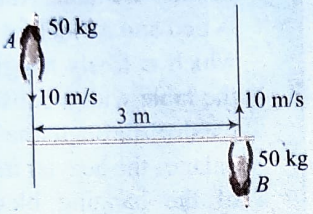
$$v = \frac{v_i r_i}{r} = \frac{(1.50\ \text{m/s})(0.300\ \text{m})}{0.100\ \text{m}} = 4.50\ \text{m/s}$$

(b) From Newton's second law, the tension is always

$$T = \frac{mv^2}{r} = \frac{(0.050\,\text{kg})(4.50\ \text{m/s})^2}{0.100\ \text{m}} = 10.1\ \text{N}$$

ILLUSTRATION 4.13

Two skaters, each of mass 50 kg, approach each other along parallel paths separated by 3 m. They have equal and opposite velocities of 10 m/s. The first skater carries a long light pole, 3 m long, and the second skater grabs the end of it as he passes (assume frictionless ice).

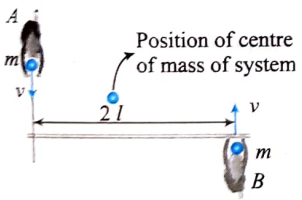


(a) Describe quantitatively the motion of the skaters after they are connected by the pole.

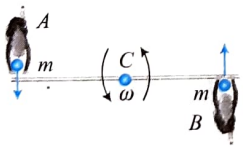
(b) By pulling on the pole the skaters reduce their distance to 1 m. What is their motion then?

Sol.

(a) If we take skater 'A', skater 'B' and pole as system there is no net external force acting on system, the linear momentum of the system will remain conserved. As the initial linear momentum of the system (skaters + pole) is zero, the centre of mass will be at rest before and after the collision. After connecting with pole, the skaters and the pole will rotate around the centre of mass (at the midpoint of the pole).



Just before collision



During collision

Applying the conservation of angular momentum about an axis through C and perpendicular to the plane of the figure,

$$mvl + mvl = I\omega \quad \text{where } I = 2m(l)^2$$

$$\omega = \frac{2mvl}{I} = \frac{v}{l} = \frac{20}{3}\ \text{rad/s}$$

(b) Now, the separation reduces to $2l' = 1$ m,

Again, applying conservation of angular momentum, we get

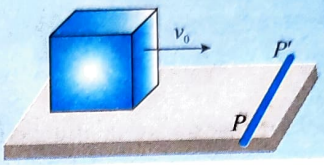
$$I\omega = I'\omega'$$

$$\omega' = \frac{I\omega}{I'} = \frac{2ml^2\omega}{2ml'^2} = \frac{l^2\omega}{l'^2} = 9\omega = 60\ \text{rad/s}$$

Therefore, angular velocity increases.

ILLUSTRATION 4.14

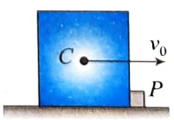
A cube of mass m and edge length l is sliding without friction on a smooth horizontal floor with velocity v_0, it strikes a long obstruction PP' of small height. The obstruction is parallel to the leading bottom edge of the cube. The leading bottom edge of the cube gets pivoted with the obstruction and the cube starts rotating about the edge. Determine angular velocity of the cube immediately after the impact.



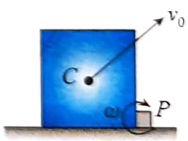
Sol. Before the impact, there is no external force in the horizontal direction and the cube slides with uniform velocity v_0, and during the impact reaction forces of the obstruction stops its leading bottom edge and cause it to rotate about its leading bottom.

During the impact external forces acting on the cube are its weight, the normal reaction from the ground and reaction from the obstruction. The weight and the normal reaction from the ground both are finite in magnitude and the impact ends in infinitesimally small time interval so their impulses and angular impulses about any axes are negligible. It is the reaction from the obstruction which has finite impulse during the impact. Its horizontal component changes the momentum of the cube during the impact, but its angular impulse about the obstruction is zero, therefore the angular momentum of the cube about an axis coincident with the leading bottom edge remain conserved.

Let the velocity of the mass center and angular velocity of the cube immediately after the impact are v_0 and ω_0. These velocities are shown in the adjacent figure.



Immediately before the impact



Immediately after the impact

We denote the angular momentum of the cube about axis coincident with the obstruction edge before and after the impact by L_{P1} and L_{P2}.

Applying principle of conservation of angular momentum about an axis coincident with the obstruction, we have

$$\vec{L}_{P1} = \vec{L}_{P2} \Rightarrow \vec{r}_{C/P} \times m\vec{v}_0 = I_P\vec{\omega}$$

Using theorem of parallel axes for moment of inertia I_P about the leading bottom edge, we get

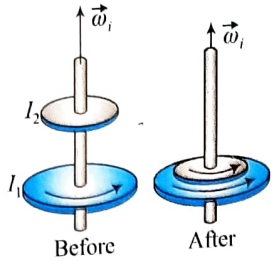
$$I_P = I_C + mr_{C/P}^2 = \frac{2}{3}ml^2$$

Substituting this in the above equation, we have

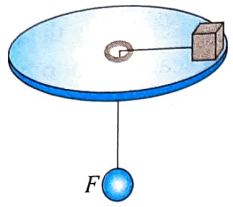
Angular velocity immediately after the impact, $\omega_0 = \dfrac{3v_0}{4l}$

CONCEPT APPLICATION EXERCISE 4.2

1. A disk with moment of inertia I_1 rotates about a frictionless, vertical axle with angular speed ω_i. A second disk, this one having moment of inertia I_2 and initially not rotating, drops onto the first disk (figure). Because of friction between the surfaces, the two eventually reach the same angular speed ω_f. calculate ω_f.

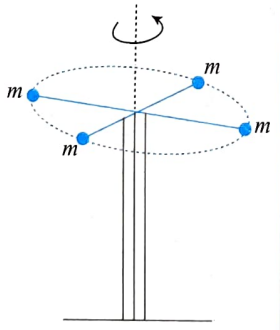


2. A small block of mass 4 kg is attached to a cord passing through a hole in a horizontal frictionless surface. The block is originally revolving in a circle of radius 0.5 m about the hole, with a tangential velocity of 4 m/s. The cord is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the cord is 600 N. What will be the radius of the circle when the cord breaks?



3. A man of mass 100 kg stands at the rim of a turntable of radius 2 m and moment of inertia 4000 kg m^2 mounted on a vertical frictionless shaft at its centre. The whole system is initially at rest. The man now walks along the outer edge of the turntable with a velocity of 1 m/s relative to the earth.
   (a) With what angular velocity and in what direction does the turntable rotate?
   (b) Through what angle will table have rotated when the man reaches his initial position on the turntable?
   (c) Through what angle will table have rotated when the man reaches his initial position relative to the earth?

4. The device shown in figure rotates on the vertical axle as shown. The frame has negligible mass as compared to the four masses each of mass m. Initial angular velocity of the system is ω_0. Due to an internal mechanism the spokes in the frame lengthen so that the radii of the masses become $2a$. Initially, it was a. What will be the new angular velocity of the system?



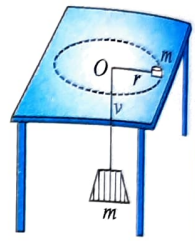
5. A girl jumps from a height h on the end of a see-saw. The see-saw consists of a uniform plank of length l pivoted at its centre. The plank is horizontal before the girl jumps. The mass of the see-saw is twice the mass of the girl. Find the angular velocity of the plank just after the girl jumps on the plank.

6. A child of mass m is standing on the periphery of a circular platform of radius R, which can rotate about its central axis. The moment of inertia of the platform is I. Child jumps off from the platform with a velocity u tangentially relative to the platform. Find the angular speed of the platform after the child jumps off.

7. Suppose in the previous problem, the child stays at rest on the platform and one of his friend throws a ball of mass m_1 towards him with a velocity of v from a direction tangential to the platform and the child on the platform catches the ball. Find the angular velocity of the platform after he catches the ball.

8. In the previous problem, initially the platform with the child is rotating with an angular speed ω_1. If the child starts walking along its periphery in opposite direction with speed u relative to the platform, what will be the new angular speed of the platform.

9. A thread is passing through a hole at the centre of a frictionless table. At the upper end a block of mass 0.5 kg is tied and a block of mass 8.0 kg is tied at the lower end which is freely hanging. The smaller mass is rotated on the table with a constant angular velocity about the axis passing through the hole so as to balance the heavier mass. If the mass of the hanging block is changed from 8.0 kg to 1.0 kg, what is the fractional change in the radius and the angular velocity of the smaller mass so that it balances the hanging mass again?



ANSWERS

1. $\frac{I_1}{I_1+I_2}\omega_i$ 2. 0.3 m

3. (a) 0.05 rad/s (b) $(2\pi/11)$ rad (c) 36°

4. $\frac{\omega_0}{4}$ 5. $\frac{6}{5}\sqrt{\frac{2gh}{l}}$ 6. $\frac{muR}{mR^2+I}$ 7. $\frac{m_1uR}{I+mR^2+m_1R^2}$

8. $\frac{(I+mR^2)\omega_1+muR}{I+mR^2}$ 9. 1, $\frac{3}{4}$

DYNAMICS OF COMBINATION OF TRANSLATION AND ROTATION

When the axis of rotation is not fixed (stationary in space), the motion of a rigid body is considered as combination of motion of centre of mass plus a rotation about centre of mass. We follow following steps for analyzing this type of motion

- For translational motion we write force equation:

$$\Sigma \vec{F}_{ext} = m\vec{a}_{CM}$$

- For rotational motion we write torque equation:

$$\Sigma \vec{\tau}_{CM} = I_{CM}\vec{\alpha}$$

Care should be taken for writing torque equation, as we can apply this equation either about center of mass or about a fixed axes or about instantaneous axis of rotation.

- When the number of variables exceed the number of equations (force and torque equations), then we write constraint equations.

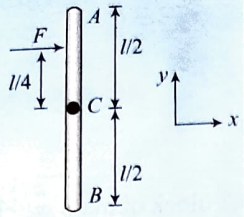
Here we are going to learn method through illustrations.

ILLUSTRATION 4.15

A thin uniform rod of mass m and length l is placed on smooth horizontal surface. It is acted upon by a force F as shown in the figure. Find:

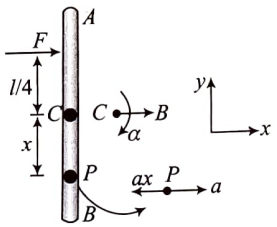
(i) the linear and angular acceleration of the rod.

(ii) the location of a point on the rod which has zero acceleration.



Sol.

(i) The external force acting on the rod in x-direction, hence the centre of mass of the rod will accelerate along x-direction. Also because of this force there will be net torque acting on the rod in clockwise sense, hence the motion of the rod will be combined rotational and translational.



Equation of motion for rod:

$$F = ma \Rightarrow a = \frac{F}{m} \quad \ldots(i)$$

$$\tau = I_c\alpha \Rightarrow F\frac{l}{4} = \frac{ml^2}{12}\alpha$$

which gives $\alpha = \dfrac{3F}{ml}$ ...(ii)

(ii) Point of zero acceleration on the rod should be between C and B, as the vectors due to acceleration due to translation and acceleration due to rotation act in opposite directions as shown in figure.

At point P, $a_P = a - \alpha x = \dfrac{F}{m} - \dfrac{3F}{ml}x$

If the acceleration of this point is zero then

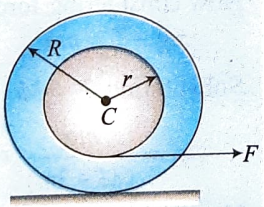
$$a_P = 0 = \frac{F}{m} - \frac{3F}{ml}x \text{ or } x = \frac{l}{3}$$

ILLUSTRATION 4.16

A stepped disc of mass m inner and outer radius r and R respectively is placed on a smooth horizontal surface. It is wrapped by a thread which is pulled by a horizontal force F. If the radius of gyration of the disc is k, find:

(i) the acceleration of CM of the disc

(ii) the angular acceleration of the disc.

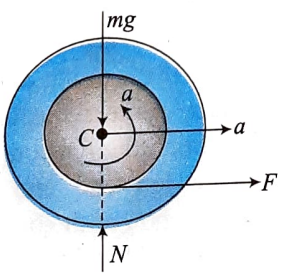


Sol. The motion of the disc is combined rotation and translation.

The disc will move in horizontal direction and it will rotate in counter clockwise sense.

(i) Force equation:

$$F = ma \Rightarrow a = \frac{F}{m}$$

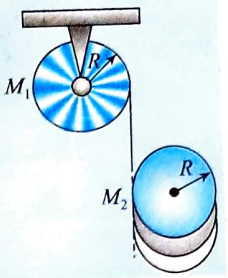


(ii) Torque equation about C, $\tau = I_c\alpha$

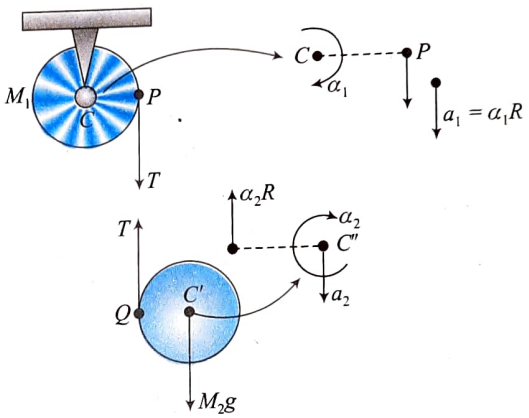
$$\Rightarrow F \cdot r = (mk^2)\alpha \Rightarrow \alpha = \frac{Fr}{mk^2}$$

ILLUSTRATION 4.17

An extensible string is wound over a rough pulley of mass M_1 and radius R and a cylinder of mass M_2 and radius R such that as the cylinder rolls down, (refer figure) the string unwound over the pulley as well the cylinder. Find the acceleration of cylinder.



Sol. Through this illustration we will learn the application of torque equation and constraint relation in more complex case here. Pulley M_1 will have only pure rotation while cylinder M_2 will have rotation and translation combined. Let us analyse step by step in the same way as the previous illustration.



Step I: Analyse the motions of the pulley and the cylinder.

Pulley: rotational acceleration α_1 (clockwise)

Cylinder: rotational acceleration α_2 (clockwise) and a linear acceleration a_2 (downward).

Step II: Equation of motion for M_2

$$M_2g - T = M_2a_2 \quad \ldots(i)$$

Step III: Torque equation for pulley: $\tau_c = I_c\alpha$

$$TR = \left(\frac{M_1R^2}{2}\right)\alpha_1 \Rightarrow \alpha_1 = \frac{2T}{M_1R} \quad \ldots(ii)$$

Torque equation for the cylinder (about centre of mass of cylinder):

$$\tau_c = I_c\alpha$$

$$TR = \left(\frac{M_2R^2}{2}\right)\alpha_2 \Rightarrow \alpha_2 = \frac{2T}{M_2R} \quad \ldots(iii)$$

Step IV: The acceleration of P and Q should be equal as both are connected with the same inextensible string.

Acceleration of P, $a_P = \alpha_1R$ (downwards) Acceleration of Q,

$a_Q = a_2 - \alpha_2R$ (downwards)

Hence, constraint relation: $\alpha_1R = a_2 - \alpha_2R$ ...(iv)

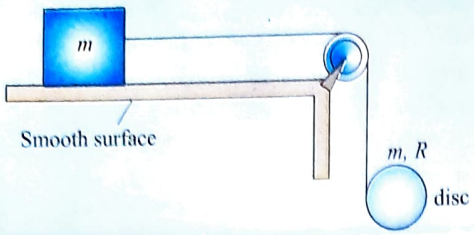
Step V: Solving equations:

After solving Eqs. (i) (ii), (iii) and (iv), we get

$$a_2 = \left[\frac{2(M_1 + M_2)}{3M_1 + 2M_2}\right]g$$

ILLUSTRATION 4.18

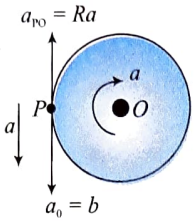
A block of mass m slides on a smooth horizontal surface. It is connected with a disc of same mass and radius R, through a light string which passes through light pulley as shown in figure. Assuming pulley to be massless and frictionless and thread inextensible, calculate acceleration of centre of the disc.



Sol. The motion of the block is pure translational and that of disc is translational and rotational. Let acceleration of block and disc be a and b respectively and angular acceleration of disc be α (clockwise). The acceleration of point P on the rim should be equal to the magnitude of the acceleration of the block.

$$\vec{a}_P = \vec{a}_{P,O} + \vec{a}_O$$

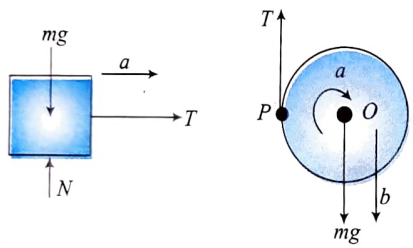
$$\Rightarrow \quad a = (-R\alpha) + b$$



Hence the net downward acceleration of centre of disc will be equal to,

$$b = (a + R\alpha) \qquad ...(i)$$

Now consider FBD of block and disc



For vertical equilibrium of the block, $N = mg$

For horizontal forces, $T = ma$ ...(ii)

Here the axis of rotation of moving, hence we can apply torque equation about centroidal axis of the disc,

$$T \cdot R = I_O \alpha$$

$$T \cdot R = \frac{mR^2}{2}\alpha \quad \text{or} \quad T = \frac{1}{2}mR\alpha \qquad ...(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$R\alpha = 2a \text{ or } b = 3a \qquad ...(iv)$$

Now analyzing vertical forces, acting on the disc,

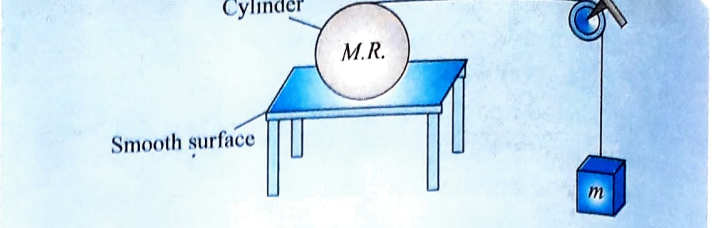
$$mg - T = mb \qquad ...(v)$$

From (ii), (iv) and (v) we get

$$a = \frac{g}{4} \text{ and } b = \frac{3g}{4}$$

ILLUSTRATION 4.19

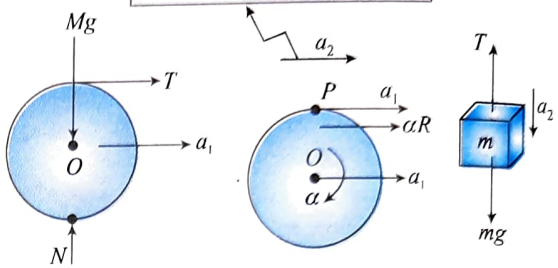
An inextensible thread is wound round a cylinder of mass $M = 7.0$ kg and radius $R = 10$ cm. The cylinder is placed over a smooth horizontal surface and thread is passed over a massless and frictionless pulley such that the part of the thread between cylinder and pulley is horizontal as shown in figure.



A block of mass $m = 1.0$ kg is attached with free end of thread. If the system is released from rest, calculate acceleration of the block. ($g = 10$ m s^{-2}).

Sol. As horizontal surface is smooth, and force is acting on the cylinder, therefore the cylinder will slip over the surface.

$$\vec{a}_P = \vec{a}_{PO} + \vec{a}_O$$
$$\Rightarrow a_P = a_2 = R\alpha + a_1$$



FBD of cylinder and block

Let acceleration of axis of the cylinder be a_1 (rightward) and its angular acceleration be α(clockwise) then downward acceleration of the block will be equal to,

$$a_2 = a_1 + R\alpha$$

Taking moment of forces acting on the cylinder about O.

$$T \cdot R = I\alpha \quad \text{where } I = \frac{MR^2}{2}$$

$$T = \frac{1}{2}MR\alpha \qquad ...(i)$$

For horizontal forces on the cylinder,

$$T = Ma_1 \qquad ...(ii)$$

From Eqs. (i) and (ii)

$$R\alpha = 2a_1$$

hence $a_2 = 3a_1$

Now considering forces acting on the block,

$$mg - T = ma_2$$

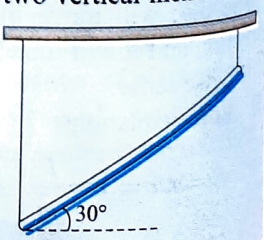
Substituting $T = Ma_1$ and $a_2 = 3a_1$

$$a_1 = \frac{mg}{(M + 3m)} \text{ or } a_1 = 1\,\text{m s}^{-2}$$

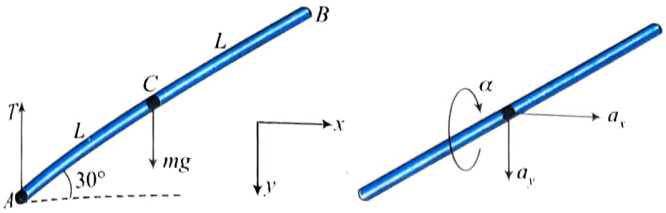
Hence, acceleration of the block, $a_2 = 3$ m s^{-2}

ILLUSTRATION 4.20

A thin uniform bar of mass m and length $2L$ is held at an angle 30° with the horizontal by means of two vertical inextensible strings, at each end as shown in figure. If the string at the right end breaks, leaving the bar to swing, determine the tension in the string at the left end and the angular acceleration of the bar immediately after string breaks.



Sol. Just after cutting the string the tension of right string will disappear and tension in left string will change and motion of the rod will be combined rotational and translational. Let us draw free body diagram of rod immediately after the string breaks.



Let a_x and a_y be the linear acceleration of centre of mass and α the angular acceleration of the rod.

As no force in horizontal direction. Then

$$\Sigma F_x = 0$$

$$\therefore \quad a_x = 0$$

$$\therefore \quad a_y = \frac{\Sigma F_y}{m} = \frac{mg - T}{m} \qquad ...(i)$$

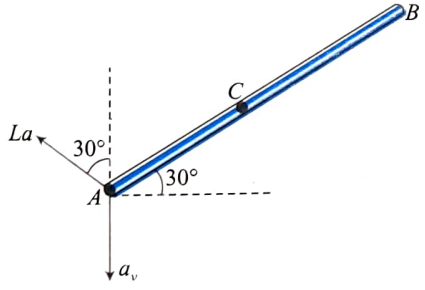
The torque equation is applicable about a fixed point or about centre of mass. In this case we have no fixed point, hence applying torque equation about center of mass

$$\tau = I_{cm}\alpha \;\Rightarrow\; T \cdot (L\cos 30°) = \frac{m(2L)^2}{12} \cdot \alpha$$

$$\Rightarrow \quad \alpha = \frac{\tau}{I_{cm}} = \frac{TL\cos 30°}{m(2L)^2/12} = \frac{3\sqrt{3}T}{2mL} \qquad ...(ii)$$

Now just after the string breaks, acceleration of point A in the vertical direction should be zero as the string is inextensible.

i.e., $\quad a_y = L\alpha\cos 30° \qquad ...(iii)$



Solving Eqs. (i), (ii), and (iii), we get

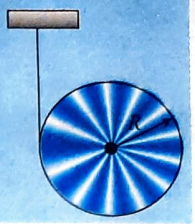
$$T = \frac{4mg}{13} \text{ and } a = \frac{9g}{13}$$

Applying Torque Equation About Instantaneous Centre of Rotation (ICR)

The torque equation is applicable about centroidal axes or about a fixed axis. In case of combined rotation and translation motion we apply torque equation about centroidal axes. We know instantaneous centre of rotation (ICR) is a point which can be consider stationary at certain instant. It means we can consider the axes passing through ICR as a fixed axis and apply torque equation. In this way we can solve the problems of combined rotational and translational motion in lesser steps. Let us learn this through following illustrations.

Illustration 4.21

A uniform disc of mass M has a thin string wrapped several times around its circumference. The string is fixed at one end and the disc is released. Determine the magnitude of the downward acceleration of the mass as it falls.

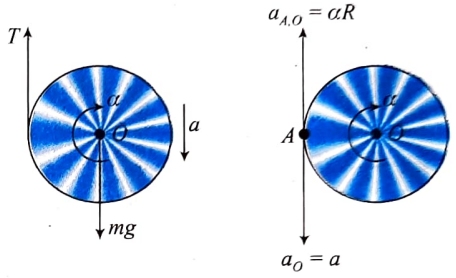


Sol. **Method 1:** The motion of the disc will be combined rotational and translational. Let a be the linear acceleration of centre of mass and α the angular acceleration of the disc.

The equation of motion of disc is

$$mg - T = ma \qquad ...(i)$$

The torque equation is applicable about a fixed axis or about centroidal axes.



Now writing torque equation for the disc (about centroidal axes of disc):

$$\tau_c = I_c\alpha \;\Rightarrow\; TR = \frac{mR^2}{2}\alpha$$

or $\quad T = \frac{1}{2}mR\alpha \qquad ...(ii)$

Let us now write constraint relation. As one end of the string is fixed, hence along the length of the string, acceleration must be zero. It means the acceleration of a point A situated on the disc should be zero i.e., the resultant of a_{CM} and $R\alpha$ is zero, i.e.,

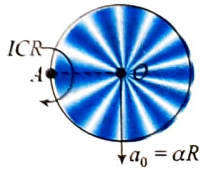
$$a = R\alpha \qquad ...(iii)$$

From (ii) and (iii),

$$T = \frac{1}{2}ma \qquad ...(iv)$$

From (i) and (iv), $a = \frac{2}{3}g$

Method 2: The string is fixed at one end it means point A, the contact point of string with disc as shown in figure, should not move. It means point A is instantaneous centre of rotation (ICR) of the disc. We can apply the torque equation about ICR. Now writing torque equation for the disc (about point A):



$$\tau_A = I_A\alpha \;\Rightarrow\; mgR = I_A\alpha \qquad ...(i)$$

In this case moment of inertia is obtained from the parallel axis theorem.

$$I_A = I_{CM} + mR^2 = \frac{mR^2}{2} + mR^2 = \frac{3}{2}mR^2 \qquad ...(ii)$$

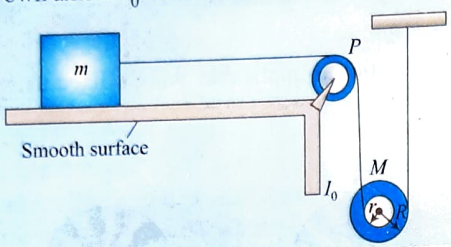
$$\Rightarrow \quad mgR = \frac{3}{2}mR^2\alpha \text{ or } \alpha = \frac{2g}{3R} \qquad ...(iii)$$

Hence the acceleration of CM,

$$a_C = R\alpha = R\left(\frac{2g}{3R}\right) = \frac{2g}{3}$$

ILLUSTRATION 4.22

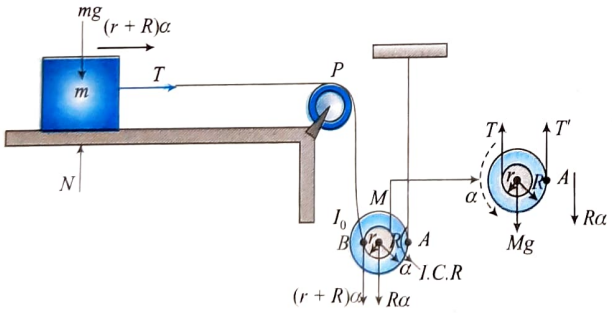
A block of mass m placed over a smooth horizontal surface is connected with a stepped pulley through a light string which passes through a massless and frictionless pulley 'P' as shown in figure. Inner and outer radii of the stepped pulley are r and R respectively. Mass of the pulley is M and its moment of inertia about its own axis is I_0. Calculate the acceleration of the block.



Sol. Since upper end of the right thread is fixed it means point A, the contact point of string with disc as shown in figure, should not move. It means point A is instantaneous centre of rotation (ICR) of the disc. Therefore, instantaneous axis of rotation will pass through A. We can apply the torque equation about ICR.

Let angular acceleration of the pulley be α (anticlockwise), then downward acceleration of its centre will be equal to $R\alpha$ and rightward acceleration of the block will be equal to $(r + R)\,\alpha$.

FBD of pulley and block will be as shown in figure.



For the block, $N = mg$

and $T = ma_{block} = m(r+R)\alpha$ ...(i)

Now writing torque equation for the disc (about point A):

$$Mg \cdot R - T(r+R) = I_A\alpha$$

where, $I_A = (I_0 + MR^2)$

$$MgR - T(r+R) = (I_0 + MR^2).\alpha \quad \text{...(ii)}$$

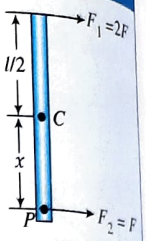
From Eqs. (i) and (ii), $\alpha = \dfrac{MgR}{I_0 + MR^2 + m(r+R)^2}$

Since, acceleration of the block is equal to $(r + R)\alpha$,

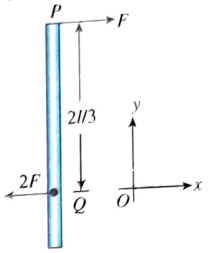
Therefore, it is equal to, $a_{block} = \dfrac{MgR(r+R)}{I_0 + MR^2 + m(r+R)^2}$

CONCEPT APPLICATION EXERCISE 4.3

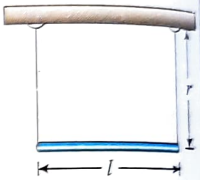
1. A uniform rod of mass m and length l is acted upon by the forces F_1 and F_2. Find that:
   (a) linear and angular acceleration of the rod.
   (b) value of x for which the point P does not accelerate.



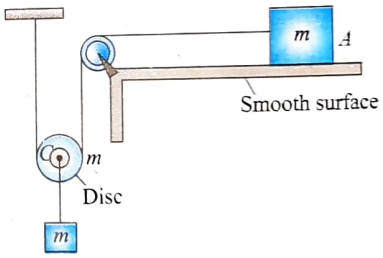
2. Find a_C and α of the smooth rod of mass m and length l.



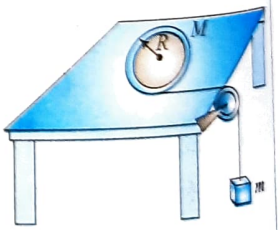
3. A uniform rod of mass M and length l is suspended by two vertical inextensible strings as shown in figure. Calculate tension in left string at the instant, when right string snaps.



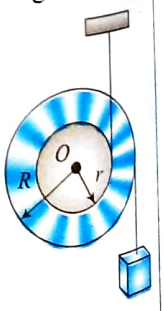
4. In the figure A and B are identical blocks, each of mass m and C is a circular disc of equal mass m. Pulley is massless and frictionless and thread is inextensible. Neglecting friction, find the acceleration of block A and B.



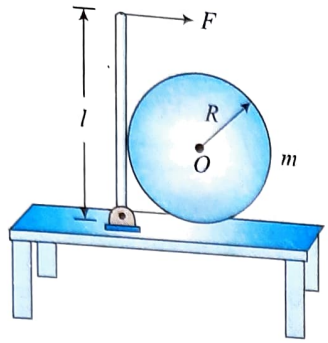
5. A light thread has been tightly wrapped around a disc of mass M and radius R. The disc has been placed on a smooth table, lying flat as shown. The other end of the string has been attached to a mass m as shown. The system is released from rest. Find the acceleration of the disc and block.



6. In the arrangement shown, the double pulley has a mass M and the two massless threads have been tightly wound on the inner (radius = r) and outer circumference (radius $R = 2r$). The block shown has a mass 4 M. The moment of inertia of the double pulley system about a horizontal axis passing through its centre and perpendicular to the plane of the figure is $I = \dfrac{Mr^2}{2}$. Find the acceleration of the center of the pulley after the system is released.



7. A uniform rod of mass m and length l which can rotate freely in vertical plane without friction, is hinged at its lower end on a table. A sphere of mass m and radius $R = l/3$ is placed in contact with the vertical rod and a horizontal force F is applied at the upper end of the rod.



(a) Find the acceleration of the sphere just after the force starts acting.

(b) Find the horizontal component of hinge reaction acting on the rod just after force F starts acting.

ANSWERS

1. (a) $\frac{3F}{m}, \frac{12F(l-x)}{ml^2}$ (b) $\frac{l}{2}$ 2. $\frac{F}{m}, \frac{8F}{ml}$ 3. $Mg/4$

4. 8g/13 5. $a_{block} = \frac{3mg}{3m+M}$ and $a_{disc} = \frac{mg}{3m+M}$

6. 6g/11 7. (a) $3F/4m$ (b) $7F/8$

DYNAMICS OF ROLLING MOTION

Before going to learn dynamics of rolling motion let us revise, what is rolling? Rolling can be viewed as a combination, or superposition, of purely translational motion (moving a wheel from one place to another with no rotation) and purely rotational motion (only rotation with no movement of the center of the wheel). The point of contact of the rolling object does not slide with respect to the surface on which it is moving. This is also called pure rolling. The situations of combined rotation and translation where point of contact slide on the surface is called rolling with slipping. There is a special connection between the translational component of the motion and the rotational component. Here we will learn the cases of dynamics of rolling into two categories

(i) Rolling of an object without net external force

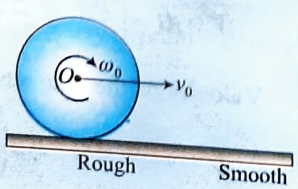
(ii) Rolling of an object under net external force

ROLLING OF AN OBJECT WITHOUT NET EXTERNAL FORCE

Let us first take rolling of an object without net external force, for understanding the concept, let us take some illustrations.

ILLUSTRATION 4.23

A rolling body of mass m and radius R is rolling on a rough surface without any pulling force as shown in figure. What happens when it enters on a smooth surface?



Sol. The point of contact of the rolling object does not slide with respect to the surface on which it is moving. It means when body is rolling on the rough surface no friction force acts on it.

When the body enters the smooth surface. In this case, the linear speed v_0 also remains constant. The body keeps on rolling on the smooth horizontal surface and the point of contact is stationary with respect to the surface.

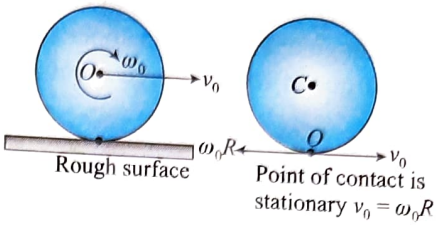
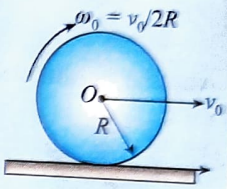
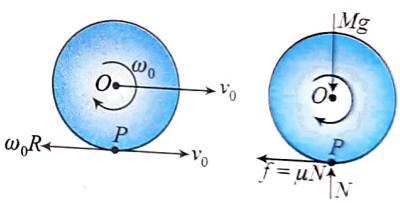


ILLUSTRATION 4.24

A sphere of mass M and radius R shown in the figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 and rotational velocity about the centre $v_0/2R$. Find the translational velocity after the sphere starts pure rolling.



Sol. **Method 1. Force/torque method:** It is given at certain instant, the velocity of the centre of mass of the disc is v_0 and the angular velocity about the centroidal axes is $v_0/2R$.



As, $v_0 > \omega_0 R$, the point of contact 'P' of sphere with ground will slips forward and thus the friction by the plane will act backward on the sphere. This friction will be kinetic in nature and the sphere will be decelerated by

$$a_{cm} = \frac{f}{M} = \frac{\mu Mg}{M} = \mu g \quad \ldots(i)$$

Hence velocity of centre of mass of the sphere at any time t is

$$v(t) = v_0 - \mu g t \quad \ldots(ii)$$

This friction will also produce a torque about the centroidal axes. This torque is clockwise and in the direction of ω_0. Hence the angular velocity of the sphere will increase. Here we can apply torque equation about centroidal axes

$$\tau_O = I_O \alpha \Rightarrow \mu Mg \times R = \frac{2}{5} MR^2 \alpha \quad \ldots(iii)$$

From Eq. (iii), the angular acceleration of the sphere, $\alpha = \frac{5\mu g}{2R}$ and the angular velocity at time t will be

$$\omega(t) = \omega_0 + \alpha t = \frac{v_0}{2R} + \frac{5\mu g}{2R} t$$

The pure rolling starts when $v(t) = R\omega(t)$

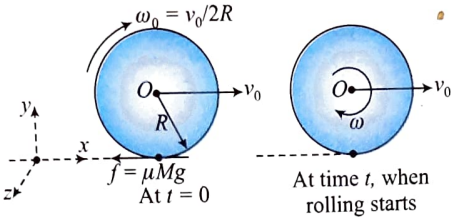
i.e., $$v_0 - \mu g t = R\left(\frac{v_0}{2R} + \frac{5\mu g}{2R} t\right) \Rightarrow t = \frac{v_0}{7\mu g}$$

On substituting the value of t in Eq. (ii), we get

$$v(t) = v_0 - \mu g\left(\frac{v_0}{7\mu g}\right) = \frac{6v_0}{7}$$

Thus, the sphere rolls with translational velocity $6v_0/7$ in the forward direction.

Method 2. By conservation of angular momentum: We can solve this problem by using conservation of angular momentum. We can conserve the angular momentum about any point situated on the ground. (As torque of friction force about any point on the ground is zero.)



Angular momentum about any point O situated on ground at $t = 0$

$$\vec{L}_{\text{initial}} = \vec{L}_{CM} + \vec{r} \times m\vec{v} = \frac{2}{5}MR^2\omega_0(-\hat{k}) + Mv_0R(-\hat{k})$$

$$= \left[\frac{2}{5}MR^2\left(\frac{v_0}{2R}\right) + Mv_0R\right](-\hat{k})$$

$$\vec{L}_{\text{initial}} = \frac{6}{5}Mv_0R(-\hat{k}) \quad \text{...(i)}$$

Angular momentum at time t when rolling starts

$$\vec{L}_{\text{final}} = \vec{L}_{CM} + \vec{r} \times m\vec{v} = \frac{2}{5}MR^2\omega(-\hat{k}) + MvR(-\hat{k})$$

When rolling starts $v = \omega R$

Hence, $\vec{L}_{\text{final}} = \left[\frac{2}{5}MR^2\left(\frac{v}{R}\right) + MvR\right](-\hat{k})$

$$\Rightarrow \quad \vec{L}_{\text{final}} = \frac{7}{5}MvR\,(-\hat{k}) \quad \text{...(ii)}$$

Conserving angular momentum, we get

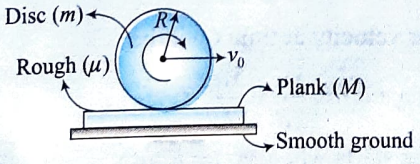
$$\vec{L}_{\text{initial}} = \vec{L}_{\text{final}}$$

$$\frac{6}{5}Mv_0R(-\hat{k}) = \frac{7}{5}MvR(-\hat{k})$$

which gives $v = \frac{6}{7}v_0$

ILLUSTRATION 4.25

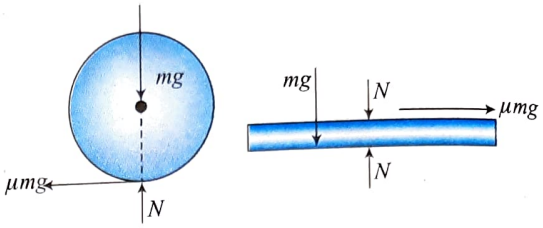
A plank of mass M, whose top surface is rough with coefficient of friction μ is placed on a smooth ground (figure). Now a disc of mass $m = M/2$ and radius r is placed on the plank. The disc is now given a velocity v_0 in the horizontal direction, at $t = 0$.



(a) Find the time when the disc starts rolling.
(b) Find the velocity of the plank and the disc up to that time.
(c) Find the distance travelled by the plank up to this instant.

Sol.

(a) Initially the point of contact of the disc will slide on the plank, hence friction will be of kinetic nature and act on the disk in backward direction. It means the friction on the plank will act in forward direction.



Equation of motion

For the plank: $\mu mg = MA$

$$\Rightarrow A = \frac{\mu mg}{M} \text{ (towards right)} \quad \text{...(i)}$$

$$\Rightarrow A = \frac{\mu g}{2} \quad (\text{as } m = M/2)$$

For the disc: $\mu mg = ma$

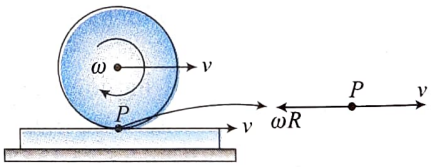
$a = \mu g$ (backwards, towards left)

Torque equation about centre of mass of the disc

$$\mu mgR = I_C\alpha = \frac{mR^2}{2}\alpha$$

$$\Rightarrow \alpha = \frac{2\mu g}{R} \text{ (clockwise, will be positive)}$$

Now let us write constraint relations. At the time of rolling, point P should not slide with respect to the plank.



$$V = v - \omega R$$

$$(0 + At) = (v_0 - at) - (0 + \alpha t)R$$

Now substituting the values of A and a, we can calculate 't'.

$$\frac{\mu mg}{M}t = (v_0 - \mu g t) - \left(\frac{2\mu g}{R}t\right)R$$

$$\left(\frac{\mu mg}{M} + \mu g + 2\mu g\right)t = v_0$$

which gives

$$t = \frac{v_0}{\mu g\left(\frac{m}{M} + 3\right)} = \frac{2v_0}{7\mu g} \quad (\text{as } m = M/2)$$

(b) Velocity of the plank when rolling starts,

$$V = At = \left(\frac{\mu g}{2}\right)\left(\frac{2v_0}{7\mu g}\right) = \frac{v_0}{7}$$

Velocity of the disc when rolling starts,

$$v = v_0 - at = v_0 - \mu g\left(\frac{2v_0}{7\mu g}\right) = \frac{5}{7}v_0$$

Angular velocity at the time of rolling,

$$\omega = \alpha t = \left(\frac{2\mu g}{R}\right)\left(\frac{2v_0}{7\mu g}\right) = \frac{4}{7}\frac{v_0}{R}$$

(c) Distance travelled by the plank,

$$S_{\text{plank}} = \frac{1}{2}At^2 = \frac{1}{2}\left(\frac{\mu g}{2}\right)\left(\frac{2v_0}{7\mu g}\right)^2 = \frac{v_0^2}{49\mu g}$$

ROLLING OF AN OBJECT UNDER NET EXTERNAL FORCE

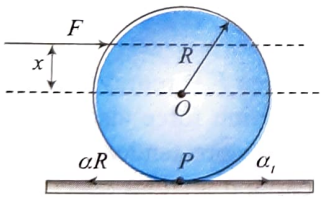
When a body rolls on a surface under external force, the frictional force on the body (if any) will be static in nature, less than its limiting value. But if the object rolls with sliding the nature of friction should be kinetic nature. Let us first learn how to find the direction of friction force in case of combined translational and rotational motion.

Direction of Friction in Case of Translation and Rotation Combined

In case of translation motion, we can decide the direction of friction force by direct observation. But the direction of friction cannot be found by direct observation in case of rotational motion, as the body is translating as well as rotating. The direction of friction force is determined after deciding the motion tendency of the point of contact of the body under consideration with the ground.

Let us take an example of analysis of direction of friction in this case.

In the following diagram (figure), a rolling object of mass M and radius R is placed on a rough horizontal surface. A force F is applied as shown.



To find the direction of friction at P, let us consider the surface to be frictionless.

Acceleration of point P due to translation only,

$$a_l = \frac{F}{M}(\rightarrow)$$

Acceleration of point P due to rotation only,

$$a_r = \alpha R = \frac{\tau}{I}R \Rightarrow \frac{FR}{I}x(\leftarrow)$$

Net acceleration of point P is

$$\vec{a}_p = \vec{a}_l + \vec{a}_r$$

$$a_p = \frac{F}{M} - \frac{FRx}{I}(\rightarrow) \qquad \ldots(i)$$

From Eq. (ii), it is clear that motion tendency at point P depends upon both x and I.

Equation (i) can be written as

$$a_p = \frac{F}{M}\left(1 - \frac{Rx}{K^2}\right) \qquad \ldots(ii)$$

The motion tendency of point P will be in forward direction if

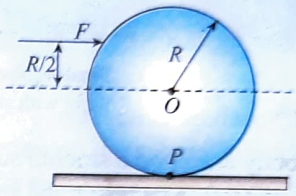
$$1 - \frac{Rx}{K^2} > 0 \Rightarrow 1 > \frac{Rx}{K^2} \Rightarrow K^2 > Rx \qquad \ldots(iii)$$

If this condition is satisfied, friction will act in backward direction.

Note: We can summarize the situation as
If $K^2 > Rx$: friction will act in backward direction.
If $K^2 = Rx$: no friction will act.
If $K^2 < Rx$: friction will act in forward direction.
If force acts in lower diametric plane, friction will act in backward direction only.

ILLUSTRATION 4.26

Find the direction of friction force in the following cases. If rolling object is



(a) a ring **(b)** a disc
(c) a solid sphere **(d)** a hollow sphere

Sol.

(a) For ring: $K^2 = R^2$ and $Rx = R\frac{R}{2} = \frac{R^2}{2}$

As $K^2 > Rx$

The friction will act in the backward direction.

(b) For disc: $K^2 = \frac{R^2}{2}$ and $x = R\frac{R}{2} = \frac{R^2}{2}$

$\Rightarrow K^2 = Rx$

No friction will act.

(c) For sphere: $K^2 = \frac{2}{5}R^2$ and $Rx = R\frac{R}{2} = \frac{R^2}{2}$

$\Rightarrow K^2 < Rx$

The friction will act in the forward direction.

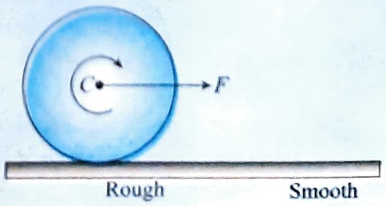
(d) For hollow sphere: $K^2 = \frac{2}{3}R^2$ and $Rx = R\frac{R}{2} = \frac{R^2}{2}$

$\Rightarrow K^2 > Rx$

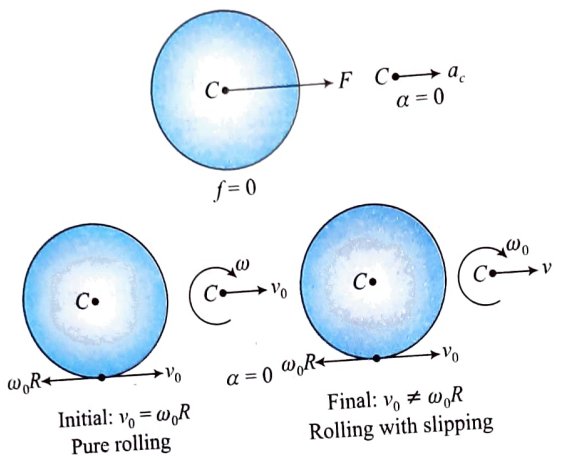
The friction will act in the backward direction.

ILLUSTRATION 4.27

A force of magnitude F is acting on a rolling body of mass m and radius R as shown in figure. What happens when it enters on a smooth surface?



Sol. When the body is rolling on the rough surface, the friction force acting on the surface acts in the backward direction. The force F increases velocity of centre of mass while friction force provides torque for providing angular acceleration. The velocity of centre of mass is always such that $v_c = \omega R$, i.e., rolling is present. When the body enters on the smooth surface, the pulling force F is still acting, i.e., $F \neq 0$.



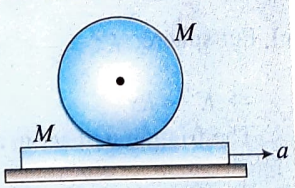
When the rolling body enters on a smooth surface, the force of friction disappears, i.e., $f = 0$. As a result, the angular acceleration of the body about its centre of mass becomes zero.

Whatever is the angular speed attained by the body, it continues to rotate with that. On the contrary, the linear acceleration of the centre of mass suddenly increases to F/m.

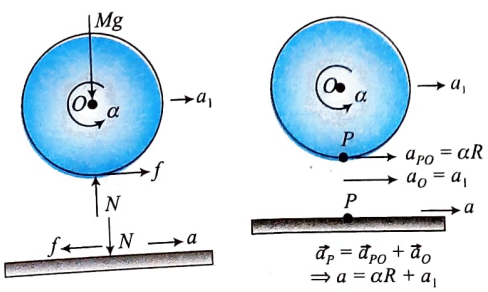
As a result, its linear speed increases rapidly and the condition $v_c = \omega R$ does not hold true and the body will not roll on the smooth surface. The point of contact slips forward ($v > \omega_0 R$).

ILLUSTRATION 4.28

A uniform cylinder (mass M) of radius R is kept on an accelerating platform (mass M) as shown in figure. If the cylinder rolls without slipping on the platform, determine the magnitude of acceleration of the centre of mass of the cylinder. Assuming the coefficient of friction μ, determine the maximum acceleration the platform may have without slip between the cylinder and the platform.



Sol. As there is no sliding between the cylinder and plank, the friction will be static in nature. For finding the direction of friction assume, there is no friction anywhere, it suggests sliding tendency of the cylinder w.r.t. plank will be in backward direction. It means friction force on the cylinder will act in forward direction. The friction acting on the cylinder will be responsible for angular acceleration of the cylinder.



From FBD of cylinder:

Force equation of cylinder, $f = Ma_1$ ...(i)

As axes of rotation is moving so we can apply torque equation about centripetal axis, $\tau_0 = I_0 \alpha$

$$\Rightarrow \quad f \cdot R = \frac{MR^2}{2}\alpha \text{ or } R\alpha = \frac{2f}{M} \qquad \text{...(ii)}$$

As the cylinder is rolling, it means point of contact of cylinder with plank will not slide. It means,

acceleration of the plank = acceleration of point P

or $a = a_1 + \alpha R$ ...(iii)

From (i), (ii) and (iii), we get

$$f = \frac{Ma}{3} \qquad \text{...(iv)}$$

As there is no sliding, it means

$$f \leq \mu N$$

$$\frac{Ma}{3} \leq \mu Mg$$

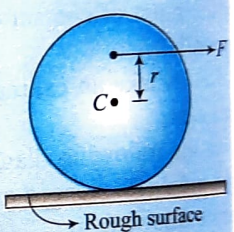
$$\Rightarrow \quad a_{max} = 3\mu g$$

ILLUSTRATION 4.29

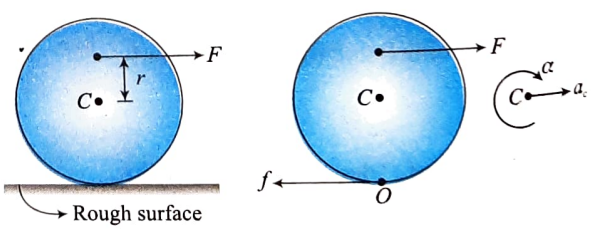
A force of magnitude F is acting on a rolling body of mass M and radius R as shown in figure. What happens if the pulling force F is removed? Discuss the different possibilities when the pulling force acts

(a) at centre **(b)** above centre

(c) below centre



Sol. If the pulling force F is removed from a body rolling on a rough surface, the friction force also disappears, because the point of contact has no more tendency to slip with respect to the surface. The linear and angular speeds of the body remain constant and the body keeps on rolling.



As body is rolling, the friction will be of static nature and unknown (both in magnitude and direction). Let friction act in the backward direction.

Equation of motion:

$$F - f = Ma_c \qquad \text{...(i)}$$

Here we can apply torque equation about centre of mass as well as about point of contact with ground (instantaneous centre of rotation)

About centre of mass: $Fr + fR = I_c\alpha$ ...(ii)

About point of contact O: $F(R+r)=I_O\alpha$

$$F(R+r)=(I_c+MR^2)\alpha \quad \text{...(iii)}$$

Taking torque equation about O, we can directly calculate the value of α.

Hence, $\alpha=\dfrac{F(R+r)}{(I_c+MR^2)}$ ...(iv)

As the body is rolling, $a_c=\alpha R=\dfrac{F(R+r)R}{(I_c+MR^2)}$

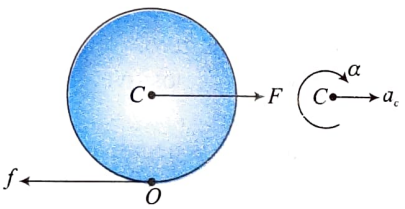
or $$a_c=\frac{F}{M}\left[\frac{\left(1+\frac{r}{R}\right)}{\left(1+\frac{I_C}{MR^2}\right)}\right] \quad \text{...(v)}$$

Substituting the value of a_c in Eq. (i), we get force of friction,

$$f=F\left[\frac{I_c-MrR}{I_c+MR^2}\right] \quad \text{...(vi)}$$

Note: We can find the values of a_c and f by using Eqs. (i) and (ii). But by taking torque equation about point of contact, the calculation steps are reduced.

(a) If force acts through C, i.e., $r = 0$:



Friction force acts backward: r is zero

From Eq. (v), we get $a_C=\dfrac{F}{M}\left[\dfrac{1}{1+\dfrac{I_C}{MR^2}}\right]$

From Eq. (vi), we get $f=F\left[\dfrac{1}{1+\dfrac{MR^2}{I_C}}\right]$

From the values of a_c and f, it is clear that

- body moves forward
- friction acts backward

(b) Force acts above C, i.e., r is positive:

From Eqs. (v) and (vi), it is clear that

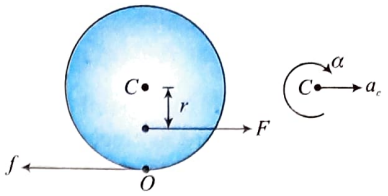
- the body moves forward
- rotation about the centre of mass is clockwise
- the friction force may act forward or backward

Backward, if $r<\dfrac{I_c}{MR}$

Forward, if $r>\dfrac{I_c}{MR}$

No friction force acts when $r=\dfrac{I_c}{MR}$

(c) Force acts below C, i.e., r is negative:

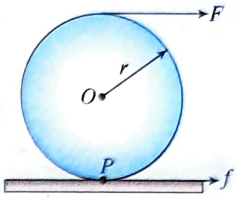


Friction force acts backward: r is negative

- The body moves forward.
- Rotation about the centre of mass is clockwise.
- Friction force acts backward ($f < F$).

ILLUSTRATION 4.30

A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of the centre of the sphere.



Sol. The situation is shown in figure. As force F rotates the sphere, the point of contact has a tendency to slip towards left so that the static friction on the sphere will act towards right. Let r be the radius of the sphere and a be the linear acceleration of the centre of the sphere. The angular acceleration about the centre of the sphere is $\alpha = a/r$, as there is no slipping.

For the linear motion of the centre,

$$F+f=ma \quad \text{...(i)}$$

and for the rotational motion about the centre,

$$Fr-fr=I\alpha=\left(\frac{2}{5}mr^2\right)\frac{a}{r}$$

or $$F-f=\frac{2}{5}ma \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$2F=\frac{7}{5}ma \text{ or } a=\frac{10F}{7m}$$

Alternative method: As the sphere is rolling, we can apply torque equation about point of contact.

i.e., $$F(2r)=I_P\alpha=\left(\frac{2}{5}mr^2+mr^2\right)\alpha$$

$$\Rightarrow \quad F(2r)=\left(\frac{7}{5}mr^2\right)\alpha$$

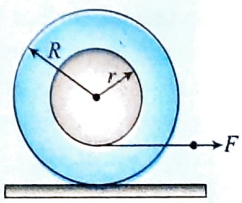
Hence, angular acceleration $\alpha=\dfrac{10F}{7mr}$ ...(iii)

As sphere is rolling, acceleration of centre of mass is

$$a=\alpha r=\frac{10F}{7m}$$

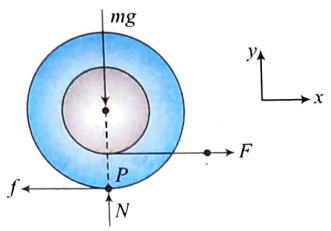
ILLUSTRATION 4.31

A cotton reel has inner radius r and outer radius R. Mass of the reel is M and moment of inertia about longitudinal rotational axis is I. A force F is applied at the free end of thread wrapped on the reel as shown in figure. If the reel rolls without sliding.



(a) Determine the frictional force exerted by the table on the reel and the direction in which it acts.
(b) In what direction does the reel begin to move?

Sol. **Method 1:** As the body rolls, point P will be instantaneous centre of rotation (ICR). There is clockwise torque about ICR and centre of mass will have acceleration in right direction. Hence the reel will rotate in clockwise direction and moves to the right. Force F exerts anticlockwise torque; therefore, friction force must exert a larger clockwise torque to produce clockwise rolling. The equations of the motion are:



$$\Sigma F_x = F - f = Ma \quad ...(i)$$

$$\Sigma F_y = N - Mg = 0 \quad ...(ii)$$

$$\Sigma \tau = f \times R - F \times r = I\alpha \quad ...(iii)$$

As the reel rolls without slipping, $a = R\alpha$
From Eqs. (i) and (iii), we get

$$\frac{F-f}{M} = \frac{R(fR - Fr)}{I} \quad \text{or} \quad f = \frac{F(I + MRr)}{I + MR^2}$$

Force of friction comes out to be positive, therefore our assumption about the direction of friction force was correct.
On substituting the expression for f in Eq. (i), we obtain

$$a = \frac{FR(R-r)}{I + MR^2}$$

As $R > r$, a positive and towards right. Similarly, from Eq. (iii), we obtain

$$\alpha = \frac{F(R-r)}{I + MR^2}$$

which is positive, i.e., net torque on the reel is clockwise.

Method 2: We can apply torque equation about ICR

$$\Rightarrow \quad F(R-r) = I_P\alpha$$

$$F(R-r) = [I + MR^2]\alpha$$

$$\Rightarrow \quad \alpha = \frac{F(R-r)}{[I + MR^2]}$$

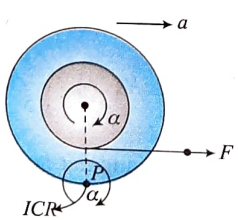
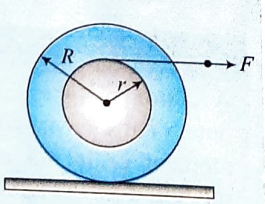


ILLUSTRATION 4.32

A wheel of radius R, mass m and moment of inertia I is pulled along a horizontal surface by application of force F to a rope unwinding from the axel of radius r as shown in figure. Friction is sufficient for pure rolling of the wheel.



(a) What is the linear acceleration of the wheel?
(b) Calculate the frictional force that acts on the wheel.

Sol. The equations of the motion of the wheel are:

$$F - f = ma \quad ...(i)$$

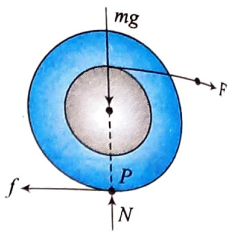
$$F \times r + f \times R = I\alpha \quad ...(ii)$$

In case of pure rolling,

$$a = R\alpha \quad ...(iii)$$

$$F \times r + (F - ma)R = \frac{I}{R}a$$

or $$a = \frac{F(r+R)R}{(I + mR^2)}$$



We can find acceleration of centre of wheel by another approach also. The point of contact will be instantaneous centre of rotation (ICR), and we apply torque equation about ICR.

$$F(R+r) = (I + mR^2)\alpha \Rightarrow \alpha = \frac{F(R+r)}{(I + mR^2)}$$

Acceleration of centre of mass, $a = \alpha R = \frac{F(R+r)R}{(I + mR^2)}$

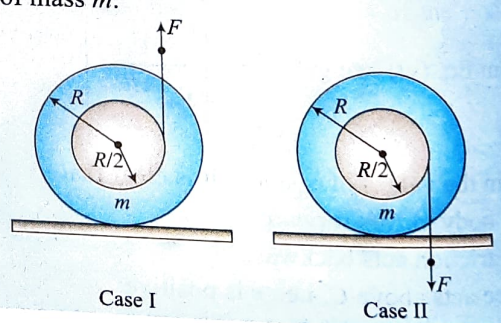
and the frictional force is

$$f = F - ma = F\left[1 - \frac{(r+R)R}{\left(\frac{I}{m} + R^2\right)}\right] = \frac{F[(I/m) - Rr]}{[(I/m) + R^2]}$$

For $I/m = Rr$, frictional force is zero.
For $I/m > Rr$, frictional force is positive, i.e., it acts in backward direction.
For $I/m < Rr$, frictional force is negative, i.e., it acts in forward direction.

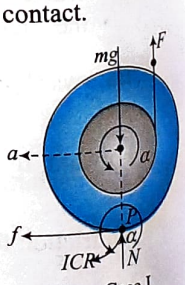
ILLUSTRATION 4.33

A force F is applied on a spool (mass m and radius R) in vertical direction as shown in the case (I) and case (II). The radius of gyration k of spool is $R/\sqrt{2}$. Assuming pure rolling, find the acceleration of the CM of the body, assuming it as a uniform cylinder of mass m.



Sol. As the cylinder is rolling. The point of contact will be instantaneous centre of rotation (ICR). The rolling body can be considered as pure rotation about the point of contact.
Case I: The force F provides anticlockwise torque about P. The cylinder will rotate in anticlockwise direction about P. The point P is instantaneous centre of rotation (ICR).

The cylinder will also rotate in anticlockwise direction about its centre of mass. The cylinder will accelerate in left direction. Hence friction will act in left direction.



Applying torque equation about P

$$F\left(\frac{R}{2}\right) = I_P\alpha = m(k^2 + R^2)\alpha$$

or $$F\left(\frac{R}{2}\right) = m\left(\frac{R^2}{2} + R^2\right)\alpha = \frac{3}{2}mR^2\alpha$$

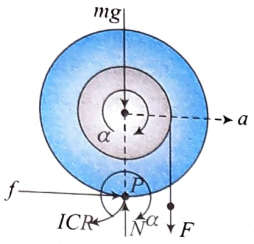
which gives $R\alpha = \dfrac{F}{3m}$

Hence acceleration of cylinder is $a = \dfrac{F}{3m}$ (towards left).

Case II: In this case the torque of F about instantaneous centre of rotation (ICR) is in clockwise direction.

The cylinder will also rotate in clockwise direction about its centre of mass. Hence, the acceleration of cylinder will be towards right and friction will act towards right.

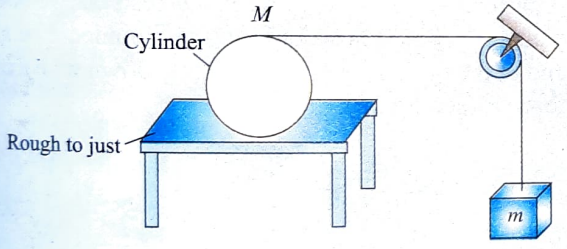
Now the torque equation about P will be same as we got in previous case. We will get the same value of acceleration, but direction will be towards right.



Case II

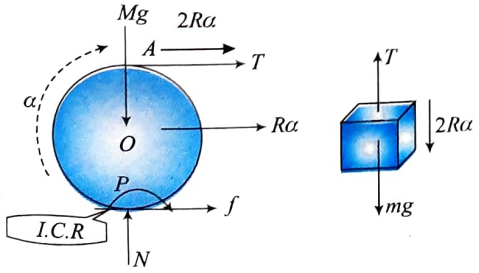
ILLUSTRATION 4.34

An inextensible thread is wound round a cylinder on mass M, the cylinder is placed over a rough horizontal surface and the thread is passed over massless and frictionless pulley such that the part of the thread between cylinder and pulley is horizontal as shown in figure.



When a block of mass m is attached with free end of the thread and system is released, it is observed that friction just prevents slipping of the cylinder over the horizontal surface. Find minimum coefficient of friction between horizontal surface and cylinder.

Sol. As the cylinder does not slip over the horizontal surface, the friction will be static in nature. Let us assume it is acting in forward direction. The instantaneous axis of rotation will pass through the lowest point P. Let angular acceleration of the cylinder is α, (clockwise) then acceleration of its centre will be equal to $R\alpha$ horizontally rightwards and that of block will be $2R\alpha$ vertically downward (refer figure).



We can apply torque equation about ICR. Writing torque equation about P,

$$T \cdot 2R = I_P \cdot \alpha$$

where I_P is moment of inertia of the cylinder about P

$$T \cdot 2R = \left(\frac{3MR^2}{2}\right)\cdot\alpha \Rightarrow T = \frac{3}{4}MR\alpha \quad \text{...(i)}$$

Now considering FBD's

Force equation of the block,

$$mg - T = m(2R\alpha) \quad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$R\alpha = \frac{mg}{\left(2m + \frac{3}{4}M\right)} \quad \text{...(iii)}$$

Force equation for cylinder,

$$T + f = M\cdot(R\alpha) \quad \text{...(iv)}$$

From (i), (iii) and (iv), we get

$$f = \frac{Mmg}{(8m + 3M)}$$

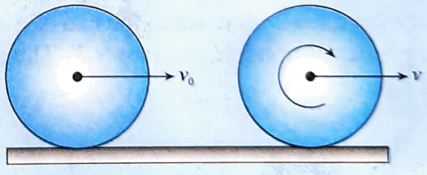
As friction is static nature,

$$f \le \mu N \Rightarrow \frac{Mmg}{(8m+3M)} \le \mu Mg$$

or $$\mu \ge \frac{m}{(8m+3M)}$$

ILLUSTRATION 4.35

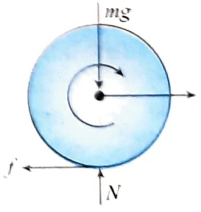
A uniform disc of mass m and radius r is projected horizontally with velocity v_0 on a rough horizontal floor so that it starts off with a purely sliding motion at $t = 0$. At $t = t_0$ seconds it acquires a purely rolling motion.



(a) Calculate the velocity of the centre of mass of the disc at $t = t_0$.

(b) Assuming coefficient of friction to be μ, calculate t_0.

Sol. When the disc is projected it starts sliding and hence there is a relative motion between the points of contact. Therefore, frictional force acts on the disc in the direction opposite to the motion.



(a) Now for translational motion:

$$\vec{a}_{cm} = -\frac{\vec{f}}{m}$$

$f = \mu N$ (as it slides) $\Rightarrow f = \mu mg$

$\Rightarrow a_{cm} = -\mu g$, negative sign indicates that a_{cm} is opposite to v_{cm}

$\Rightarrow v_{cm}(t) = v_0 - \mu g t$

At $t = t_0$, $t_0 = \dfrac{(v_o - v)}{\mu g}$, where $v_{cm}(t_0) = v$ ... (i)

For rotational motion about centre,

$\tau_f = I_{cm}\alpha$

$\mu mgr = \dfrac{mr^2}{2}\alpha \Rightarrow \alpha = \dfrac{2\mu g}{r}$ ...(ii)

Therefore $\omega(t_0) = 0 + \dfrac{2\mu g}{r}t_0$,

Using (i), $\omega = \dfrac{2(v_0 - v)}{r}$ ...(iii)

For pure rolling $v_{cm} = \omega r \Rightarrow v = 2(v_0 - v)$ using (iii)

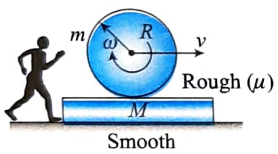
$\Rightarrow v = \dfrac{2}{3}v_0$

(b) Putting the value of v in Eq. (i), we get

$t_0 = \dfrac{v_o}{3\mu g}$

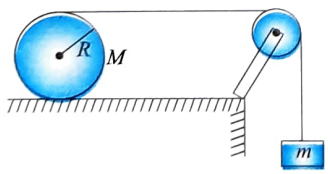
CONCEPT APPLICATION EXERCISE 4.4

1. A cylinder of mass m and radius R rolls on a stationary plank of mass M. The lower surface of the plank is smooth and the upper surface is sufficiently rough with a coefficient of friction μ. A man is to hold the plank stationary with respect to the ground, as shown figure.



Find the force exerted by the man to keep the plank stationary.

2. A load of mass m is attached to the end of a string wound on a cylinder of mass M and radius R. The string passes round a pulley.

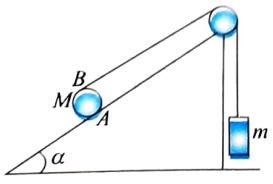


(a) Find the acceleration of the cylinder M when it rolls.

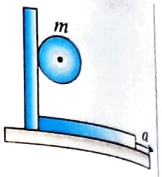
(b) If the coefficient of friction is μ. Find the acceleration of the cylinder when it slips and rolls.

(c) Find the minimum value of coefficient of friction for which the cylinder rolls always.

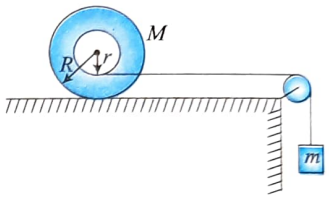
3. Find the tension in the tape and the linear acceleration of the cylinder up the incline, assuming there is no slipping. (Take $M = 4m$ and $g = 10$ m/s^2)



4. A uniform solid sphere rolls down a vertical surface without sliding. If the vertical surface moves with an acceleration $a = g/2$, find the minimum coefficient of friction between the sphere and vertical surfaces so as to prevent relative sliding.

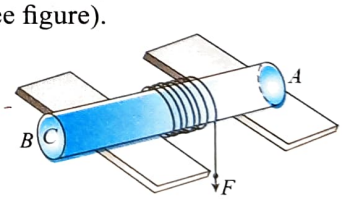


5. A spool (consider it as a double disc system joined by a short tube at their centre) is placed on a horizontal surface as shown in figure. A light string wound several times over the short connecting tube leaves it tangentially and passes over a light pulley. A weight of mass m is attached to the end of the string. The radius of the connecting tube is r and mass of the spool is M and radius is R. Find the acceleration of the falling mass m. Neglect the mass of the connecting tube and slipping of the spool.



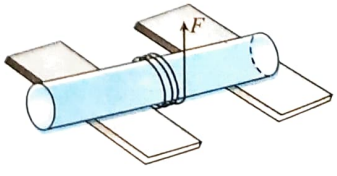
6. A solid cylinder wheel of mass M and radius R is pulled by a force F applied to the centre of the wheel at 37° to the horizontal. If the wheel is to roll without slipping, what is the maximum value of $|F|$? The coefficients of static and kinetic friction are $\mu_S = 0.40$ and $\mu_K = 0.30$. (sin 37° = 3/5).

7. A uniform solid cylinder of mass m rests on two horizontal planks. A thread is wound on the cylinder. The hanging end of the thread is pulled vertically down with a constant force F (see figure).

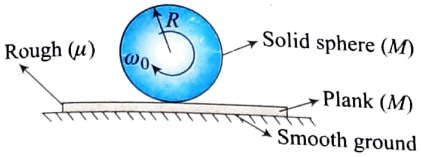


Find the maximum magnitude of the force F which may be applied without bringing about any sliding of the cylinder, if the coefficient of friction between the plank and the cylinder is equal to μ. What is the maximum acceleration of the centre of mass over the planks?

8. A uniform solid cylinder of mass m rests on two horizontal planks as shown in figure. A thread is wound on the cylinder. The hanging end of the thread is pulled vertically up with a force F. What is the maximum magnitude of the force F which still does not bring about any sliding of the cylinder, if the coefficient of friction between the cylinder and the planks is equal to μ. What is the maximum acceleration of the axis of the cylinder?



9. A solid sphere of mass M is placed on the top of a plank of the same mass, after giving an angular velocity ω_0 at $t = 0$. Find the velocity of the plank and the sphere when the sphere starts rolling.



ANSWERS

1. zero **2.** (a) $\dfrac{mg}{m+\frac{3}{8}M}$ (b) $\dfrac{3mg-\mu Mg}{M+3m}$ (c) $\dfrac{M}{8m+3M}$

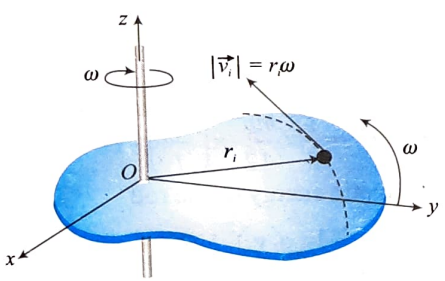
3. 12 N, 4 m/s^2 **4.** $\dfrac{4}{7}$

5. $\dfrac{2mg}{2m+3M[R/(R-r)]^2}$ **6.** $0.79Mg$ **7.** $\dfrac{3\mu mg}{2-3\mu}, \dfrac{2\mu g}{2-3\mu}$

8. $\dfrac{3\mu mg}{2+3\mu}, \dfrac{2Mg}{2+3\mu}$ **9.** $\dfrac{2}{9}\omega_0 R$

KINETIC ENERGY OF ROTATING BODY ABOUT A FIXED AXIS

We define the kinetic energy of an object as the energy associated with its motion through space. An object rotating about a fixed axis remains stationary in space, so there is no kinetic energy associated with translational motion. The individual particles making up the rotating object, however, are moving through space; they follow circular paths. Consequently, there is kinetic energy associated with rotational motion.



Let us consider an object as a system of particles and assume it rotates about a fixed z axis with an angular speed ω. Figure shows the rotating object and identifies one particle on the object located at a distance r_i from the rotation axis. If the mass of the i^{th} particle is m_i and its tangential speed is v_i, its kinetic energy is

$$K_i = \frac{1}{2}m_i v_i^2 \qquad ...(i)$$

To proceed further, recall that although every particle in the rigid object has the same angular speed ω, the individual tangential speeds depend on the distance r_i from the axis of rotation. The total kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:

$$K_R = \sum_i K_i = \sum_i \frac{1}{2}m_i v_i^2 = \frac{1}{2}\sum_i m_i r_i^2 \omega^2$$

We can write this expression in the form

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \qquad ...(ii)$$

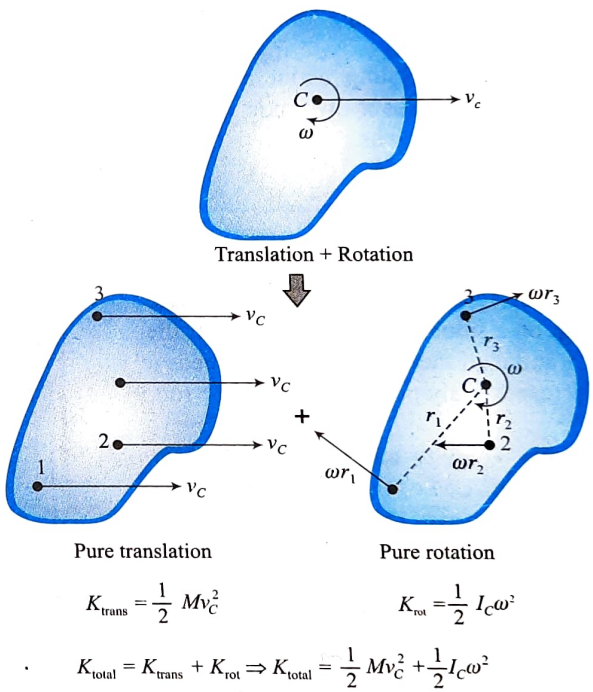
where we have factored w^2 from the sum because it is common to every particle. We recognize the quantity in parentheses as the moment of inertia of the object about the rotational axes. Therefore, Eq. (ii) can be written

$$K_R = \frac{1}{2}I\omega^2 \qquad ...(iii)$$

Although we commonly refer to the quantity $\frac{1}{2}I\omega^2$ as **rotational kinetic energy**, it is not a new form of energy. It is ordinary kinetic energy because it is derived from a sum over individual kinetic energies of the particles contained in the rigid object hence total kinetic energy of the body.

KINETIC ENERGY OF A BODY IN COMBINED ROTATION AND TRANSLATION

If during rotation, the center of mass of the rigid body undergoes translational motion, it will have both translational and rotational kinetic energy.



$$K_{trans} = \frac{1}{2}Mv_C^2 \qquad K_{rot} = \frac{1}{2}I_C\omega^2$$

$$K_{total} = K_{trans} + K_{rot} \Rightarrow K_{total} = \frac{1}{2}Mv_C^2 + \frac{1}{2}I_C\omega^2$$

In pure translation, each and every particle of the body moves with same velocity as centre of mass, we can define kinetic energy of the object in case of pure translation.

$$K_{trans} = \frac{1}{2}Mv_C^2 \qquad ...(i)$$

In case of pure rotation, the body rotate about its axes passing through its centre of mass, we define kinetic energy of the object as rotational kinetic energy

$$K_{trans} = \frac{1}{2}Mv_C^2 \qquad ...(ii)$$

where I_C = moment of inertia of the object about an axis passing through its centre of mass.

Hence total kinetic energy of the object

$$K_{total} = K_{trans} + K_{rot}$$

$$\Rightarrow \quad K_{total} = \frac{1}{2}Mv_C^2 + \frac{1}{2}I_C\omega^2$$

The first term signifies the translational KE of the rigid body which can be understood as the KE of the centre of mass of the body. The second term gives the KE of rotational of the body about the centre of mass. Hence, we can call the term "$\frac{1}{2}I_C\omega^2$" as rotational kinetic energy of the rigid body.

Important Points:

- **KE for pure translation:** $K_{translational} = \frac{1}{2}mv_C^2$

  As a perfectly translating body has KE formula identical to that of a point mass, we can say that a translating body behaves as a point mass.

- **KE for fixed axis rotation:** $K = \frac{1}{2}I_P\omega^2$

  I_P = moment of inertia of body about fixed axis

- **Kinetic energy for combined motion:**

  $$K_{total} = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2$$

  If a body is rotating about a fixed axis and spinning (or rotating) about it own axis we write the total kinetic energy of the body as

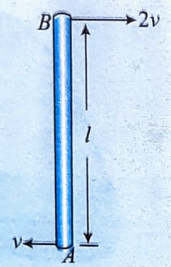
  $$K_{total} = K_{orbital} + K_{spin}$$

ILLUSTRATION 4.36

A uniform rod of mass m = 2 kg and length l is kept on a smooth horizontal plane. If the ends A and B of the rod move with speeds v and $2v$ respectively perpendicular to the rod, find the:

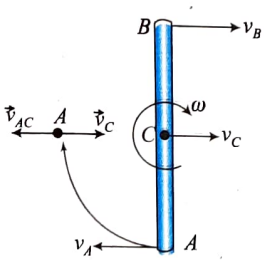
(a) angular velocity of the rod and linear velocity of CM of the rod.

(b) kinetic energy of the rod



Sol.

(a) As the velocities of the ends A and B of the rod are given as $2v$ and v respectively, we need to find velocity of centre of mass of rod and its angular velocity as follows.



Since $\vec{v}_A = \vec{v}_{AC} + \vec{v}_C$

We know $\vec{v}_{AC} = -\frac{l}{2}\omega\hat{i}$

and $\vec{v}_C = v_C\hat{i}$

But we have velocity of A;

$$v_A(=-v) = v_C - \frac{l}{2}\omega \qquad ...(i)$$

Similarly, we have

$$v_B(=2v) = v_C + \frac{l}{2}\omega \qquad ...(ii)$$

Solving Eqs. (i) and (ii), we have

$$\omega = \frac{3v}{l} \text{ and } v_C = \frac{v}{2}$$

(b) Then substituting v_C and ω in the equation

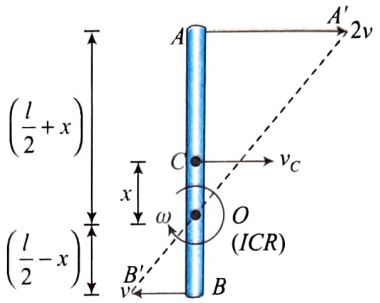
$$K = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2$$

We have $K = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}\left(\frac{ml^2}{12}\right)\left(\frac{3v}{l}\right)^2$

This gives $K = \frac{1}{2}mv^2$

Alternative: We can calculate angular velocity of the rod and velocity of centre of mass by ICR method.

We can find the ICR as shown in figure.



The triangles $AA'O$ and $BB'O$ are similar triangles

$$\frac{AA'}{AO} = \frac{BB'}{BO} \Rightarrow \frac{2v}{(l/2+x)} = \frac{v}{(l/2-x)} = \omega$$

which gives $x = \frac{l}{6}$

Hence angular velocity $\omega = \frac{v}{\frac{l}{2}-x} = \frac{v}{\frac{l}{2}-\frac{l}{6}} \Rightarrow \omega = \frac{3v}{l}$

and velocity of centre of mass; $v_C = \omega x = \frac{3v}{l} \times \frac{l}{6} = \frac{v}{2}$

Rotational kinetic energy of the rod

$$K_{rot} = \frac{1}{2}I_C\omega^2 = \frac{1}{2} \times \frac{ml^2}{12} \times \left(\frac{3v}{l}\right)^2 \Rightarrow K_{rot} = \frac{3}{8}mv^2$$

Translational kinetic energy of the rod

$$K_{trans} = \frac{1}{2}mv_C^2 = \frac{1}{2}m\left(\frac{v}{2}\right)^2 = \frac{1}{8}mv^2$$

Total kinetic energy $K_{total} = K_{trans} + K_{rot}$

$$\Rightarrow \quad K_{total} = \frac{1}{8}mv^2 + \frac{3}{8}mv^2 = \frac{1}{2}mv^2$$

We can calculate total kinetic energy of the rod by using the fact that the rod has pure rotation about ICR. It means total kinetic energy of the rod should be equal to rotational kinetic energy about ICR.

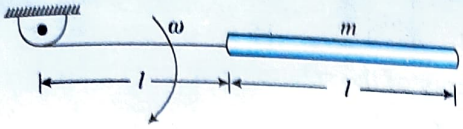
$$K_o = \frac{1}{2}I_o\omega^2 = \frac{1}{2}[I_C + mx^2]\omega^2$$

$$= \frac{1}{2}\left[\frac{ml^2}{12} + m\left(\frac{l}{6}\right)^2\right]\left(\frac{3v}{l}\right)^2$$

$$\Rightarrow \quad K_o = \frac{1}{2}mv^2$$

ILLUSTRATION 4.37

A rod of mass m and length l is connected with a light rod of length l. The composite rod is made to rotate with angular velocity ω as shown in the figure. Find the



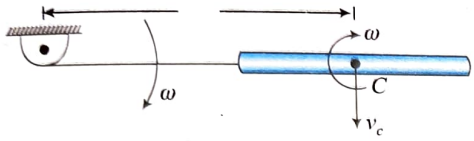
(a) translational kinetic energy.
(b) rotational kinetic energy.
(c) total kinetic energy of rod.

Sol.

(a) **Method 1:** Translational kinetic energy of rod,

$$K_{\text{translational}} = \frac{1}{2}mv_c^2$$

Velocity of centre of mass $v_c = \omega\left(\frac{3}{2}l\right)$



Hence $K_{\text{translational}} = \frac{1}{2}m\left[\frac{3\omega l}{2}\right]^2 = \frac{9}{8}m\omega^2 l^2$

(b) Rotational kinetic energy of rod,

$$K_{\text{rotational}} = \frac{1}{2}I_c\omega^2 = \frac{1}{2}\left[\frac{ml^2}{12}\right]\omega^2 = \frac{1}{24}m\omega^2 l^2$$

(c) Total kinetic energy,

$$K_{\text{total}} = K_{\text{translationl}} + K_{\text{rotational}}$$

$$= \frac{9}{8}m\omega^2 l^2 + \frac{1}{24}m\omega^2 l^2 = \frac{7}{6}m\omega^2 l^2$$

Method 2: Total energy of a body rotating about fixed axis = rotational kinetic energy about the fixed axis.

$$K_{\text{total}} = \frac{1}{2}I_P\omega^2$$

Now using parallel axis theorem,

$$I_P = I_C + m\left(\frac{3}{2}l\right)^2$$

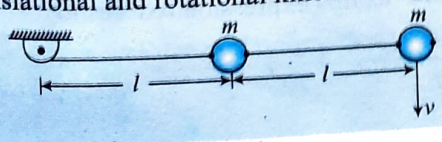
$$= \frac{ml^2}{12} + \frac{9}{4}ml^2 = \frac{7}{3}ml^2$$

Hence total kinetic energy

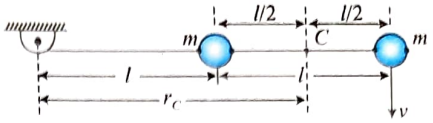
$$K_{\text{total}} = \frac{1}{2}\left(\frac{7}{3}ml^2\right)\omega^2 = \frac{7}{6}m\omega^2 l^2$$

ILLUSTRATION 4.38

Two beads each of mass m are welded at the ends of two light rigid rods each of length l. if the pivots are smooth, find the ratio of translational and rotational kinetic energy of system.



Sol. We know translational kinetic energy $K_t = \frac{1}{2}mv_C^2$ and rotational kinetic energy $K_r = \frac{1}{2}I_C\omega^2$,



Distance of centre of mass from pivot end

$$r_C = \frac{ml + m(2l)}{m+m} = \frac{3}{2}l$$

Moment of inertia of system about centre of mass

and $I_C = m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{2}$

Angular velocity of rod $\omega = \frac{v}{2l}$

Velocity of centre of mass

$$v_c = \omega\left(\frac{3}{2}l\right) = \left(\frac{v}{2l}\right)\left(\frac{3}{2}l\right) = \frac{3}{4}v$$

Translational kinetic energy of system

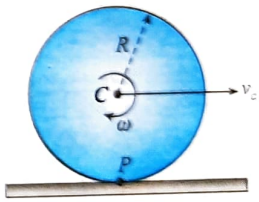
$$K_t = \frac{1}{2}M_{\text{total}}\cdot v_c^2 = \frac{1}{2}(2m)\left(\frac{3}{4}v\right)^2 = \frac{9}{16}mv^2$$

$$K_r = \frac{1}{2}I_C\omega^2 = \frac{1}{2}\left(\frac{ml}{2}\right)^2\left(\frac{v}{2l}\right)^2 = \frac{1}{16}mv^2$$

This gives $\frac{K_t}{K_r} = 9$

KINETIC ENERGY OF A BODY ROLLING ON A FIXED SURFACE

Let a body of mass m rolling on a fixed horizontal surface. The rolling motion is the combined rotational and translational motion. Here we can define kinetic energy of the rolling body



$$K_{\text{total}} = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2 \quad \text{...(i)}$$

Here m is the mass of the body and I_c is the moment of inertia of the body about an axis passing through centre of mass.

$I = mk^2$, k = radius of gyration

As the body is rolling, $v_c = \omega R$ or $\omega = \frac{v_c}{R}$ ...(ii)

From (i) and (ii)

$$K_{\text{total}} = \frac{1}{2}mv_c^2 + \frac{1}{2}(mk^2)\left(\frac{v_c}{R}\right)^2$$

$$\Rightarrow \quad K_{\text{total}} = \left(1 + \frac{k^2}{R^2}\right)\cdot\frac{1}{2}mv_c^2 = (1+c)\cdot\frac{1}{2}mv_c^2 \quad \text{...(iii)}$$

Here c is a constant depends upon the shape of the rolling body
From (i) and (ii) we can write

$$K_{\text{total}} = \frac{1}{2}m(\omega R)^2 + \frac{1}{2}I_c\omega^2 = \frac{1}{2}(mR^2 + I_c)\omega^2$$

Using parallel axis theorem, we can write, $mR^2 + I_c = I_P$

I_P is the moment of inertia the rolling body about point of contact with the ground, which is also instantaneous centre of rotation (ICR).

Hence $K_{total} = \frac{1}{2} I_P \omega^2$ ...(iv)

Therefore, we conclude that the combined effects of translation of centre of mass of a body and its rotation about an axis passing through the CM are equivalent to its pure rotation about an axis passing through the point of contact P, of the rolling body.

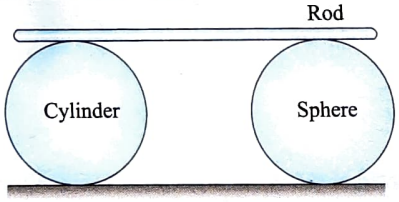
$$\% \text{ of energy of translation} = \frac{KE_t}{KE} \times 100\%$$

$$= \frac{\frac{1}{2}mv_c^2}{\frac{1}{2}mv_c^2(1+c)} \times 100\% = \frac{1}{1+c} \times 100\%$$

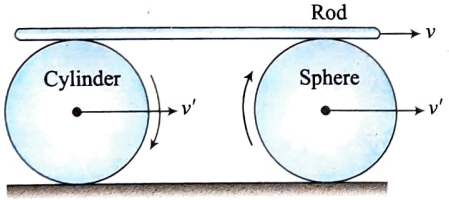
$$\text{Similarly, \% of energy of rotation} = \frac{c}{1+c} \times 100\%$$

ILLUSTRATION 4.39

A rod of mass m is kept on a cylinder and sphere each of radius R. The masses of the sphere and cylinder are m_1 and m_2 respectively. If the speed of the rod is v, find the KE of the system (rod + cylinder + sphere). Assume that the surfaces do not slide relative to each other.



Sol. Kinetic energy in case of rolling $K = (1+c)\frac{mv^2}{2}$



where, $c = \frac{k^2}{R^2}$; k = radius of gyration and R = radius

For cylinder, $c = \frac{\frac{1}{2}R^2}{R^2} = \frac{1}{2}$

For sphere, $c = \frac{\frac{2}{5}R^2}{R^2} = \frac{2}{5}$

In case of rolling on ground or on a fixed surface ,the velocity of the top point of the rolling body is double the velocity of its centre, hence $v' = \frac{v}{2}$, here v' is the velocity of the centre of mass of the cylinder and sphere.

Kinetic energy of the cylinder,

$$K_{cylinder} = \left(1+\frac{1}{2}\right).\frac{1}{2}m_1 v'^2$$

$$\Rightarrow K_{cylinder} = \frac{3}{4}m_1\left(\frac{v}{2}\right)^2 = \frac{3}{16}m_1 v^2$$

Kinetic energy of the sphere,

$$K_{sphere} = \left(1+\frac{2}{5}\right).\frac{1}{2}m_2 v'^2$$

$$\Rightarrow K_{sphere} = \frac{7}{10}m_2\left(\frac{v}{2}\right)^2 = \frac{7}{40}m_2 v^2$$

The motion of the rod is pure translational, hence kinetic energy of rod is,

$$K_{rod} = \frac{1}{2}mv^2$$

Then total kinetic energy of the system

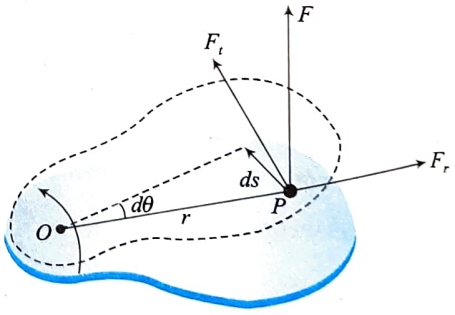
$$K_{total} = K_{rod} + K_{cylinder} + K_{sphere} = \frac{1}{2}mv^2 + \frac{3}{16}m_1 v^2 + \frac{7}{40}m_2 v^2$$

ROTATIONAL WORK AND POWER

Consider the rigid object pivoted at O in figure. It is acted on by an external force F that has a radial component F_r and a tangential component F_t. In an infinitesimal time interval dt, the point of application of force moves in a circular arc of length $r\,d\theta$. Therefore, the work done is

$$dW_{rot} = F_t(r\,d\theta) = (F_t r)\,d\theta = \tau\,d\theta$$

Notice that the radial component vector of $\vec{F}$ does no work on the object because it is perpendicular to the displacement of the point of application of $\vec{F}$.



For a finite angular displacement,

$$W_{rot} = \int_{\theta_i}^{\theta_f} \tau\,d\theta \quad \text{...(i)}$$

The rate at which work is being done by $\vec{F}$ as the object rotates about the fixed axis through the angle $d\theta$ in a time interval dt is

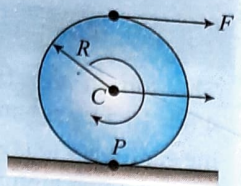
$$\frac{dW}{dt} = \tau\frac{d\theta}{dt}$$

Because dW/dt is the instantaneous power P delivered by the force and $d\theta/dt = \omega$, this expression reduces to

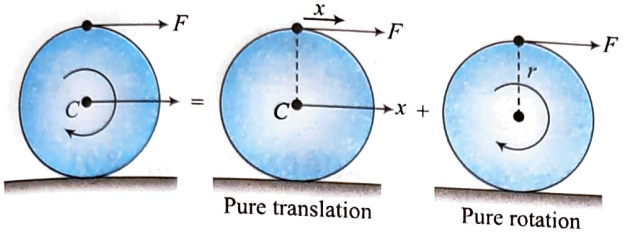
$$P = \frac{dW}{dt} = \tau\omega$$

ILLUSTRATION 4.40

A disc of mass m and radius R is placed on a smooth horizontal surface. A horizontal force F is acting at the top most position of the disc. Find the work done by the force F in rotating and translating the disc, if its centre of mass moves through distance x and the body rotates through an angle θ.



Sol. The motion of the disc is combined translation and rotation, here we can use method of superposition.



The work done by the force F in translating the disc

$$W_{\text{translation}} = F \cdot x$$

Work done by the force F in rotating the disc

$$W_{\text{rotation}} = \tau \cdot \theta = (F \cdot R)\theta = FR\theta$$

Total work done by F in combined rotation and translation

$$W_{\text{total}} = Fx + FR\theta$$

ILLUSTRATION 4.41

Calculate the torque developed by an airplane engine whose output is 2000 HP at an angular velocity of 2400 rev/min.

Sol. Given angular velocity, $\omega = 2\pi(2400/60) = 80\pi$ rad/s

Power output of airplane engine = 2000 HP = 2000 × 746 W

Work done by torque = (torque) × (angular displacement)

$$\text{Power} = \text{work done per second} = \tau \frac{\Delta\theta}{\Delta t} = \tau\omega$$

$$\text{As } P = \tau\omega \Rightarrow \tau = \frac{2000 \times 746}{80\pi} = 5937 \text{ Nm}$$

WORK–KINETIC ENERGY THEOREM FOR ROTATIONAL MOTION

Like the work–kinetic energy theorem in translational motion, this theorem states that the net work done on the system of the rigid object, which represents a transfer of energy across the boundary of the system that appears as an increase in the object's rotational kinetic energy.

To prove that fact, let us begin with the rigid object under a net torque model, whose mathematical representation is given by,

$$\tau_{\text{ext}} = I\alpha = I\frac{d\omega}{dt} \quad \ldots(\text{i})$$

Using the chain rule from calculus, we can express the net torque as

$$\tau_{\text{ext}} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega \quad \ldots(\text{ii})$$

Rearranging this expression and noting that $\tau_{\text{ext}}\, d\theta = dW$ gives

$$\tau_{\text{ext}} d\theta = I\omega d\omega$$

Integrating this expression, we obtain for the work W done by the net external force acting on a rotating system

$$W_{\text{rotation}} = \int_{\omega_i}^{\omega_f} I\omega\, d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 \quad \ldots(\text{iii})$$

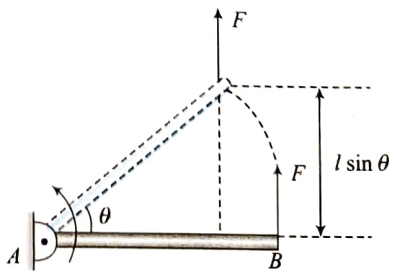
Important Points:

- The work done by a force acting on a rigid body may be classified into two parts: translational work and rotational work.
- A force can do translational work if the centre of mass of the body is free to move.
  Mathematically, it is defined as
  $$W_{\text{trans}} = \int F ds_c$$
  where ds_c is the infinitesimal displacement of the centre of mass.
- A force can do rotational work if the body is free to rotate.
  Mathematically, it is defined as
  $$W_{\text{rot}} = \int \tau_c d\theta$$
  where $d\theta$ is the infinitesimal angular displacement about the centre of mass and τ_c is the torque produced by the force about the centre of mass of the body.
- In an inertial frame, the work–energy theorem may be stated as
  $$W_C + W_{NC} + W_{\text{oth}} = \Delta K$$
  where W_C is the work done by conservative forces, W_{NC} is the work done by non-conservative forces, W_{oth} is the work done by all other forces which are not included in the above two categories. Each term of work includes both the translational and rotational work. In the right-hand side term of the equation the kinetic energy also includes both the translational and rotational kinetic energies.

ILLUSTRATION 4.42

A thin horizontal uniform rod AB of mass m and length l can rotate freely about a vertical axis passing through its end A. At a certain moment, the end B starts experiencing a constant force F which is always perpendicular to the original position of the stationary rod and directed in a horizontal plane. Find the angular velocity of the rod as a function of its rotation angle θ measured relative to the initial position.

Sol. Work done by the torque,



$$W_{\text{rotation}} = \int \tau\, d\theta = \int_0^{\theta} Fl\cos\theta\, d\theta$$

$$\Rightarrow W_{\text{rotation}} = Fl\sin\theta$$

or we can calculate the work done by force as

$W_{\text{rotation}} = F \times$ displacement in the direction of force $= Fl\sin\theta$

Now using work energy theorem, $W_{\text{rotation}} = \Delta K_{\text{rotational}}$

$$\therefore \quad Fl\sin\theta = \left[\frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2 - 0\right],$$

which gives, $\omega = \sqrt{\dfrac{6F\sin\theta}{ml}}$

Important Points:

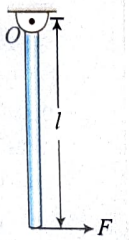
- The rotational kinetic energy about a fixed axis is the total kinetic energy. It includes both translational as well as rotational kinetic energy about centroidal axes.
- Care should be taken in calculating the work done. Either calculate work done by rotational method (torque method) or by translational method (force method).

ILLUSTRATION 4.43

A uniform rod of mass m and length l is pivoted smoothly at O. A horizontal force acts at the bottom of the rod.

(a) Find the angular velocity of the rod as the function of angle of rotation θ.

(b) What is the maximum angular displacement of the rod?



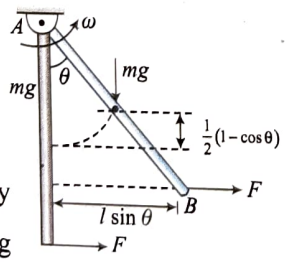
Sol. Here we can calculate work done in rotating the rod either by torque method or by force method, but in this case, we can easily use force method for calculating the work by force and gravity.

(a) Using work energy theorem

$$W_F + W_{gr} = \Delta K$$

Hence, $W_F + W_{gr} = \left(\dfrac{1}{2}I_P\omega^2 - 0\right)$

$\dfrac{1}{2}I_P\omega^2$ = Rotational kinetic energy about the rotational axes passing through hinge, it is also total kinetic energy of the rod



Work done by 'F': $W_F = F \times l\sin\theta = Fl\sin\theta$

Work done by gravity: $W_{gr} = -mg \times \dfrac{l}{2}(1-\cos\theta)$

$$\Rightarrow Fl\sin\theta - mg\frac{l}{2}(1-\cos\theta) = \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{6F}{ml}\sin\theta - \frac{3g}{l}(1-\cos\theta)} \qquad \ldots(i)$$

(b) At maximum angular displacement put $\omega = 0$ in Eq. (i)

$$0 = \sqrt{\frac{6F}{ml}\sin\theta - \frac{3g}{l}(1-\cos\theta)}$$

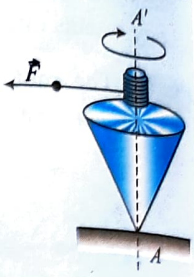
$$\Rightarrow \frac{6F}{ml}\sin\theta = \frac{3g}{l}(1-\cos\theta)$$

$$\Rightarrow \frac{2F}{m}\left(2\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)\right) = g\left(2\sin^2\left(\frac{\theta}{2}\right)\right)$$

$$\Rightarrow \tan\frac{\theta}{2} = \frac{2F}{mg} \Rightarrow \theta = 2\tan^{-1}\left(\frac{2F}{mg}\right)$$

ILLUSTRATION 4.44

The top in the figure has a moment of inertia equal to 4.0×10^{-4} kg m^2 and is initially at rest. It is free to rotate about the stationary axis AA'. A string wrapped around a peg along the axis of the top is pulled in such a manner as to maintain a constant tension of 10.0 N. If the string does not slip while it is unwound from the peg, what is the angular speed of the top after 80.0 cm of string has been pulled off the peg?



Sol. Work done $= F\Delta r = (10.0\,\text{N})(0.80\,\text{m}) = 8.0\,\text{J}$

and work $= \Delta K = \dfrac{1}{2}I\omega_f^2 - \dfrac{1}{2}I\omega_i^2$

Thus, $8.0 = \dfrac{1}{2}(4.0\times10^{-4}\,\text{kg}\times\text{m}^2)\omega_f^2 - 0$

(The last term is zero because the top starts from rest.)

and from this $\omega_f = 200$ rad/s.

CONSERVATION OF MECHANICAL ENERGY

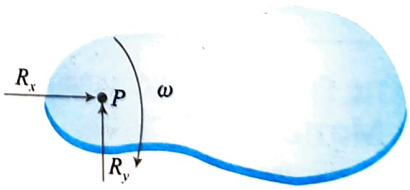
In an inertial frame, the work–energy theorem may be stated as

$$W_C + W_{NC} = \Delta K$$

If a rigid body does not experience any non-conservative force or even though a non-conservative force like force of static friction acts, if it does not perform any work (as described in pure rolling etc.) i.e., $W_{NC} = 0$ the work–energy equation assumes the form $W_C = \Delta K$.

Since conservative forces perform work at the expense of their potential energy, we can write $W_C = -\Delta U$.

Then, we have $-\Delta U = \Delta K$. This gives $\Delta K + \Delta U = 0$.

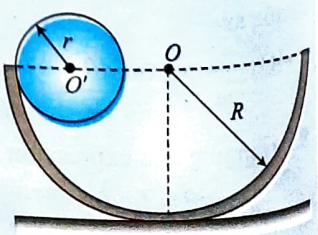


When a rigid body moves, the constraint forces are reactions at the pivot. The normal reaction sometimes does not perform work if the point of application of the forces does not move. For instance, when a rigid body rotates about a fixed axis, the reaction forces R_x and R_y do not perform work.

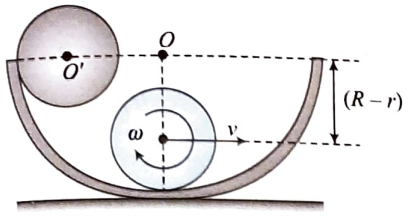
In such situations also, we can apply conservation of mechanical energy principle.

ILLUSTRATION 4.45

A solid sphere of mass m and radius r is released from rest from the given position. If it rolls without sliding on the circular track of radius R, find its speed when it reaches its lowest position.



Sol. As the sphere is rolling, the static friction does not perform any work. Hence mechanical energy is conserved



According to conservation of energy, $\Delta K + \Delta U = 0$.

or $(K_f - K_i) + \Delta U = 0$ ...(i)

When the sphere reaches its lowest position, it loses gravitational potential energy by ΔU and gains a kinetic energy by ΔK, where

$$\Delta U = -mg(R-r)$$

The sphere is released from rest, its initial kinetic energy $K_i = 0$.

And kinetic energy when the sphere reaches at its lowest position

$$K_f = (1+c)\frac{1}{2}mv^2$$

For sphere, $c = \frac{k^2}{R^2} = \frac{2/5\,R^2}{R^2} = \frac{2}{5}$

$$K_f = \frac{1}{2}mv^2\left(1+\frac{2}{5}\right) = \frac{7}{10}mv^2$$

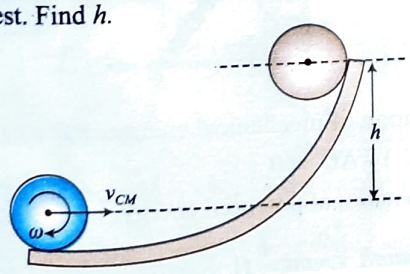
Hence from (i),

$$\left(\frac{7}{10}mv^2 - 0\right) + (-mg(R-r)) = 0$$

We get, $v = \sqrt{\frac{10}{7}g(R-r)}$

ILLUSTRATION 4.46

A ball of radius R and mass m is rolling without slipping on a horizontal surface with velocity of its centre of mass v_o. It then rolls without slipping up a hill to a height h before momentarily coming to rest. Find h.



Sol. Mechanical energy is conserved because static friction does not perform any work. As the ball is rolling, then initial kinetic energy of the ball,

$$K_{\text{initial}} = (1+c)\frac{1}{2}mv^2 = \left(1+\frac{2}{5}\right)\frac{1}{2}mv^2 = \frac{7}{10}mv_0^2$$

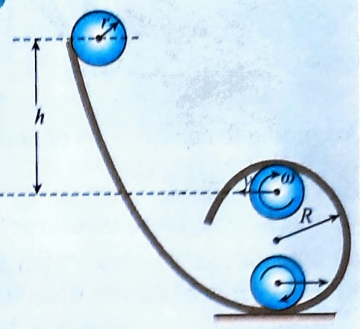
As mechanical energy is conserved, the initial kinetic energy will be converted to potential energy

$$\Delta K + \Delta U = 0$$

$$\Rightarrow \quad \left(0 - \frac{7}{10}mv_0^2\right) + (mgh) = 0 \quad \text{or} \quad h = \frac{7v_0^2}{10g}$$

ILLUSTRATION 4.47

A spherical ball of mass m and radius r is released on the path as shown in figure. If friction is sufficient to prevent the sliding of the ball on every point on the track, find the minimum height h (above the top position in the loop) that will permit the ball to maintain constant contact with the surface of the loop.

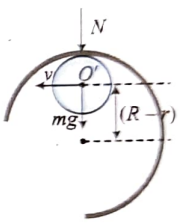


Sol. According to conservation of mechanical energy, the kinetic energy of the sphere at the top position in the loop is equal to the decrease in potential energy (mgh) as it falls from the initial position to this position.

As the ball is rolling, the kinetic energy of the ball,

$$K = (1+c)\frac{1}{2}mv^2 = \left(1+\frac{2}{5}\right)\frac{1}{2}mv^2 = \frac{7}{10}mv^2 \qquad \text{...(i)}$$

In the critical case when the ball just negotiates the loop, the force exerted by the loop on the ball is zero when the latter reaches the top of the loop. The centripetal force needed for the circular motion of the ball is supplied entirely by gravity:



$$\frac{mv^2}{(R-r)} = mg$$

$$\Rightarrow \quad v^2 = (R-r)g \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$K = \frac{7}{10}m(R-r)g \qquad \text{...(iii)}$$

As mechanical energy is conserved, then $\Delta K + \Delta U = 0$

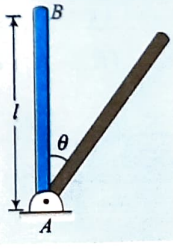
$$\left(\frac{7}{10}m(R-r)g - 0\right) + (-mgh) = 0$$

$$\Rightarrow \quad h = \frac{7(R-r)}{10}$$

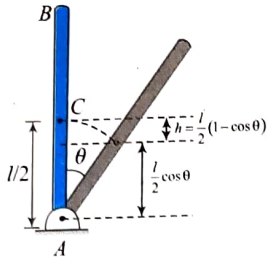
Hence $h = 7R/10$ is the minimum initial height required.

ILLUSTRATION 4.48

A rod of length l is pivoted about a horizontal, frictionless hinge through one end. The rod is released from rest in a vertical position. Find the velocity of the CM of the rod when the rod is inclined at an angle θ from the vertical.



Sol. The rod is rotating about frictionless hinge, its mechanical energy should be conserved.



According to conservation of energy,

$$\Delta K + \Delta U = 0$$

$$\Rightarrow \quad (K_f - K_i) + \Delta U = 0 \qquad \text{...(i)}$$

Initially the rod is released from rest $K_i = 0$.

Kinetic energy of the rod when it makes an angle θ with vertical

$K_f = \frac{1}{2} I_A \omega^2$ = Rotational kinetic energy about the rotational axes passing through hinge, it is also total kinetic energy of the rod.

The fall in position of the CM of the rod, $h = \frac{l}{2}(1-\cos\theta)$.

Hence change in potential energy of the rod,

$$\Delta U = -mg\frac{l}{2}(1-\cos\theta)$$

From (i), $\left(\frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2 - 0\right) + \left(-mg\frac{l}{2}(1-\cos\theta)\right) = 0$

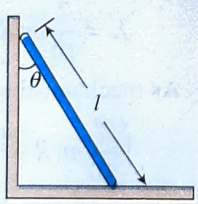
$$\therefore \quad \omega = \sqrt{\frac{3g}{l}(1-\cos\theta)}$$

The velocity of the CM of the rod, $v_{cm} = \omega R$ is

$$v_{cm} = \sqrt{\frac{3g}{l}(1-\cos\theta)} \times \frac{l}{2} = \sqrt{\frac{3gl(1-\cos\theta)}{4}}$$

ILLUSTRATION 4.49

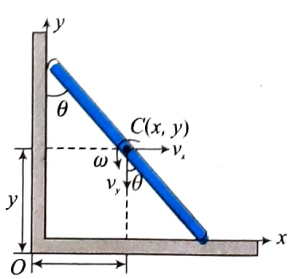
A uniform rod of length l is released from rest. In consequence, it rotates in vertical plane and its centre of mass moves down under the action of gravity and the reaction forces offered by horizontal and vertical smooth surfaces. Find the angular speed of the rod as the function of angle θ made by it with the vertical.



Sol. **Method 1:** The rod is sliding without sfriction, its mechanical energy should be conserved. According to conservation of energy,

$$\Delta K + \Delta U = 0$$

$$\Rightarrow \quad (K_f - K_i) + \Delta U = 0 \qquad \text{...(i)}$$



As the rod rotates through an angle θ its centre of mass falls through a vertical distance

$$h = \frac{l}{2}(1-\cos\theta).$$

Hence $\Delta U = -mgh = -mg\frac{l}{2}(1-\cos\theta)$

Initially the rod is released from rest, $K_i = 0$.

If the rod spins with an angular velocity ω and horizontal and vertical velocities of centre of mass of the rod be v_x and v_y respectively, the kinetic energy of the rod is given as

$$K_f = K_{\text{translational}} + K_{\text{rotational}} = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2$$

$$K_f = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}\left(\frac{ml^2}{12}\right)\omega^2 \qquad \text{...(ii)}$$

Since $x = \frac{l}{2}\sin\theta$, we have

$$v_x = \frac{l}{2}\cos\theta\frac{d\theta}{dt} = \frac{l\omega}{2}\cos\theta$$

Similarly, as $y = \frac{l}{2}\cos\theta$, we have

$$v_y = \frac{l}{2}\left(-\sin\theta\frac{d\theta}{dt}\right) = -\frac{l\omega}{2}\sin\theta$$

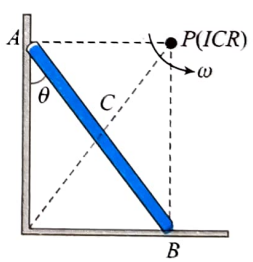
Substituting v_x and v_y in Eq. (ii), we have

$$K_f = \frac{ml^2\omega^2}{6}$$

Finally substituting ΔU and ΔK in the Eq. (i), we have

$$\omega = \sqrt{\frac{3g}{l}(1-\cos\theta)}$$

Method 2: Instantaneous axis of rotation (IAR) passes through P. There will be pure rotation of the rod about P.



From conservation of mechanical energy, $\Delta K + \Delta U = 0$.

$$\Rightarrow \quad (K_f - K_i) + \Delta U = 0 \qquad \text{...(i)}$$

Change in gravitational potential energy

$$\Delta U = -mgh = -mg\frac{l}{2}(1-\cos\theta)$$

Initially the rod is released from rest, $K_i = 0$.

When the rod makes an angle θ with vertical let it spins with an angular velocity ω about IAR. It's kinetic energy at this position

$$K_f = \frac{1}{2}I_P\omega^2$$

Here, $I_P = I_C + m(PC)^2 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}$

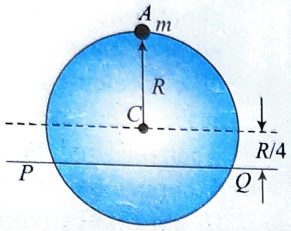
Hence, $K_f = \frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2$

Finally substituting ΔU and ΔK in the Eq. (i), we have

$$\omega = \sqrt{\frac{3g}{l}(1-\cos\theta)}$$

ILLUSTRATION 4.50

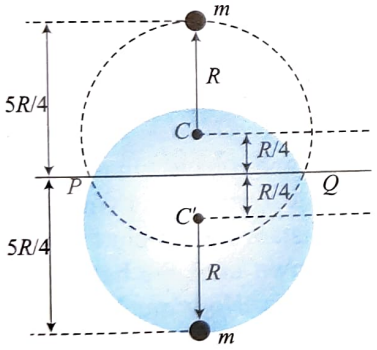
A uniform circular disc has radius R and mass m. A particle, also of mass m, is fixed at point A on the edge of the disc as shown in figure. The disc can rotate freely about a fixed horizontal chord PQ that is at a distance $R/4$ from the centre C of the disc. The line AC is perpendicular to PQ. Initially, the disc is held vertical with point A at its highest position. It is then allowed to fall so that it starts rotating about PQ. Find the linear speed of the particle as it reaches its lowest position.



Sol. The system (disc + particle) is rotating about fixed axis without friction, its mechanical energy should be conserved. According to conservation of energy, $\Delta K + \Delta U = 0$

$$\Rightarrow \quad (K_f - K_i) + \Delta U = 0 \qquad \ldots(i)$$

During the fall, the disc-particle system gains rotational kinetic energy. This is at the expense of potential energy.



Change in gravitational potential energy,

$$\Delta U = -\left(mg2\left(R+\frac{R}{4}\right) + mg2\left(\frac{R}{4}\right)\right) = -3mgR$$

Change in kinetic energy, $\Delta K = \left(\frac{1}{2}I_{PQ}\omega^2 - 0\right)$

where I_{PQ} = MI of the disc + MI of particle about PQ

$$I_{PQ} = (I_{\text{disc}})_{PQ} + (I_{\text{par}})_{PQ} = \left[\frac{mR^2}{4} + m\left(\frac{R}{4}\right)^2\right] + m\left(\frac{5R}{4}\right)^2$$

$$= \frac{15mR^2}{8}$$

Therefore, $\Delta K = \frac{1}{2}\left(\frac{15mR^2}{8}\right)\omega^2$

Finally substituting ΔU and ΔK in the Eq. (i), we have

$$\frac{1}{2}\left(\frac{15mR^2}{8}\right)\omega^2 + (-3mgR) = 0$$

$$\Rightarrow \quad \omega = \sqrt{\frac{16g}{5R}}$$

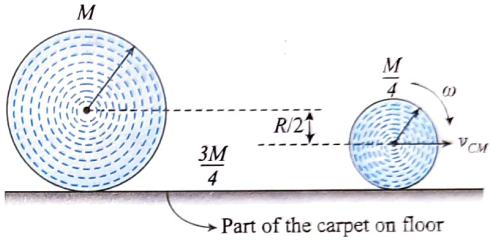
Let v be the velocity of mass m at the lowest point of rotation

$$v = \omega\left(R + \frac{R}{4}\right) = \sqrt{\frac{16g}{5R}} \times \frac{5R}{4} = \sqrt{5gR}$$

ILLUSTRATION 4.51

A carpet of mass M made of inextensible material is rolled along its length in the form of a cylinder of radius R and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when a negligibly small push is given to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius reduces to $R/2$.

Sol. Figure shows the forces acting on the carpet in two cases:



(i) when the radius of the roll is R

(ii) when the radius is $R/2$

Mass of the roll of radius $R/2 = M/4$

(Mass is proportional to the area of cross section)

The mass of the part of the carpet spread on the floor = $(3/4)M$.

The mechanical energy of the system will remain constant. The part of the carpet spread on the ground will have no kinetic energy.

According to conservation of energy, $\Delta K + \Delta U = 0$.

$$\Rightarrow \quad (K_f - K_i) + \Delta U = 0. \qquad \ldots(i)$$

As the roll of the carpet is not sliding, we can consider that the roll of carpet as a rolling cylinder. We can calculate the kinetic energy of the moving cylindrical roll by the relation

$$K = (1+c)\frac{mv^2}{2} = \left(1+\frac{1}{2}\right)\frac{mv^2}{2} = \frac{3mv^2}{4}$$

M = mass of the roll at any time

Hence the kinetic energy of the roll when its radius is $R/2$

$$K_f = \frac{3(M/4)v^2}{4} = \frac{3Mv^2}{16}$$

Change in kinetic energy,

$$\Delta K = K_f - K_i = \frac{3Mv^2}{16}$$

Change in gravitational potential energy,

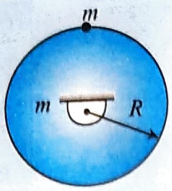
$$\Delta U = -MgR + \frac{1}{4}Mg\frac{R}{2} = -\frac{7}{8}MgR$$

Finally substituting ΔU and ΔK in the Eq. (i), we have

$$\Rightarrow \quad \frac{3Mv^2}{16} + \left(-\frac{7}{8}MgR\right) = 0$$

$$\Rightarrow \quad v = \sqrt{\frac{14}{3}gR}$$

ILLUSTRATION 4.52



A bead of mass m is welded at the periphery of the smoothly pivoted disc mass m and radius R. Find the speed of the bead at its lowest position.

Sol. When the bead comes to its lowest position

$$\Delta U = -mgh = -mg(2R) = -2mgR \quad \ldots(i)$$

$$\Delta K' = \Delta K'_{\text{bead}} + \Delta K_{\text{disc}} = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\omega^2$$

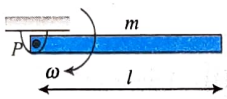
$$= \frac{mv^2}{2} + \frac{mv^2}{4} = \frac{3}{4}mv^2 \quad \left(\text{because } \omega = \frac{v}{R}\right) \quad \ldots(ii)$$

Substituting ΔU from Eq. (i), ΔK from Eq. (ii) in the equation, $\Delta U + \Delta K = 0$, we have

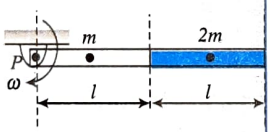
$$-2mgR + \frac{3}{4}mv^2 = 0 \text{ or } v = \sqrt{\frac{8}{3}gR}$$

CONCEPT APPLICATION EXERCISE 4.5

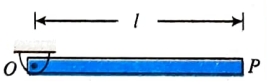
1. A rod of mass m spins with an angular speed $\omega = \sqrt{g/l}$, Find its
   (a) kinetic energy of rotation.
   (b) kinetic energy of translation
   (c) total kinetic energy.



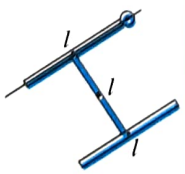
2. Find KE of the rotating composite rod comprising two rods of mass m and $2m$ and each of length $l = 1$ m. Assume $\omega =$ 10 rad s^{-1} and $m = 1$ kg.



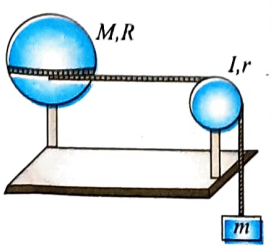
3. A uniform rod smoothly pivoted at one of its ends is released from rest. If it swings in vertical plane, find the maximum speed of the end P of the rod.



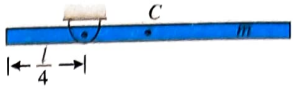
4. A rigid body is made of three identical thin rods, each with length l, fastened together in the form of letter H. The body is free to rotate about a horizontal axis that runs along the length of one of the arms of H. The body is allowed to fall from rest from a position in which the plane of the H is horizontal. What is the angular speed of the body when the plane of H is vertical?



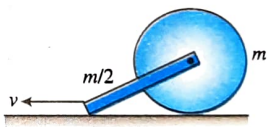
5. A uniform spherical shell of mass M and radius R rotates about a vertical axis on frictionless bearing. A massless cord passes around the equator of the shell, over a pulley of rotational inertia I and radius r and is attached to a small object of mass m that is otherwise free to fall under the influence of gravity. There is no friction of pulley's axle; the cord does not slip on the pulley. What is the speed of the object after it has fallen a distance h from rest? Use work–energy considerations.



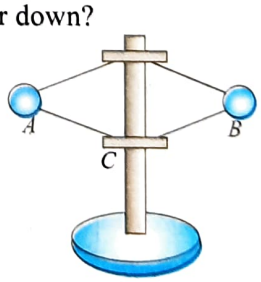
6. A uniform rod of mass m and length l is kept vertical with the lower end clamped. It is slightly pushed to let it fall down under gravity. Find its angular speed when the rod is passing through its lowest position. Neglect any friction at the clamp. What will be the linear speed of the free end at this instant?
7. A uniform rod of length l is released from rest such that it rotates about a smooth pivot. Find the angular speed of the rod when it becomes vertical.



8. A solid uniform disk of mass m and radius R is pivoted about a horizontal axis through its centre and a small body of mass m is attached to the rim of the disk. If the disk is released from rest with the small body, initially at the same level as the centre, find the angular velocity when the small body reaches the lowest point of the disk.
9. A copper ball of mass $m = 1$ kg with a radius of $r =$ 10 cm rotates with angular velocity $\omega = 2$ rad/s about an axis passing through its centre. What work should be performed to increase the angular velocity of rotation of the ball two fold?
10. A uniform disc of mass m is fitted (pivoted smoothly) with a rod of mass $m/2$. If the bottom of the rod is pulled with a velocity v, it moves without changing its angle of orientation and the disc rolls without sliding. Find the kinetic energy of the system (rod + disc).



11. The steel balls A and B have a mass of 500 g each and are rotating about the vertical axis with an angular velocity of 4 rad/s at a distance of 15 cm from the axis. Collar C is now forced down until the balls are at a distance of 5 cm from the axis. How much work must be done to move the collar down?



ANSWERS

1. (a) $\frac{mgl}{24}$ (b) $\frac{mgl}{8}$ (c) $\frac{mgl}{6}$ 2. 250 J 3. $\sqrt{3gl}$

4. $\sqrt{\frac{9g}{4l}}$ 5. $\left(\frac{mgh}{\frac{m}{2} + \frac{I}{2r^2} + \frac{M}{3}}\right)^{1/2}$ 6. $\sqrt{\frac{6g}{l}}, \sqrt{6gl}$

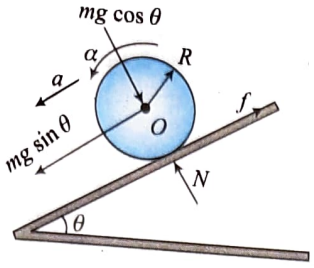
7. $2\sqrt{\frac{6g}{7l}}$ 8. $\sqrt{\frac{12g}{11r}}$ 9. 2.4×10^{-2} J

10. mv^2 11. 1.44 J

ROLLING OF A BODY ON AN INCLINED PLANE

A rigid body of radius of gyration k and radius R rolls (without slipping) down a plane inclined at an angle θ with the horizontal.

When a body is placed on an inclined plane, it tries to slip down and hence a static friction f acts upwards. This friction provides a torque which causes the body to rotate. Let a be the linear acceleration of centre of mass and α be the angular acceleration of the body.



Acceleration of the rolling body can be determined by considering its pure rolling motion as superposition of rotational (about the centre of mass) and translational motion.

From force diagram: for linear motion parallel to the plane,

$$mg\sin\theta - f = ma \quad \text{...(i)}$$

For rotation around the axis through centre of mass, net torque

$$\tau = I\alpha$$

$$\Rightarrow \quad fR = (mk^2)\alpha \quad \text{...(ii)}$$

As there is no slipping, the point of contact of the body with the plane is instantaneously at rest.

$$\Rightarrow \quad v = R\omega \text{ and } a = R\alpha \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii) for a and f,

$$a = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)} = \frac{g\sin\theta}{(1+c)}$$

and $$f = \frac{mg\sin\theta}{\left(1+\frac{R^2}{k^2}\right)} = \frac{mg\sin\theta}{\left(1+\frac{1}{c}\right)} = \frac{c}{(1+c)}mg\sin\theta$$

where c is a constant depends upon the shape of the rolling object.

We can also derive the condition for pure rolling (rolling without slipping), to avoid slipping, $f \le \mu_s N$.

$$\left(\frac{c}{1+c}\right)mg\sin\theta \le \mu_s\, mg\cos\theta \Rightarrow \mu_s \ge \left(\frac{c}{1+c}\right)\tan\theta$$

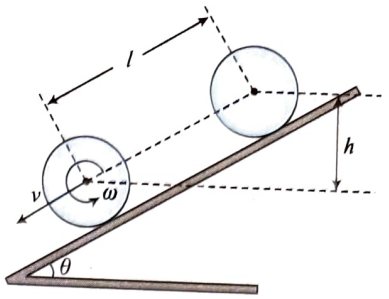
This is the condition of μ_s so that the body rolls without slipping.

Important Points:

- If $\mu_s = 0$, then the body will perform pure sliding and its acceleration will be maximum.
  $a_{max} = g\sin\theta$
- If $0 < \mu_s < \left(\frac{c}{1+c}\right)\tan\theta$, then the motion will be rolling with sliding and then $\frac{g\sin\theta}{c+1} < a < g\sin\theta$
- If $\mu_s \ge \left(\frac{c}{1+c}\right)\tan\theta$, the motion will be pure rolling and the acceleration will be minimum.
  $a_{min} = \frac{g\sin\theta}{c+1}$

APPLICATION OF CONSERVATION OF MECHANICAL ENERGY FOR FINDING THE ACCELERATION OF ROLLING OBJECT

Let the rolling object is released from rest, after moving down the height h its velocity be v.



As the object is rolling, mechanical energy will remain conserved applying conservation of mechanical energy principle,

$$\Delta K + \Delta U = 0$$

$$\Rightarrow \quad (K_f - K_i) + \Delta U = 0$$

$$\Rightarrow \quad \left((1+c)\frac{1}{2}mv^2 - 0\right) + (-mgh) = 0$$

or $$v^2 = \frac{2gh}{1+c} \Rightarrow v = \sqrt{\frac{2gh}{1+c}} \quad \text{...(i)}$$

Equation (i) gives the velocity of the rolling object when its CM moves down the height h.

As $h = l\sin\theta \Rightarrow v^2 = \frac{2gl\sin\theta}{1+c}$ ...(ii)

Differentiating Eq. (i) w.r.t. time

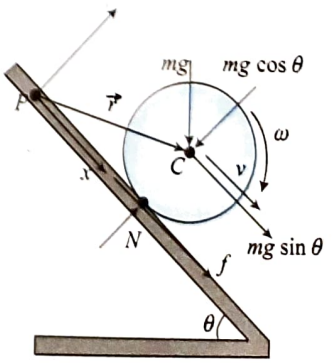
$$2v\left(\frac{dv}{dt}\right) = \frac{2g\sin\theta}{1+c}\left(\frac{dl}{dt}\right) \Rightarrow 2va = \frac{2g\sin\theta}{1+c}v$$

or $$a = \frac{g\sin\theta}{1+c} \qquad \left(\text{As } \frac{dv}{dt} = a \text{ and } \frac{dl}{dt} = v\right)$$

APPLICATION OF ANGULAR MOMENTUM PRINCIPLE FOR FINDING THE ACCELERATION OF ROLLING OBJECT

Let a rolling object with radius of gyration is rolling on an inclined plane. Let P is a fixed point on inclined plane. The angular momentum of the rolling body at any time about P is

$$\vec{L}_P = \vec{r}\times m\vec{v} + mk^2\vec{\omega}$$
$$= -mvR\hat{k} - mk^2\omega\hat{k}$$
$$= -(mvR + mk^2\omega)\hat{k},$$



where $v = R\omega$ for rolling,

or $$(\vec{L}_P)_i = -m(R^2 + k^2)\omega\hat{k}$$

Since $(\vec{L}_P)_{initial} = 0$,

Change in angular momentum in time t

$$\Delta\vec{L}_P = (\vec{L}_P)_t - 0 = -m(R^2 + k^2)\omega\hat{k} \quad \text{...(i)}$$

The net torque about P is

$$\vec{\tau}_P = Rmg\sin\theta(-\hat{k}) \quad \text{...(ii)}$$

Substituting, $\Delta\vec{L}_P$ from Eq. (i) and τ_P from Eq. (ii) in $\int\vec{\tau}_P dt = \Delta\vec{L}_P$, we have

$$-\int mgR\sin\theta\,\hat{k}\,dt = -m(R^2+k^2)\omega\hat{k}$$

Differentiating both sides, we have

$$mgR\sin\theta = m(R^2+k^2)\alpha\hat{k}$$

This gives $\alpha = \dfrac{gR\sin\theta}{R^2+k^2}$

But acceleration of CM of the rolling object

$$a = \alpha R = \left(\frac{gR\sin\theta}{R^2+k^2}\right)R = \frac{g\sin\theta}{1+\dfrac{k^2}{R^2}} = \frac{g\sin\theta}{(1+c)}$$

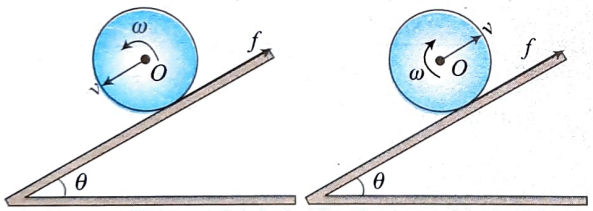
Comparison of pure rolling and pure sliding

	Pure Rolling	Pure Sliding
Acceleration	$a = \dfrac{g\sin\theta}{1+c}$	$a = g\sin\theta$
Velocity	$v = \sqrt{\dfrac{2gh}{1+c}}$	$v = \sqrt{2gh}$

The comparison reveals that $a_{\text{rolling}} < a_{\text{sliding}}$ and $v_{\text{rolling}} < v_{\text{sliding}}$

Important Points:

- If a body rolls up an inclined plane the direction of friction force in this case also in upward direction. As body rolls up velocity of centre of mass decreases, hence torque of friction should decrease angular velocity.

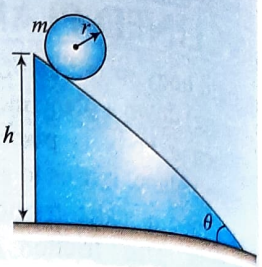


Rolling down Rolling up

- The acceleration of the bodies rolling on inclined plane depends on radius of gyration (k). We can say the bodies with same value of k will have the same acceleration.
- For example, two uniform solid spheres made of different material, unequal mass and unequal radii are released from the same inclined plane if they roll, they will have same accelerations.
- If a solid sphere, a hollow sphere, a disc and a ring of same mass and radius are released from top of incline and they roll. The solid sphere takes the least time while the ring will take maximum time to reach the bottom.
- If all the four objects are released from the top of a smooth incline. All the four bodies will reach the bottom at the same time.
- If all the rolling objects are released on a rough incline, such that friction is not sufficient for pure rolling, still the objects reach the bottom of incline at the same time because acceleration of centre of mass is the same in each case. Each object will have the same velocity of centre of mass but different ω.

ILLUSTRATION 4.53

Consider a uniform solid cylinder of mass m and radius r rolling (without slipping) down at an angle θ with the horizontal, as shown in figure. Find the time taken and speed of its centre of mass when the cylinder reaches the bottom.



Sol. We can use the formula which we derived

$a = \dfrac{g\sin\theta}{(1+c)}$; for cylinder, $c = \dfrac{k^2}{R^2} = \dfrac{R^2/2}{R^2} = \dfrac{1}{2}$

Hence, acceleration of the cylinder $a = \dfrac{2}{3}g\sin\theta$

Let v_{CM} be the velocity of the centre of mass at the bottom. From equation $v^2 = u^2 + 2as$.

$$v_{CM}^2 = 0^2 + 2\left(\frac{2}{3}g\sin\theta\right)\left(\frac{h}{\sin\theta}\right) = \frac{4}{3}gh$$

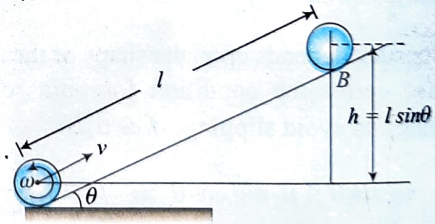
$$\Rightarrow \quad v_{CM} = \sqrt{\frac{4gh}{3}}$$

Let time to reach bottom by cylinder is t, using $v = u + at$

$$\Rightarrow \quad t = \frac{v-u}{a} = \frac{\sqrt{\dfrac{4gh}{3}} - 0}{\dfrac{2}{3}g\sin\theta} = \sqrt{\frac{3h}{g}}\frac{1}{\sin\theta}$$

ILLUSTRATION 4.54

A solid sphere is projected up along an inclined plane of inclination $\theta = 30°$ with a speed $v = 2$ m s^{-1}. If it rolls without slipping, find the maximum distance traversed by it. ($g = 10$ m s^{-2}).



Sol. Let the solid sphere of mass m and radius r traverse a distance l along the inclined plane. Since it rolls without sliding its initial KE is given as

$$KE_i = (1+c)\frac{1}{2}mv^2$$

where for solid sphere $c = \dfrac{2}{5}$

$$KE_i = \left(1+\frac{2}{5}\right)\frac{1}{2}mv^2 = \frac{7}{10}mv^2$$

As it comes to rest attaining a height $h = l\sin\theta$, its final $KE = 0$. Therefore,

$$\Delta KE = -\frac{7}{10}mv^2 \qquad \text{...(i)}$$

Change in gravitational potential energy, $\Delta U = mgh$ ...(ii)

Conservation of energy yields $\Delta K + \Delta U = 0$

$$\left(-\frac{7}{10}mv^2\right)+(mgh)=0$$

$$h=\frac{7v^2}{10g}=\frac{7\times(2)^2}{10\times10}=0.28\text{ m}$$

$$l=h\,\text{cosec}\,\theta=0.28\,\text{cosec}\,30^\circ=0.56\text{ m}$$

ILLUSTRATION 4.55

A ring of mass m and radius R is released on a rough inclined plane inclined at angle θ with horizontal. The ring rolls with sliding on the inclined plane. The coefficient of the friction between ring and plane is $\mu=\frac{\tan\theta}{4}$. Find out work done by kinetic friction if centre of mass of the ring has displaced 'l' length along the inclined plane.

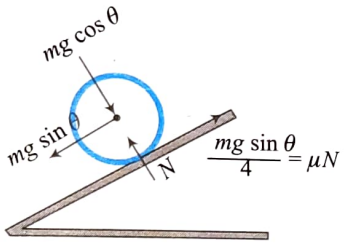
Sol. As the ring rolls with sliding it means the friction is not sufficient to provide rolling. In this case kinetic friction will act on it in upward direction.

For translatory motion:

$$mg\sin\theta-\mu N=ma\ [N=mg\cos\theta]$$

$$mg\sin\theta-\left(\frac{\tan\theta}{4}\right)mg\cos\theta=ma$$

$$\Rightarrow a=\frac{3g\sin\theta}{4}\qquad\ldots(i)$$



For rotatory motion, applying torque equation about centroid axis: $\tau_{cm}=I_{cm}\alpha$

$$\left[\frac{mg\sin\theta}{4}\right]R=mR^2\alpha$$

$$\alpha=\frac{g\sin\theta}{4R}\qquad\ldots(ii)$$

Time required to reach bottom of incline

$$s=ut+\frac{1}{2}at^2\Rightarrow l=\frac{1}{2}\left[\frac{3g\sin\theta}{4}\right]t^2\Rightarrow t=\sqrt{\frac{8l}{3g\sin\theta}}$$

At this time velocity and angular velocity of ring are

$$v=u+at=\sqrt{\frac{3gl\sin\theta}{2}}$$

and $\omega=\omega_i+\alpha t=\sqrt{\frac{gl\sin\theta}{6R^2}}$

By work energy theorem,

$K_f-K_i=$ work done by friction + work done by gravity

$$\frac{1}{2}mv^2+\frac{1}{2}mR^2\omega^2=W_{fr}+W_g$$

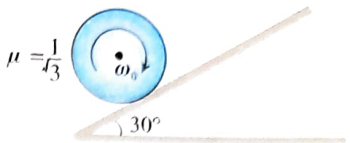
$$\Rightarrow\frac{1}{2}m\frac{3gl\sin\theta}{2}+\frac{1}{2}mR^2\frac{gl\sin\theta}{6R^2}=W_{fr}+W_g$$

$$\frac{1}{2}mgl\sin\theta\left[\frac{3}{2}+\frac{1}{6}\right]=W_{fr}+mgl\sin\theta$$

$$\Rightarrow W_{fr}=\frac{-1}{6}mgl\sin\theta$$

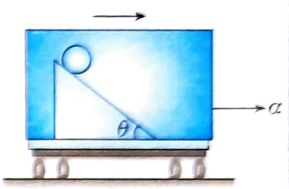
CONCEPT APPLICATION EXERCISE 4.6

1. A cylinder of mass m is rotated about its axis by an angular velocity ω and lowered gently on an inclined plane as shown in figure. Then:



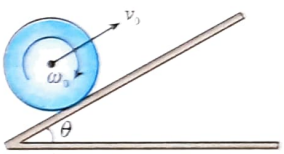
(a) it will start going upward
(b) it will first go upward and then downward
(c) it will go downward just after it is lowered
(d) it can never go upward

2. Figure shows a smooth inclined plane fixed in a car accelerating on a horizontal road. The angle of incline θ is released to the acceleration 'a' of the car as $a=g\tan\theta$. If the sphere is set in pure rolling on the incline:



(a) it will continue pure rolling
(b) it will slip down the plane
(c) its linear velocity will increase
(d) its linear velocity will decrease

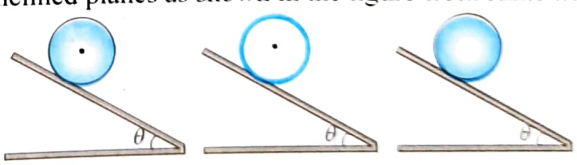
3. A sphere is moving on a smooth surface with linear speed v_0 and angular velocity ω_0. It finds a rough inclined surface and it starts climbing up:



(a) if $v_0>R\omega_0$, friction force will act downwards
(b) if $v_0<R\omega_0$, friction force will act downwards
(c) if $v_0=R\omega_0$, no friction force will act
(d) in all above case friction will act upwards

PASSAGE

A sphere, a ring and a disc of same mass and radius are allowed to roll down three similar sufficiently rough inclined planes as shown in the figure from same height.



4. Which of the following order is true for final KE of the bodies?
(a) sphere > disc > ring (b) ring > disc > sphere
(c) disc > ring > sphere (d) disc = ring = sphere

5. Which of the following order is true for final linear velocity of the bodies?
(a) sphere > disc > ring (b) ring > disc > sphere
(c) disc > ring > sphere (d) disc = ring = sphere

6. Which of the following order is true for time taken by the bodies to reach the bottom of incline?
(a) sphere > disc > ring (b) ring > disc > sphere
(c) disc > ring > sphere (d) disc = ring = sphere

ANSWERS

1. (d)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (a)
6. (b)				

ANGULAR IMPULSE

The sum of torques of all external forces acting on a system of particles (or rigid body) about any fixed point is equal to the rate of change in angular momentum about that point

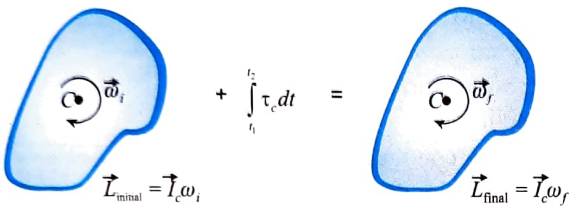
As $\vec{\tau} = \dfrac{d\vec{L}}{dt}$

Hence the change in angular momentum during time dt is given by

$$d\vec{L} = \vec{\tau} \cdot dt$$

It means the change in angular momentum of the rigid body during a time internal $\Delta t(t_2 - t_1)$ is given by

$$\Delta\vec{L} = \int d\vec{L} = \int_{t_1}^{t_2} \vec{\tau} \cdot dt = \text{Angular impulse}$$

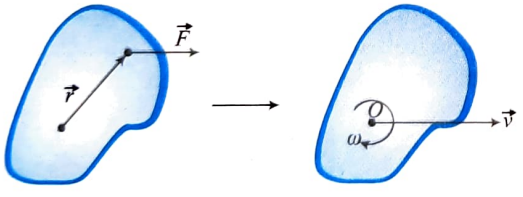


Initial angular momentum about CM + Net angular impulse about CM = final angular momentum about CM

Hence, "the net angular impulse about any fixed point or CM of the rigid body is equal to the change in angular momentum measured about that reference point." This statement is called **angular impulse–momentum theorem**.

CALCULATION OF ANGULAR IMPULSE

Let us assume a force $\vec{F}$ is acting at position P in a rigid body. Apart from translational effect, this force also produces rotational effect, that is, torque of the force.



The angular impulse of the force $\vec{F}$ related to the centre of mass can be given as the time integral of the torque taken relative to centre of mass (CM).

Then, angular impulse of the force $\vec{F}$ about C is

$$\Delta\vec{L} = \int \vec{\tau}_{P,C} \cdot dt = \int (\vec{r}_{P,C} \times \vec{F}) dt$$

$$\Rightarrow \quad \Delta\vec{L} = \vec{r}_{P,C} \times \int \vec{F} \cdot dt \qquad ...(i)$$

The term $\int \vec{F} \cdot dt$ is the linear impulse of the force $\vec{F}$. Now Eq. (i) can be write as

$$\Delta\vec{L} = \vec{r}_{P,C} \times \vec{J} \qquad ...(ii)$$

From Eq. (ii) we can say that angular impulse is the moment of linear impulse. It means we can calculate the angular impulse in the same way as we calculate the torque

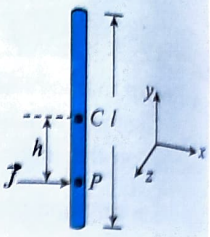
Hence, we can write the magnitude of angular impulse as

$$|\Delta\vec{L}| = \text{Linear impulse} \times \text{lever arm}$$

Hence lever arm is the perpendicular distance from the reference point and line of action of the linear impulse.

ILLUSTRATION 4.56

A uniform rigid rod of mass m, length l is placed as a smooth horizontal surface. An impulse $\vec{J}$ is give at point P which is at a distance h from the CM of the rod. Find:
(i) velocity of CM
(ii) angular velocity of the rod.

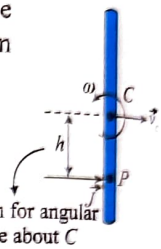


Sol.

(i) In case of rigid body, the linear impulse acting on it changes the linear momentum associated with its centre of mass.
Applying impulse momentum equation,

$$\vec{J} = \Delta\vec{p} \Rightarrow J(\hat{i}) = m\vec{v}_C - 0$$

or $\vec{v}_C = \dfrac{J}{m}(\hat{i})$



Lever arm for angular impulse about C

(ii) For finding angular velocity of the rod just after applying the impulse, we use angular impulse momentum equation

Angular impulse $\Delta\vec{L} = I_C\vec{\omega}_{final} - I_C\vec{\omega}_{initial}$ ...(i)

The magnitude of angular impulse

$|\Delta\vec{L}|$ = Linear impulse × lever arm = $J \times h$

For finding direction of angular impulse we use right hand rule

$\Rightarrow \Delta\vec{L} = Jh(\hat{k})$ ...(ii)

From (i) and (ii), we get

$$Jh(\hat{k}) = \frac{ml^2}{12}\vec{\omega} - 0 \text{ or } \vec{\omega} = \frac{12Jh}{ml^2}(\hat{k})$$

Hence the angular velocity of the rod will be $\dfrac{12Jh}{ml^2}$ in counter clockwise sense.

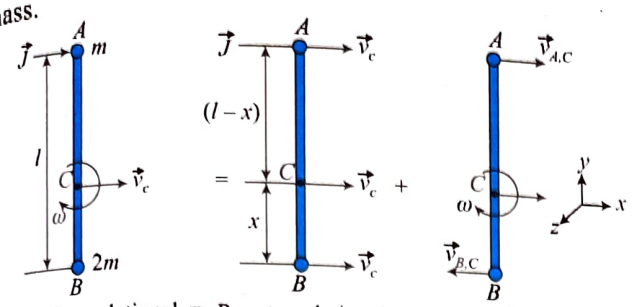
ILLUSTRATION 4.57

Two particles of masses m and $2m$ are attached at the ends A and B of a light rigid rod of length l. The system (m + rod + $2m$) is placed on a smooth horizontal surface. An impulse $\vec{J}$ is given at the end A of the rod perpendicular to its length as shown in figure. Find the
(i) velocity of centre of mass of the system
(ii) angular velocity ($\vec{\omega}$) of the angular velocity of the rod
(iii) velocity of end A and B of the rod

Sol.

(i) Since, line of action of impulse does not pass through centre of mass of the system, therefore, just after application of impulse, the system starts to move, both translationally

and rotationally. The linear impulse acting on the system changes the linear momentum associated with its centre of mass.



Combined translational and rotational motion = Pure translational motion + Pure rotational motion

(ii) Let us calculate the angular velocity of the rod, for this we need to calculate angular impulse on the system about CM:

$$\Delta\vec{L} = J(l-x)(-\hat{k}) \qquad ...(i)$$

x is the distance of centre of mass of the system from end B

$$x = \frac{2m(0)+m(l)}{2m+m} = \frac{l}{3}$$

Hence, $\Delta\vec{L} = J\left(l-\frac{l}{3}\right)(-\hat{k}) = \frac{2}{3}Jl(-\hat{k})$

But $\Delta\vec{L} = I_{CM}\vec{\omega}_{\text{final}} - I_{CM}\vec{\omega}_{\text{initial}}$

I_{CM} is the moment of inertia of the system about CM

$$I_{CM} = m(l-x)^2 + 2mx^2$$

$$= m\left(\frac{2}{3}l\right)^2 + 2m\left(\frac{l}{3}\right)^2 = \frac{2}{3}ml^2$$

Hence, $\frac{2}{3}Jl(-\hat{k}) = \frac{2}{3}ml^2\vec{\omega} - 0 \Rightarrow \vec{\omega} = \frac{J}{ml}(-\hat{k})$

It means the system will start rotating with angular velocity J/ml is clockwise sense.

(iii) Velocity of end A; $\vec{v}_A = \vec{v}_{A,C} + \vec{v}_C$

$$|\vec{v}_{A,C}| = \omega(l-x) = \frac{J}{ml}\times\frac{2}{3}l$$

$$\Rightarrow \vec{v}_{A,C} = \frac{2J}{3m}(\hat{i})$$

$$\vec{v}_A = \frac{2J}{3m}(\hat{i}) + \frac{J}{3m}(\hat{i}) = \frac{J}{m}\hat{i}$$

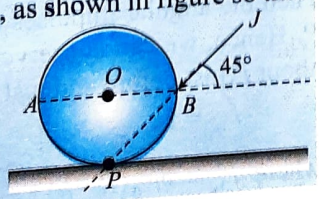
Similarly, velocity of end B; $\vec{v}_B = \vec{v}_{B,C} + \vec{v}_C$

Hence, $\vec{v}_B = \omega x(-\hat{i}) + \frac{J}{3m}(\hat{i})$

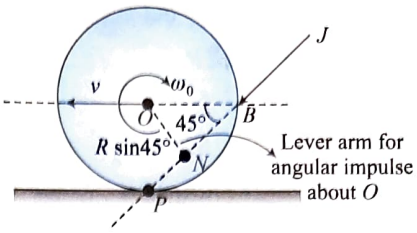
$$\Rightarrow \vec{v}_B = \frac{J}{ml}\times\frac{l}{3}(-\hat{i}) + \frac{J}{3m}(\hat{i}) = 0$$

ILLUSTRATION 4.58

AB is a horizontal diameter of a ball of mass $m = 0.4$ kg and radius $R = 0.10$ m. At time $t = 0$, a sharp impulse is applied at B at angle of 45° with the horizontal, as shown in figure so that the ball immediately starts to move with velocity $v_0 = 10$ m s^{-1}. Calculate the impulse and angular velocity of ball just after impulse provided.



Sol. Since, line of action of impulse does not pass through centre of mass of the sphere, therefore, just after application of impulse, the sphere starts to move, both translationally and rotationally. Translation motion is produced by horizontal component of the impulse, while rotational motion is produced by moment of the impulse. Let the impulse applied be J.



Then its horizontal component provides horizontal motion.

$J\cos 45°$ where gives $J = 4\sqrt{2}$ kg ms^{-1}

Moment of inertia of ball about centroidal axis is

$$I = \frac{2}{5}mR^2 = 1.6\times10^{-3}\ \text{kg m}^2$$

Initial angular momentum of ball (about centre) = $J(R\sin 45°)$

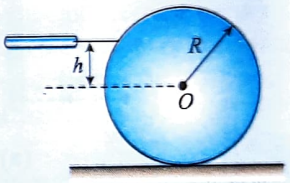
Angular impulse will change angular momentum of the ball.

$$JR\sin 45° = \Delta L = (I\omega_0 - 0)$$

which gives $\omega_0 = 250$ rad s^{-1} (clockwise)

ILLUSTRATION 4.59

A billiard ball (of radius R), initially at rest is given a sharp impulse by a cue. The cue is held horizontally at a distance h above the central line as shown figure. The ball leaves the cue with a speed v_0. It rolls and slides while moving forward and eventually acquires a final speed of $\frac{9}{7}v_0$. Show that $h = \frac{4}{5}R$.



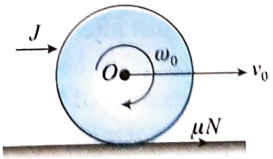
Sol. Let v_0 be the linear velocity and ω_0 be the angular velocity imparted.

Linear impulse = $J = mv_0$

Angular impulse = $Jh = I\omega_0$

$$h = \frac{(2/5\,mR^2)\omega_0}{mv_0} \Rightarrow v_0 = \frac{2R^2\omega_0}{5h} \qquad ...(i)$$

As linear velocity increases to $9/7v_0$, friction must be in forward direction and hence it opposes angular motion.



For linear motion: friction (μmg) increases the velocity. Hence,

$$\mu mg = ma \Rightarrow a = \mu g \qquad ...(ii)$$

Let t be the time when rolling starts.

For rotation: frictional torque (μmgR) opposes rotation and hence decreases ω.

$$\alpha = -\frac{\mu mgR}{I} \Rightarrow \omega = \omega_0 - \frac{\mu mgR}{\left(\frac{2}{5}mR^2\right)}t \quad \text{...(iii)}$$

After time t, $v = \frac{9}{7}v_0 = R\omega$ and $\frac{9}{7}v_0 = v_0 + \mu gt$ ...(iv)

Combining all the equations, we get

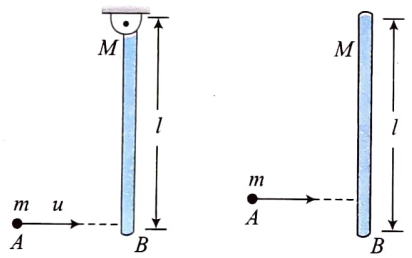
$$\frac{9}{7}v_0 - v_0 = -\left(\frac{9v_0}{7R} - \omega_0\right)\frac{2R}{5} \quad \text{(eliminating } \mu gt\text{)}$$

Substituting for v_0 from Eq. (i), we get

$$\frac{2}{7}\left(\frac{2}{5}\frac{R^2\omega_0}{h}\right) = \left(\omega_0 - \frac{9}{7}\frac{2R\omega_0}{5h}\right)\frac{2R}{5} \Rightarrow h = \frac{4R}{5}$$

CONSERVATION OF ANGULAR MOMENTUM IN CASE OF COLLISION OF RIGID BODIES

In chapter "Impulse and Collision", we have discussed head on and oblique collision. Our analysis was limited to find the velocities of the particles or bodies just after collision. In this section we will discuss the different cases of collision of two rigid bodies in which during or after collision rotational motion of the bodies is also considered.



As a particle collides with a rigid body, an impulsive force acts on the body and the particle which can change the state of motion of the body and particle.

Depending on the fact that whether the body is restricted to rotate about fixed axis of rotation or is free to rotate in any way, we are here classifying this topic into two ways:

(a) When the axis of rotation is fixed [Fig. (a)].

(b) When the body is free to rotate as well as translate (Fig. (b)).

We perform following steps while analyzing the collision in case of rigid bodies.

Step 1: Application of linear momentum. If there is no external impulse acting on the system, we can conserve linear momentum of the system just before and just after collision.

In the figures shown, if we take the particle A and rod B as system we cannot use conservation of linear momentum in case of Fig. (a) as there will be external impulse from hinge connected. But we can use conservation of linear momentum in the case of Fig. (b).

Step 2: Application of angular momentum. We can apply conservation of angular momentum if no external torque or angular impulse acts on the system about certain point.

In Fig. (a), there is impulse at hinge during collision. We can conserve angular momentum of the system (rod and particle) about hinge O. In Fig. (b) there is no external impulse, so we can conserve angular momentum of the system just before and just after collision.

Step 3: Application of impulse momentum. In the situations where we cannot use conservation of angular momentum, we can use impulse momentum equation

$$\vec{L}_{\text{initial}} + \int \vec{\tau}\, dt = \vec{L}_{\text{final}}$$

$$\int \vec{\tau}\, dt = \text{angular impulse}$$

Step 4: Application of coefficient of restitution equation. The ratio of velocity of separation after collision to the velocity of approach before collision is equal to coefficient of restitution.

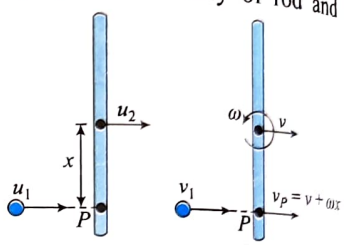
$$\vec{v}_{1,2} = -e\vec{u}_{1,2}$$

$\vec{u}_{1,2}$ and $\vec{v}_{1,2}$ are the relative velocities of the colliding points along the line of impact.

For example, in the figure a particle collides with a rod as shown in figure. Just after collision the velocity of rod and particle are given as

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{(v + \omega x) - v_1}{u_1 - u_2}$$

Here we are going to discuss different categories of rigid body collision through illustrations.

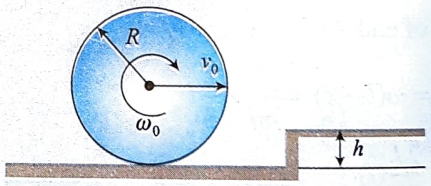


Just Before Collision Just After Collision

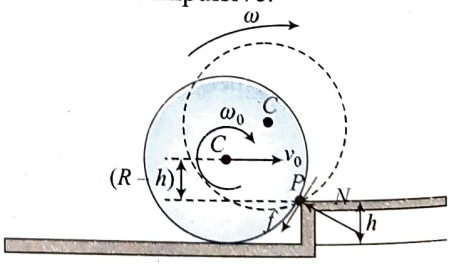
CASE 1: COLLISION OF A RIGID BODY WITH A FIXED SURFACE

ILLUSTRATION 4.60

A uniform solid sphere of radius R is rolling without sliding on a horizontal surface with a velocity v_0. It collides with an obstacle of height h ($< R$) inelastically. Find the angular speed of the sphere just after the collision.



Sol. During collision normal contact force N acting at the point of impact P is impulsive in nature hence friction will also be impulsive. The gravitational force (weight of the sphere) is ignored because it is non-impulsive.



If we consider point P, as the impulsive forces f and N are passing through this point, they cannot produce torques about P. Then the angular impulses of both forces f and N are zero about P. Hence, we can conserve the angular momentum of the body about P just before and just after the impact.

$$\vec{L}_{1_P} = \vec{L}_{2_P}$$

where $\vec{L}_{1_P} = mv_0(R-h) + \frac{2}{5}mR^2\omega_0$

and $\vec{L}_{2_P} = \left(\frac{2}{5}mR^2 + mR^2\right)\omega$

because the body rotates perfectly about P just after collision as it does not slide and bounce.

Then, we have

$$v_0(R-h) + \frac{2}{5}R^2\omega_0 = \frac{7}{5}R^2\omega$$

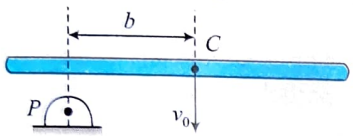
As the body was rolling before the impact, $v_0 = R\omega_0$

Putting $v_0 = R\omega_0$ in the above expression.

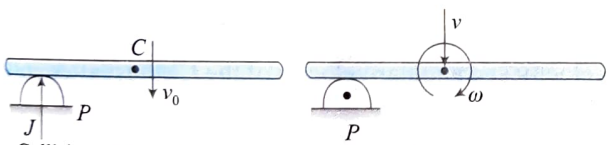
we have $\omega = \left(1 - \frac{5h}{7R}\right)\frac{v_0}{R}$

ILLUSTRATION 4.61

A uniform rod of length l moving with a downward velocity v_0 strikes a peg P. If the coefficient of restitution of impact is e, find the velocity of CM of the rod just after impact.



Sol. **Method 1:** The forces acting on the rod at the time of collision are N (upward) and mg (downward). The normal contact force N acting at the point of impact P is impulsive in nature. The gravitational force (weight of the sphere) is ignored because it is non-impulsive. Then, we disregard the effect of gravity during the impact.



Collision of the rod with peg — Just after collision

From impulse momentum equation

$$mv_0 - J = mv \quad \ldots(i)$$

The angular impulse of N about centre of mass of the rod

$$J.b = \frac{ml^2}{12}\omega \text{ or } J = \frac{ml^2}{12b}\omega \quad \ldots(ii)$$

Adding Eqs. (i) and (ii), we have

$$v_0 - v = \frac{l^2\omega}{12b} \quad \ldots(iii)$$

Newton's empirical formula gives us

Velocity of separation = e (Velocity of approach)

$$v_P = -ev_0 \quad \ldots(iv)$$

The velocity of P is $\vec{v}_P = \vec{v}_{PC} + \vec{v}_C$

Substituting $v_{PC} = -b\omega$ and $v_C = v$, we have

$$v_P = v - b\omega \text{ or } v - b\omega = -ev_0 \quad \ldots(v)$$

Now we have two Eqs. (iii) and (v) for two unknowns v and ω. Solving these two equations, we have

$$v = \frac{12b^2 - el^2}{12b^2 + l^2}v_0$$

Method 2: If we choose a fixed point at P, the angular impulse about P is zero because the impulse J passes through P during impact.

Hence the angular momentum of the rod just before and just after the collision about P remains constant.

$$\vec{L}_1 = \vec{L}_2, \text{ where } \vec{L}_1 = mv_0b(-\hat{k})$$

and $\vec{L}_2 = \frac{ml^2}{12}\omega(-\hat{k}) + mvb(-\hat{k})$

Then, $mv_0b = mvb + \frac{ml^2}{12}\omega$

This gives $v_0 - v = \frac{l^2\omega}{12b}$

The other equations remain unchanged.

Solving the above equations, we will obtain the same value of v.

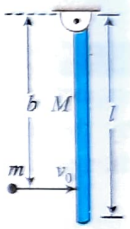
CASE 2: COLLISION OF A PARTICLE WITH RIGID BODY WITH THE AXIS OF ROTATION IS FIXED

ILLUSTRATION 4.62

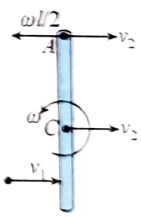
A ball of mass m collides elastically with a smooth hanging rod of mass M and length l.

(a) If $M = 3m$, find the value of b for which no horizontal reaction occurs at the pivot.

(b) For what value of m/M, the ball falls dead just after the collision assuming $e = (1/2)$?



Sol. According to problem no horizontal reaction occurs at pivot end it means there is no impulse on the system in horizontal direction. We can use conservation of linear momentum just before collision and just after collision in horizontal direction.



(a) Let the velocities of particle and rod just after collision is v_1 and v_2 respectively.

$$mv_0 = mv_1 + 3mv_2$$

or $v_0 = v_1 + 3v_2$ ...(i)

Now applying conservation of angular momentum about pivot end just before collision and just after collision. Let angular velocity of the rod about pivot end is ω.

$$mv_0 \cdot b = mv_1 \cdot b + \frac{3ml^2}{3}\omega$$

or $v_0 = v_1 + \frac{\omega l^2}{b}$ ...(ii)

From (i) and (ii),

$$3v_2 = \frac{\omega l^2}{b} \quad \ldots(iii)$$

For no impulse at end, there should not be movement of end A.

$$\vec{v}_A = \vec{v}_{A,C} + \vec{v}_C$$

$$v_A = 0 = -\frac{\omega l}{2} + v_2 \Rightarrow v_2 = \frac{\omega l}{2} \quad \ldots(iv)$$

From (iii) and (iv), $b = \frac{2l}{3}$

(b) If the ball falls dead just after the collision $v_1 = 0$
Hence, from conservation of angular momentum about pivot gives

$$mv_0 b = 0 + \frac{Ml^2}{3}\omega \Rightarrow mv_0 \cdot b = \frac{Ml^2}{3}\omega \quad ...(i)$$

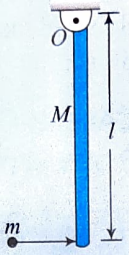
Again, applying Newton's restitution equation at the point of collision.

$$e = \frac{1}{2} = \frac{\omega b - 0}{v_0 - 0} \Rightarrow \omega = \frac{v_0}{2b} \quad ...(ii)$$

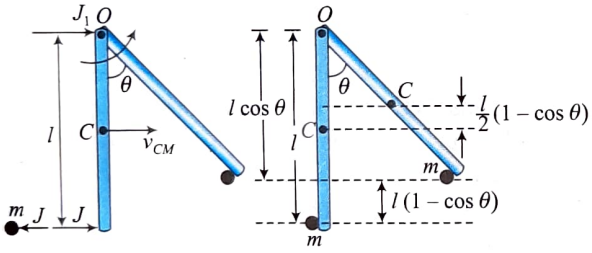
From Eqs. (i) and (ii), $\frac{m}{M} = \frac{l^2}{6b^2}$

ILLUSTRATION 4.63

A vertically oriented uniform rod of mass M and length l can rotate about its upper end. A horizontally flying bullet of mass m strikes the lower end of the rod and gets stuck in it; as a result, the rod swings through an angle θ. Assuming $m << M$, find the velocity of the flying bullet.



Sol. The impulse acting on the rod–bullet system during the impact is shown in figure. J_1 is the impulse acting on the rod by the hinge. Hence, we cannot apply conservation of linear momentum.



During the impact, the moment of impulse acting on the rod–bullet system about the hinge is zero. Angular momentum about the hinge will remain constant. Let ω be the angular velocity of the system about the hinge.

[Angular momentum of the system]$_{initial}$ = [Angular momentum of the system]$_{final}$

$$mv \times l = I_0 \times \omega \quad ...(i)$$

where I_0 is the moment of inertia of the rod–bullet system after collision about O

$$I_0 = I_{rod} + I_{bullet} = \frac{1}{3}Ml^2 + ml^2 \quad ...(ii)$$

$$mvl = \left(\frac{1}{3}Ml^2 + ml^2\right)\omega \Rightarrow \omega = \frac{3mv}{l(M+3m)} \quad ...(iii)$$

As the system swings about hinge, the mechanical energy of the system will remain constant, $\Delta K + \Delta U = 0$.

$$\left[0 - \frac{1}{2}\left(\frac{1}{3}Ml^2 + ml^2\right)\omega^2\right] + \left(Mg\frac{l}{2}(1-\cos\theta) + mgl(1-\cos\theta)\right) = 0 \quad ...(iv)$$

Substituting this value of ω in Eq. (iv), we get

$$\frac{1}{2}\left(\frac{1}{3}Ml^2 + ml^2\right)\left(\frac{3mv}{l(M+3m)}\right)^2 = gl\frac{(1-\cos\theta)}{2}(M+2m)$$

$$\Rightarrow v^2 = \frac{gl}{3}\frac{(M+2m)(M+3m)}{m^2}(1-\cos\theta)$$

Now for $m << M$, $(M+3m)(M+2m) \approx M^2$

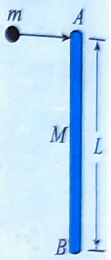
and $(1-\cos\theta) = 2\sin^2\frac{\theta}{2}$

Thus, $v^2 = \frac{2}{3}gl\frac{M^2}{m^2}\sin^2\frac{\theta}{2} \Rightarrow v = \frac{M}{m}\sqrt{\frac{2}{3}gl}\sin\frac{\theta}{2}$

CASE 3: COLLISION OF A PARTICLE WITH RIGID BODY WHEN THE BODY IS FREE TO ROTATE AS WELL AS TRANSLATE

ILLUSTRATION 4.64

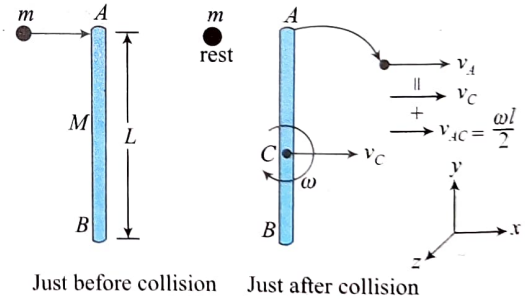
A rod AB of mass M and length L is lying on a horizontal frictionless surface. A particle of mass m travelling along the surface hits the end A of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is completely elastic. After the collision the particle comes to rest.



(a) Find the ratio m/M.
(b) A point P on the rod is at rest immediately after the collision. Find the distance AP.
(c) Find the linear speed of a point P at time $\pi L/3v_0$ after the collision.

Sol.

(a) If we take rod and particle as a system, the impulse between them will be internal. We can conserve the linear momentum of rod + particle's system. Let v_c and ω be the velocity and angular velocity of the CM just after collision.



Just before collision Just after collision

Using conservation of linear momentum, we have

$$mv_0 = 0 + Mv_c \quad ...(i)$$

Here in this case we can conserve angular momentum about any point on the rod.
And by conservation of angular momentum about the entire mass of the rod, we get

$$mv_0\frac{L}{2} = I\omega \text{ or } mv_0\frac{L}{2} = \left(\frac{ML^2}{12}\right)\omega \quad ...(ii)$$

From Eqs. (i) and (ii),

$$v_c = \frac{mv_0}{M} \text{ and } \omega = \frac{6mv_0}{ML}$$

We can apply conservation of angular momentum about point of collision.

$$\vec{L}_{initial} = \vec{L}_{final}$$

$$0 = I_0\omega(-\hat{k}) + Mv_c\frac{L}{2}(\hat{k})$$

$$\frac{ML^2}{12}\omega = \frac{Mv_cL}{2}$$

which gives $\omega L = 6v_c$ ...(iii)

Solving Eqs. (i) and (iii), we get the same results.

Since collision is completely elastic, therefore KE before collision is equal to KE after collision

$$\text{or } \frac{1}{2}mv_0^2 + 0 = \frac{1}{2}Mv_c^2 + \frac{1}{2}I\omega^2$$

$$\text{or } \frac{1}{2}mv_0^2 = \frac{1}{2}M\left(\frac{mv_0}{M}\right)^2 + \frac{1}{2}\frac{ML^2}{12}\left(\frac{6mv_0}{ML}\right)^2$$

which gives $\frac{m}{M} = \frac{1}{4}$ and $v_c = \frac{v_0}{4}$

Alternative method: We can apply coefficient of restitution equation at the point of collision, we have

$$e = \left(\frac{v_2 - v_1}{u_1 - u_2}\right)$$

Here u_1, u_2, v_1 and v_2 are the velocities of particles and point on the rod at collision point.

$$u_1 = v_0,\ u_2 = 0$$

$$v_1 = 0 \text{ and } v_2 = v_c + \frac{\omega l}{2}$$

As collision in elastic $e = 1$.

Substituting the values in restitution equation, we get

$$1 = \frac{\left(v_c + \frac{\omega L}{2}\right) - 0}{v_0 - 0} \Rightarrow v_0 = \left(\frac{mv_0}{M}\right) + \left(\frac{6mv_0}{ML}\right)\frac{L}{2}$$

which gives $\frac{m}{M} = \frac{1}{4}$

(b) Velocity of point P immediately after collision is zero, let it be at a distance y from CM

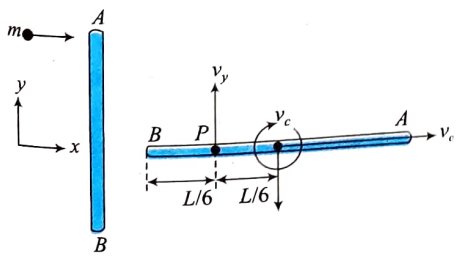
$$\therefore\ v_c - \omega y = 0 \text{ or } \frac{mv_0}{M} - \left(\frac{6mv_0}{ML}\right)y = 0$$

which gives $y = \frac{L}{6}$

$$\therefore\ AP = \frac{L}{2} + \frac{L}{6} = \frac{2L}{3}$$

(c) Angle rotated by the rod at time $\frac{\neq L}{3v_0}$,

$$\theta = \omega t = \left(\frac{6mv_0}{ML}\right)\times\frac{\pi L}{3v_0} = \frac{\pi}{2}$$



The rod turns through $\pi/2$ in this interval of time. The velocity of point P in y-direction will be

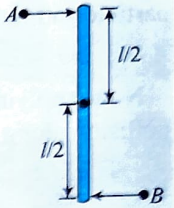
$$v_y = \omega y = \left(\frac{6mv_0}{ML}\right)\times\frac{L}{6} = \frac{v_0}{4}$$

The resultant velocity of point P,

$$v = \sqrt{v_c^2 + v_y^2} = \sqrt{\left(\frac{v_0}{4}\right)^2 + \left(\frac{v_0}{4}\right)^2} \Rightarrow \frac{v_0}{2\sqrt{2}}$$

ILLUSTRATION 4.65

Two particles A and B of mass m each hit the ends of a rigid bar of mass M and length l inelastically from opposite sides the speeds v and perpendicular to the rod. The bar is kept on a smooth horizontal plane (as shown). Find the linear and angular speed of the system (bar + particle).



Sol. There is no external force acting on the system along x-axis. Therefore, the linear momentum of the system before and after the system remains constant.

The momentum of the system before impact, $p_i = mv - mv = 0$

The linear momentum of the system after the impact must be zero. Therefore, the centre of mass of the system does not translate.

$$\Rightarrow \quad v_{cm} = 0$$

Since the impact forces are internal force, the net torque about a perpendicular axis passing through CM of the system is zero. Therefore, the angular momentum of the system before and after the impact will be equal.

$$\Rightarrow \quad L_{initial} = L_{final} \text{ where } L_{initial} = \left|\frac{ml}{2}v + \frac{ml}{2}v\right| = mvl$$

Let the system rotate about its CM 'O' with an angular speed ω

$$\Rightarrow \quad L_{final} = (I_{system})\omega$$

where I_{system} = MI of the system about O,

$$I_{system} = (I_{rod})_0 + (I_A)_0 + (I_B)_0$$

$$\Rightarrow \quad I_{system} = \frac{Ml^2}{12} + m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = \left(\frac{M+6m}{12}\right)l^2$$

Using the above equations, we get

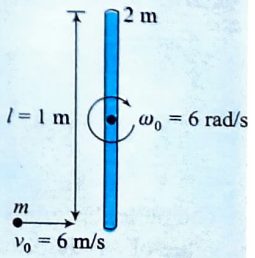
$$\left(\frac{M+6m}{12}\right)l^2\omega = mvl \Rightarrow \omega = \frac{12mv}{(M+6m)l}$$

ILLUSTRATION 4.66

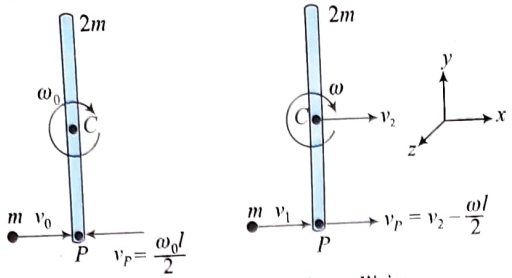
A particle of mass $m = 1$ kg collides with velocity $v_0 = 6$ m s^{-1} to a spinning rod of mass $2m$ and length $l = 1$ m at the end of the rod. If the coefficient of restitution of collision $e = (2/3)$, find the:

(a) velocity of the particle.

(b) angular velocity of the rod just after the impact.



Sol. If we take rod and particle as system we find no external impulse acting on the system.



Just before collision Just after collision

Hence, we can use conservation of linear momentum just before collision and just after collision. Let the velocities of particle and rod just after collision is v_1 and v_2 respectively.

$$mv_1 + 2mv_2 = mv_0 \text{ or } v_1 + 2v_2 = 6 \quad ...(i)$$

Now applying Newton's restitution equation at the point of collision along the line of impulse.

$$e = \frac{2}{3} = \frac{\left(v_2 - \frac{\omega l}{2}\right) - v_1}{v_0 - \left(-\frac{\omega_0 l}{2}\right)}$$

$$\Rightarrow \quad -2v_1 + 2v_2 = 12 + \omega \quad ...(ii)$$

Conserving angular momentum about the point of collision.

$$\vec{L}_i = \frac{2ml^2}{12}\omega_0(-\hat{k}) + 0$$

$$\vec{L}_f = \frac{2ml^2}{12}\omega(-\hat{k}) + 2mv_2 \times \frac{l}{2}(-\hat{k})$$

As $\vec{L}_i = \vec{L}_f$

$$\frac{2ml^2}{12}\omega_0(-\hat{k}) = \frac{2ml^2}{12}\omega(-\hat{k}) + mv_2 l(-\hat{k})$$

$$\text{or} \quad \omega_0 = \omega + \frac{6v_2}{l} \Rightarrow \omega = 6 - 6v_2 \quad ..(iii)$$

From (ii) and (iii), we get

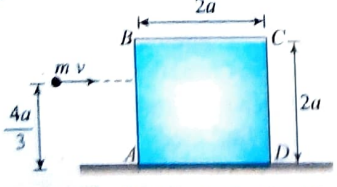
$$-v_1 + 4v_2 = 9 \quad ...(iv)$$

From (i) and (iv), $v_1 = 1$ m/s , $v_2 = 2.5$ m/s and $\omega = -9$ rad/s

It means the just after collision the rod will starts rotating with angular velocity 9 rad/s in counter clockwise sense.

ILLUSTRATION 4.67

A solid cube of wood of side $2a$ and mass M is resting on a horizontal surface. The cube is constrained to rotate about an axis passing through D and perpendicular to face $ABCD$. A bullet of mass m and speed v is shot at a height of $4a/3$ as shown in the figure.



The bullet becomes embedded in the cube. Find the minimum value of v required to topple the cube. Assume $m << M$.

Sol. Let ω be the angular velocity of cube just after collision about an axis passing through D. From conservation of the angular momentum about an axis passing through D

$$mv\left(\frac{4a}{3}\right) = I_D.\omega$$

Here $I_D = I_{cm} + mr^2$

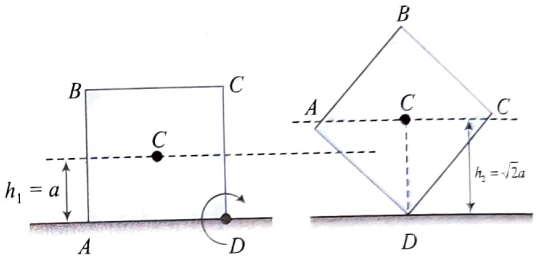
where r = perpendicular distance of axis of rotation passing through D from centre of mass (CM) of the cube $= \sqrt{2}a$

$$\text{or} \quad \frac{4mva}{3} = \left[M\left(\frac{4a^2 + 4a^2}{12}\right) + M(2a^2)\right]\omega$$

$$\text{or} \quad \frac{4mva}{3} = \frac{8}{3}Ma^2.\omega$$

$$\omega = \frac{1}{2}\frac{mv}{Ma} \quad ...(i)$$

The cube will topple if its CM is just able to reach in a vertical height h_2 as shown in Fig. (b).



Hence, applying conservation of mechanical energy

$$\frac{1}{2}I_D\omega^2 = Mg(h_2 - h_1) \qquad \left(\therefore I_D = \frac{8}{3}Ma^2\right)$$

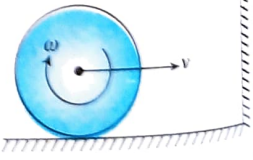
$$\text{or} \quad \frac{1}{2}\left(\frac{8}{3}Ma^2\right)\cdot\left\{\frac{1}{2}\frac{mv}{Ma}\right\}^2 = Mg(\sqrt{2} - 1)a$$

$$\text{or} \quad \frac{1}{3}\left(\frac{m}{M}\right)^2 v^2 = ag(\sqrt{2} - 1)$$

$$\text{or} \quad v = \frac{M}{m}[3ag(\sqrt{2} - 1)]^{\frac{1}{2}}$$

CONCEPT APPLICATION EXERCISE 4.7

1. A sphere rolling on a horizontal rough surface collides elastically with a smooth vertical wall, as shown in figure. State which of the following statements is true or false (T/F).



(a) After collision, the velocity of the centre of mass gets reversed.

(b) Angular momentum of the sphere about the point of contact with the wall is conserved.

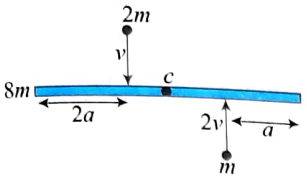
(c) Angular momentum of the sphere about a stationary point on the horizontal surface is conserved.

(d) Just after collision the point of contact with the horizontal surface is moving towards the wall.

(e) After collision the friction force acts on the sphere such that it decreases the linear speed and increases the angular speed.

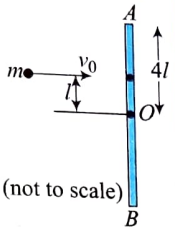
(f) Finally, when the sphere starts rolling, it is moving away from the wall.

2. A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses m and $2m$ moving in opposite direction in the same horizontal plane with speed $2v$ and v, respectively, strike the bar and stick to the bar after collision. Calculate



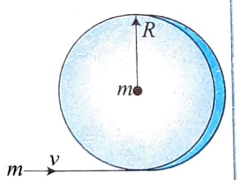
(a) the velocity of the centre of mass after collision
(b) the angular velocity of the centre of mass after collision
(c) total energy

3. A rod AB of mass M and length $8l$ lies on a smooth horizontal surface. A particle, of mass 'm' and velocity v_0, strikes the rod perpendicular to its length, as shown in figure. As a result of the collision, the centre of mass of the rod attains a speed of $v_0/8$ and the particle rebounds back with a speed of $v_0/4$. Find the following:

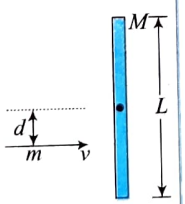


(a) The ratio M/m.
(b) The angular velocity of the rod about O.
(c) The coefficient of restitution 'e' for the collision.
(d) The velocities of the ends 'A' and 'B' of the rod, namely, v_A and v_B, respectively.

4. A circular wooden hoop of mass m and radius R rests flat on a frictionless surface. A bullet, also of mass m and moving with a velocity v, strikes the hoop and gets embedded in it. The thickness of the hoop is much smaller than R. Find the angular velocity with which the system rotates after the bullet strikes the hoop.



5. A stick of length l and mass m lies on a frictionless horizontal surface on which it is free to move in anyway. A ball of mass m moving with speed v collides elastically with the stick. What must be the mass of the ball so that it remains at rest immediately after collision?



6. A thin uniform rod of length l is initially at rest with respect to an inertial frame of reference. The rod is tapped at one end perpendicular to its length. How far the centre of mass translates while the rod completes one revolution about its centre of mass. Neglect gravitational effect.

7. A thin spherical shell of radius R lying on a rough horizontal surface is hit sharply and horizontally by a cue. Where should it be hit so that the shell does not slip on the surface?

8. A wheel rolling along a rough horizontal surface with an angular velocity ω strikes a rough vertical wall, normally. Find the initial angular velocity, as it tends to roll up the wall (neglect any slippage) due to impulse.

ANSWERS

1. (a) T (b) T (c) F (d) F (e) T

2. (a) zero (b) $\frac{v}{5a}$ (b) $\frac{3mv^2}{5}$

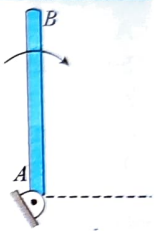
3. (a) 10 (b) $\frac{3v_0}{128l}$ (c) $\frac{51}{128}$ (d) $\frac{7}{32}v_0, \frac{v_0}{32}$ **4.** $\frac{v}{R}$

5. $\frac{ML^2}{L^2+12d^2}$ **6.** $\frac{\neq l}{3}$ **7.** $\frac{2R}{3}$ **10.** $\frac{\omega_0}{3}$

Solved Examples

EXAMPLE 4.1

A uniform slender bar AB of mass m and length L supported by a frictionless pivot at A is released from rest at its vertical position as shown in the figure. Calculate the reaction at pivot when the bar just acquires the horizontal position shown dotted.



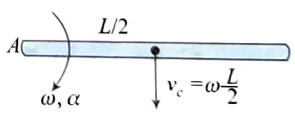
Sol. As there is no friction the mechanical energy will remain constant. Let us apply the conservation of mechanical energy between the vertical position (A) and the horizontal position (B).

Using $\Delta K + \Delta U = 0$

$$\left(\frac{1}{2}I_A\omega^2 - 0\right) + \left(-mg\frac{L}{2}\right) = 0$$

which gives $\omega = \sqrt{\frac{3g}{L}}$

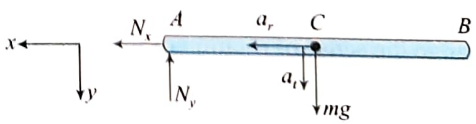
Considering the free-body diagram of the bar at this instant, the velocity of CM of bar is given below.



Let α be the angular acceleration of bar about A at the instant when it becomes horizontal.

$$\alpha = \frac{\tau_A}{I_A} = \frac{mg\frac{L}{2}}{\frac{mL^2}{3}} = \frac{3}{2}\frac{g}{L}$$

Let N_x and N_y are the components of reaction at A.



Then in horizontal direction $N_x = m a_r = m\omega^2\left(\frac{L}{2}\right)$

in vertical direction

$mg - N_y = ma_t$ ...(i)

a_t = acceleration of CM in y-direction = $(L/2)a$

or $N_y = mg - m\left(\frac{L}{2}\alpha\right) = mg - \frac{3mg}{4} = \frac{mg}{4}$ ...(ii)

Therefore, resultant reaction at support A is

$$N = \sqrt{N_x^2 + N_y^2}$$

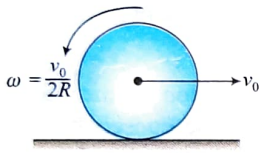
$$N = \sqrt{\left(\frac{3mg}{2}\right)^2 + \left(\frac{mg}{4}\right)^2} = \frac{\sqrt{37}}{4}mg$$

$$\tan\alpha = \frac{N_y}{N_x} = \frac{mg/4}{3mg/2} = \frac{1}{6} \Rightarrow \alpha = \tan^{-1}\left(\frac{1}{6}\right)$$

Therefore, reaction at pivot is $\frac{\sqrt{37}}{4}mg$ at an angle $\alpha = \tan^{-1}\left(\frac{1}{6}\right)$ with horizontal.

EXAMPLE 4.2

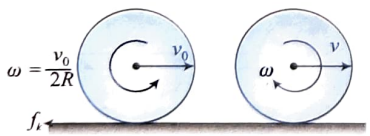
A solid sphere of radius R is set into motion on a rough horizontal surface with a linear speed v_0 in forward direction and an angular velocity $\omega_0 = v_0/2R$ in counter clockwise direction as shown in figure. If coefficient of friction is m, then find:



(a) the time after which sphere starts pure rolling,
(b) the linear speed of sphere when it starts pure rolling, and
(c) the work done by friction over a long time.

Sol.

(a) Because point of contact is slipping in forward direction relative to the surface, kinetic friction f_k will act in backward direction.



$f_k = \mu mg$

$\therefore$ Retardation, $a = \frac{f_k}{m} = \mu g$

and $\alpha = \frac{f_k R}{I_{cm}} = \frac{\mu mgR}{\frac{2}{5}mR^2} = \frac{5\mu g}{2R}$

The friction will decrease the translational velocity of CM and change the sense of rotation of the sphere, then increase the angular velocity of the sphere.

Now, let when sphere starts pure rolling its velocity be v and its angular velocity be ω.

Velocity of CM at any time, $v = v_0 - \mu gt$

Angular velocity of the sphere at any time,

$\omega = -\omega_0 + \alpha t$

$v = \omega R$, for pure rolling

$$v_0 - \mu gt = (-\omega_0 + \alpha t)R$$

$$v_0 - \mu gt = \left(-\frac{v_0}{2R} + \frac{5\mu gt}{2R}\right)R$$

$$\frac{3v_0}{2} = \frac{7}{2}\mu gt \Rightarrow t_0 = \frac{3v_0}{7\mu g}$$

(b) Linear speed at time $t = \frac{3v_0}{7\mu g}$ is

$$v = v_0 - \mu g\frac{3v_0}{7\mu g} = \frac{4v_0}{7}$$

Alternate method

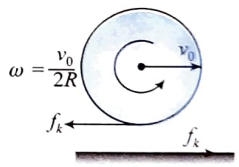
Since net torque about the axis fixed to the ground and passing through the point of contact is zero, using conservation of angular momentum about this axis,

$$-\frac{2}{5}mR^2\omega_0 + mv_0R = \frac{2}{5}mR^2\omega + mvR$$

$$-\frac{1}{5}mv_0R + mv_0R = \frac{7}{5}mvR \Rightarrow \frac{4}{5}mv_0R = \frac{7}{5}mvR$$

$\therefore \quad v = \frac{4v_0}{7}$

(c) Friction will act till the pure rolling starts. Once the pure rolling motion starts, friction will be zero. Hence, friction does work during time interval $0 < t < \frac{3v_0}{7\mu g}$ and after $t = \frac{3v_0}{7\mu g}$ work done by friction will be zero ($\because f = 0$).



Now, work done by friction over a long time is

$\Delta W_{fr} = K_f - K_i$

$$= \frac{7}{10}mv^2 - \left(\frac{1}{2}mv_0^2 + \frac{1}{2}\times\frac{2}{5}mR^2\omega_0^2\right)$$

$$= \frac{7}{10}m\times\frac{16v_0^2}{49} - \frac{11mv_0^2}{20} = \frac{8mv_0^2}{35} - \frac{11mv_0^2}{20}$$

$$= \frac{32mv_0^2 - 77mv_0^2}{140}$$

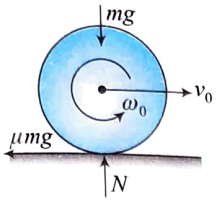
$$\Delta W_{fr} = -\frac{45mv_0^2}{140} = -\frac{9mv_0^2}{28}$$

EXAMPLE 4.3

A hoop of mass m is projected on a floor with linear velocity v_0 and reverse spin ω_0. The coefficient of friction between the hoop and the ground is μ.

(a) Under what condition will the hoop return back?
(b) How long will it continue to slip after its centre of mass becomes stationary?
(c) What is the velocity of return?

Sol. (a) The acceleration of hoop,



$$a = -\mu g$$

Angular acceleration of wheel $= -\dfrac{\mu g}{R}$

Linear velocity at time t, $v = v_0 - \mu g t$ ...(i)

Angular velocity at time t, $\omega = \omega_0 - \dfrac{\mu g}{R} t$ ...(ii)

Let linear velocity be zero at t.

So, $0 = v_0 - \mu g t_1$ or $t_1 = \dfrac{v_0}{\mu g}$

Angular velocity of the ring at t is

$$\omega = \omega_0 - \left(\frac{\mu g}{R}\right)\left(\frac{v_0}{\mu g}\right) = \omega_0 - \frac{v_0}{R} \quad \text{...(iii)}$$

The hoop will return if it has reverse spin at the instant its linear velocity is zero. So, the condition for hoop to return is

$$\omega_0 - \frac{v_0}{R} > 0 \text{ or } \omega_0 > \frac{v_0}{R}$$

(b) Let t_2 be the time up to which the hoop slips on the floor and v be the return velocity.

Then $v = \mu g t_2$ and $\omega' = \omega - \dfrac{\mu g}{R} t_2$

For no slip condition is achieved when

$$\frac{\mu g}{R} t_2 = \omega - \frac{\mu g}{R} t_2 \Rightarrow \frac{2\mu g}{R} t_2 = \omega \quad \text{...(iv)}$$

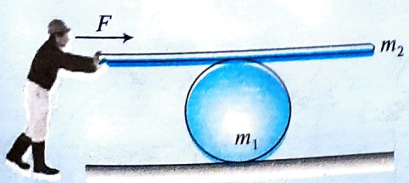
From Eqs. (iii) and (iv),

$$\frac{2\mu g}{R} t_2 = \omega = \omega_0 - \frac{v_0}{R} \text{ or } t_2 = \frac{R}{2\mu g}\left(\omega_0 - \frac{v_0}{R}\right)$$

(c) $v = \mu g t_2 = \dfrac{\omega_0 R}{2} - \dfrac{v_0}{2}$

EXAMPLE 4.4

A man pushes a cylinder of mass m_1 with the help of a plank of mass m_2 as shown in figure. There is no slipping at any contact. The horizontal component of the force applied by the man is F.

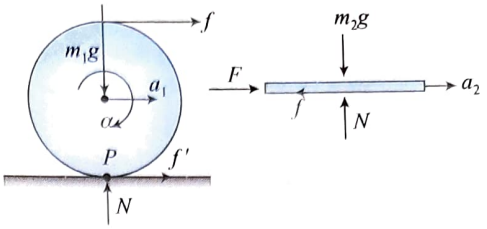


Find:

(a) the acceleration of the plank and the centre of mass of the cylinder;

(b) the magnitudes and directions of frictional forces at contact points.

Sol. Analysis of the direction of friction force (f): The man applies a force F in the horizontal direction on the plank as shown in figure. Therefore, the point of contact of the plank with the cylinder will try to move towards the right. Therefore, the friction force f will act towards the left on the plank.



To each and every action, there is an equal and opposite reaction. Therefore, a frictional force f will act on the top of the cylinder towards the right. For finding the direction of f', first assume f' is not acting.

A force f is acting on the cylinder. This force is trying to move the point of contact towards right by an acceleration $a_{CM} = \dfrac{f}{M_1}$ acting towards the right.

At the same time, the force f is trying to rotate the cylinder about its centre of mass in the clockwise direction.

$$f \times R = I \times \alpha \Rightarrow \alpha = \frac{f \times R}{I} = \frac{f \times R}{\frac{1}{2} M_1 R^2} = \frac{2f}{M_1 R}$$

Therefore, acceleration of the point of contact,

$$a_p = a_{CM} - \alpha R = \frac{f}{m_1} - \frac{2f}{m_1 R} \times R = \frac{f}{m_1}$$

i.e., towards the left.

Therefore, the point of contact of the cylinder with the ground moves towards the left. Therefore, friction force acts towards the right on the cylinder.

Here $k^2 < Rx$ as $k^2 = \dfrac{R^2}{2}$

and $Rx = R.R = R^2$

Hence, friction should act in forward direction.

Applying Newton's law on the plank, we get

$$F - f = m_2 a_2 \quad \text{...(i)}$$

Also $a_2 = 2a_1$ ...(ii)

because a_2 is the acceleration of the topmost point of the cylinder and there is no slipping. Applying Newton's law on the cylinder

$$m_1 a_1 = f + f' \quad \text{...(iii)}$$

The torque equation for the cylinder is

$$f \times R - f' \times R = I\alpha = \frac{1}{2} M_1 R^2 \times \left(\frac{a_1}{R}\right)$$

$\because$ $I = m_1 R^2$ and $R\alpha = a_1$

$\therefore$ $(f - f')R = \dfrac{1}{2} m_1 R a_1$

$\therefore$ $f - f' = \dfrac{1}{2} m_1 a_1$ ...(iv)

Solving Eqs. (iii) and (iv), we get

$$f = \frac{3}{4} m_1 a_1 \quad ...(v)$$

and $$f' = \frac{1}{4} m_1 a_1 \quad ...(vi)$$

From Eqs. (i) and (iii),

$$F - f = 2m_2 a_1$$

$$\Rightarrow \quad F = \frac{3}{4} m_1 a_1 = 2m_2 a_1$$

$$\therefore \quad a_1 = \frac{4F}{3m_1 + 8m_2}$$

$$\therefore \quad a_2 = \frac{8F}{3m_1 + 8m_2}$$

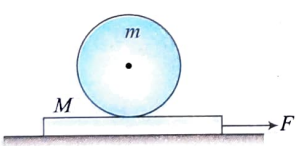
From Eqs. (v) and (vi), we get

$$f = \frac{3}{4} m_1 \times \frac{4F}{3m_1 + 8m_2} = \frac{3Fm}{3m_1 + 8m_2}$$

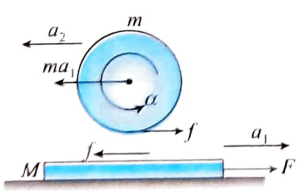
and $$f' = \frac{1}{4} m_1 \times a_1 = \frac{3Fm_1}{3m_1 + 8m_2}$$

EXAMPLE 4.5

A plank of mass M is placed on a smooth surface over which a cylinder of mass m (= M) and radius R = 1 m is placed as shown in figure. Now the plank is pulled towards the right with an external force F (= $2Mg$). If the cylinder does not slip over the surface of the plank, find the linear acceleration of the plank and the cylinder and the angular acceleration of the cylinder. (Take g = 10 m/s^2)



Sol. Here we are given that cylinder does not slip over the plank surface, it is the case of pure rolling. The friction between the plank and cylinder should be static. We can assume friction on cylinder in any direction, let it is acting on cylinder towards the right. As friction is acting on the cylinder towards the right, it must be towards the left on the plank as shown in the force diagram of figure.



Let the plank move towards the right with an acceleration a_1. If we consider the motion of the cylinder w.r.t. plank, the cylinder will experience a pseudo force ma_1 in the left direction, due to which it will roll towards the left with respect to the plank with an acceleration a_2. Let its angular acceleration during rolling be α, we have $a_2 = R\alpha$.

For translational motion of the plank, we have

$$F - f = Ma_1 \quad ...(i)$$

For translational motion of the cylinder with respect to the plank, we have

$$ma_1 - f = ma_2 \quad ...(ii)$$

For rotational motion of the cylinder with respect to the plank, we have

$$fR = I_{cm}\alpha \Rightarrow fR = \left(\frac{1}{2} mR^2\right)\left(\frac{a_2}{R}\right),$$

which gives $$f = \frac{1}{2} ma_2 \quad ...(iii)$$

From Eqs. (i) and (ii), we get

$$ma_1 - \frac{1}{2} ma_2 = ma_2$$

or $$a_1 = \frac{3}{2} a_2 \quad ...(iv)$$

Using Eqs. (i), (iii) and (iv), we get

$$F - \frac{1}{2} ma_2 = \frac{3}{2} Ma_2$$

$$a_2 = \frac{2F}{3M + m} = 10 \text{ m/s}^2$$

From Eq. (iv), $$a_1 = \frac{3F}{3M + m} = 15 \text{ m/s}^2$$

As we have already discussed that the value of a_2 is relative to the plank, the net acceleration of the cylinder will be given as $a_1 - a_2$.

Hence, the acceleration of the cylinder is

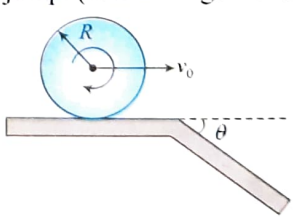
$$a_{\text{cylinder}} = a_1 - a_2 = 15 - 10 = 5 \text{ m/s}^2$$

The angular acceleration of the cylinder is

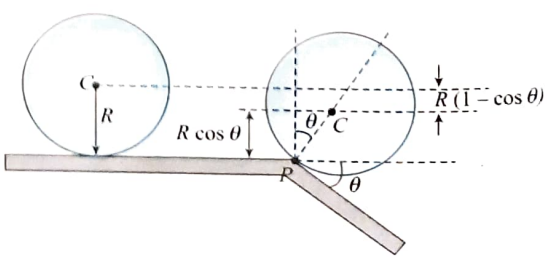
$$\alpha = a_2/R = 10 \text{ rad/s}^2$$

EXAMPLE 4.6

A uniform solid cylinder of radius R rolls over a horizontal plane passing into an inclined plane forming an angle θ with the horizontal. Find the maximum value of the velocity v_0 which still permits the cylinder to roll onto the inclined plane section without a jump. (The sliding is assumed to be absent.)



Sol. The cylinder rolling on horizontal surface and keep on rolling on the inclined surface without jumping on the edge. As there is no loss of energy we can apply conservation of mechanical energy.



The kinetic energy of a rolling object is given by

$$K_{rolling} = (1+c)\frac{1}{2}mv^2$$

Here v is the velocity of CM

and $c = \frac{K^2}{R^2}$

For cylinder $K^2 = \frac{R^2}{2}$, hence $c = \frac{1}{2}$.

It means the kinetic energy of a rolling cylinder

$$K = \left(1+\frac{1}{2}\right)\frac{1}{2}mv^2 = \frac{3}{4}mv^2$$

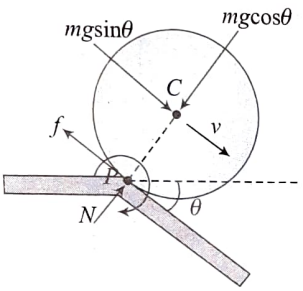
Now applying conservation of mechanical energy

$$\Delta K + \Delta U = 0$$

$$\left(\frac{3}{4}mv^2 - \frac{3}{4}mv_0^2\right) + \left(-mgR(1-\cos\theta)\right) = 0$$

or $$v^2 = v_0^2 + \frac{4}{3}gR(1-\cos\theta) \quad ...(i)$$

The cylinder rotates about the edge; from FBD of the cylinder rotating at edge.



$$mg\cos\theta - N = \frac{mv^2}{R}$$

or $$N = mg\cos\theta - \frac{mv^2}{R}$$

For no jumping $N \geq 0$

$$mg\cos\theta \geq \frac{mv^2}{R}$$

or $$v^2 \leq gR\cos\theta \quad ...(ii)$$

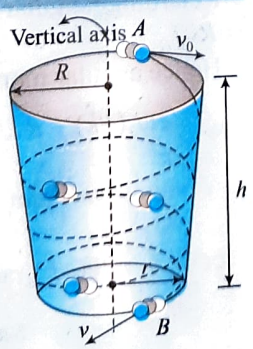
From (i) and (ii) we get

$$v_0 \leq \sqrt{\frac{gR}{3}\cos\theta - \frac{4}{3}gR}$$

or $$v_0 \leq \sqrt{\frac{gR}{3}(7\cos\theta - 4)}$$

EXAMPLE 4.7

A conical frustum of height h has been placed with its axis vertical. A particle of mass m is projected horizontally with velocity v_0 along its upper brim. The radii of its upper and lower cross sections are R and $r(=R/2)$ respectively. The particle slides along the inner surface in spiral path without friction and leaves the lower face at point B.

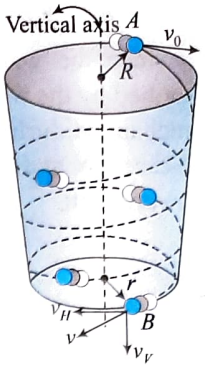


(a) Find the horizontal and vertical component of velocity of the particle as it leaves the frustum at B.

(b) Find minimum value of h for which the particle will never come out of the frustum.

Sol. The forces acting on the particle are: (i) normal reaction of the inner surface of the frustum wall (ii) weight of the particle.

Both the above forces do not produce any torque about the vertical axis of the frustum. It means, the angular momentum of the particle about the vertical axis will remain conserved.



(a) Let the horizontal and vertical component of particle velocity at point B are v_H and v_v respectively. Applying conservation of angular momentum principle

$$mv_0R = mv_Hr$$

$$\Rightarrow v_H = v_0\frac{R}{r} = 2v_0 \quad ...(i)$$

As the particle moves without friction, the mechanical energy of the particle will be conserved. Applying conservation of mechanical energy

$$\Delta K + \Delta U = 0$$

$$\left(\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2\right) + (-mgh) = 0$$

$$\therefore \quad v^2 = v_0^2 + 2gh$$

$$\Rightarrow v_H^2 + v_V^2 = v_0^2 + 2gh \quad ...(ii)$$

From (i) and (ii) we get

$$v_V^2 = v_0^2 + 2gh - 4v_0^2$$

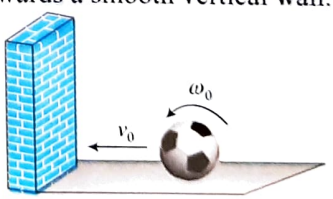
$$\Rightarrow v_V = \sqrt{2gh - 3v_0^2} \quad ...(iii)$$

(b) If the particle does not leave the frustum, its component of velocity in vertical direction should become zero before reaching the bottom of the frustum. Substituting $v_V = 0$ in Eq. (iii)

$$2gh = 3v_0^2 \Rightarrow h = \sqrt{\frac{3v_0^2}{2g}}$$

EXAMPLE 4.8

A ball of radius $R = 20$ cm has mass $m = 0.75$ kg and moment of inertia (about its diameter) $I = 0.0125$ kg m^2. The ball rolls without sliding over a rough horizontal floor with velocity $v_0 = 10$ m s^{-1} towards a smooth vertical wall.



If coefficient of restitution between the wall and the ball is $e = 0.7$, calculate velocity v of the ball long after the collision. ($g = 10$ m s^{-2}).

Sol. As the ball is rolling on ground, the lowest point of the ball should be at rest for this,

$$v_0 = \omega_0 R$$

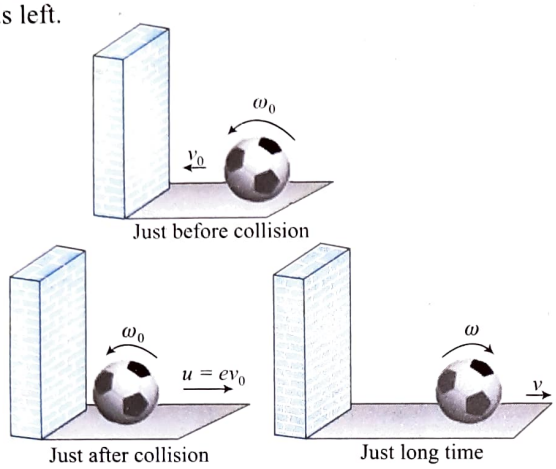
hence the angular velocity (ω_0) of the ball just before collision with wall is,

$$\omega_0 = \frac{v_0}{R} = \frac{10}{0.2} = 50 \text{ rad s}^{-1} \quad \text{(in counterclockwise sense)}$$

Now the ball collides with wall. The velocity of centre of ball, just after collision should be

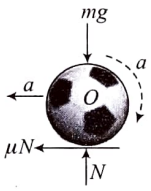
$$u = ev_0 = 0.7 \times 10 = 7\ \text{ms}^{-1} \qquad \text{(Rightwards)}$$

Since the wall is smooth, therefore no tangential force is applied by the wall. It means, angular velocity ω_0 of ball remains unchanged during collision. Now surface of ball slides on floor to the right as shown in the figure. Hence the friction should act towards left.



Let coefficient of friction between ball and the floor be μ.

From free-body diagram of ball while sliding:



For vertical forces, $N = mg$

For horizontal forces,

$$\mu N = ma \Rightarrow a = \mu g$$

Applying torque equation about centre of ball (about O) of force acting on the ball,

$$\mu NR = I\alpha \Rightarrow \alpha = \frac{\mu(mg)R}{I} = \frac{\mu \times (0.75 \times 10) \times 0.20}{0.0125}$$

or $\alpha = 120\,\mu$ (in clockwise sense)

Long after the collision, there will be no sliding, or it will be pure rolling. Let sliding stop after a time t after collision, then final translational velocity,

$v = (7 - at)$ or $v = 7 - 10\,\mu t$

and final angular velocity,

$\omega = (-\omega_0) + \alpha t$

$\omega = (120\,\mu t - 50)\ \text{rad s}^{-1}$ (clockwise)

But at that instant, $v_{cm} = \omega R$

$(7 - 10\,\mu t) = 0.2(120\,\mu t - 50)$

$\mu t = 0.5 \Rightarrow v = 2\ \text{ms}^{-1}$

EXAMPLE 4.9

A spherical ball of radius R = 5.5 cm is rotating about its one of the horizontal diameters with angular velocity ω_0 = 120 rad/s. It is released from a height so that it falls h = 1.8 m freely and then collides with the horizontal rough floor. Calculate the angular velocity of the ball after collision and the distance between the points where the ball strikes the floor for the first and second time.
(Given: Coefficient restitution is e = 5/6, coefficient of friction between the ball and the ground is ω = 0.2 and g = 10 ms^{-2}).

Sol. When ball falls freely it translational velocity increases because of acceleration due to gravity while its angular velocity ω_0 remains constant as there is no torque about its centre of mass.

The velocity of CM just before collision with the floor is,

$$v_0 = \sqrt{2gh} = \sqrt{2 \times 10 \times 1.8} = 6\ \text{ms}^{-1}$$

As, the coefficient of restitution is e = 5/6, therefore, it rebounds vertically upwards with velocity,

$$v_y = ev_0 = 5\ \text{ms}^{-1}$$

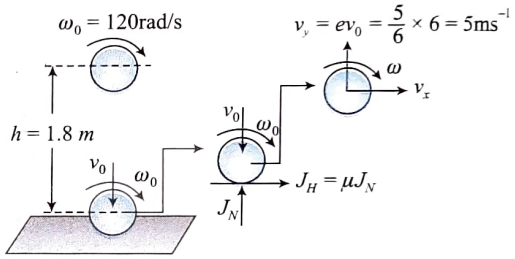
Normal impulsive reaction exerted by the floor is responsible for change in linear momentum in vertical direction. Impulse momentum equation:

$$mv_0 - J_N = -mv_y$$

$$m \times 6 - J_N = -m \times 5 \Rightarrow J_N = 11m\ \text{Ns}$$

where m is mass of the ball.

As the floor is rough and the ball is rotating about horizontal diameter, therefore its surface of the ball slips over the floor Hence, a horizontal frictional impulse is exerted by the floor to retard its angular velocity of the ball and due to this impulse the ball acquires a linear momentum in horizontal direction. It means, the ball moves like a projectile after rebounce.



That frictional impulse, $J_H = \mu . J_N = 2.2m$ Ns

Horizontal velocity of ball, just after collision is,

$$v_x = \frac{J_H}{m} = 2.2\ \text{ms}^{-1}$$

Moment of Inertial of the ball about the diameter is,

$$I = \frac{2}{5}mR^2 \text{ (Where } R \text{ is radius of the ball)}$$

Initial angular momentum of the ball,

$$\vec{L}_i = I\vec{\omega}_0 = \left(\frac{2}{5}mR^2\right)\vec{\omega}_0$$

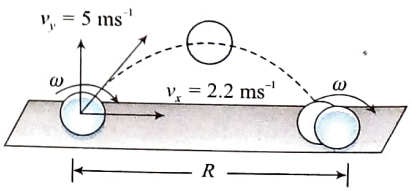
Retarding angular impulse produced by frictional impulse $= R \cdot J_H$

Hence angular momentum of ball just after collision is

$$I\omega_0(-\hat{k}) + R.J_H(\hat{k}) = I.\vec{\omega}$$

$$\left[-\left(\frac{2}{5}mR^2\right)\omega_0 + R \times 2.2m\right](\hat{k}) = \left(\frac{2}{5}mR^2\right).\vec{\omega}$$

or $\vec{\omega} = 20(-\hat{k})$ rad/s, in clockwise direction

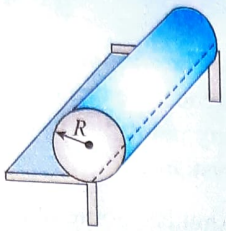


After striking with ground the ball will move in parabolic path. The horizontal range of projectile motion,

$$R = \frac{2v_x v_y}{g} = \frac{2\times 2.2\times 5}{10} = 2.2\,\text{m}$$

EXAMPLE 4.10

A solid homogeneous cylinder of radius R is placed horizontally at rest with its length parallel to the edge of a table as shown in figure. Initially the axis of the cylinder and the edge of the table are in the same vertical plane.



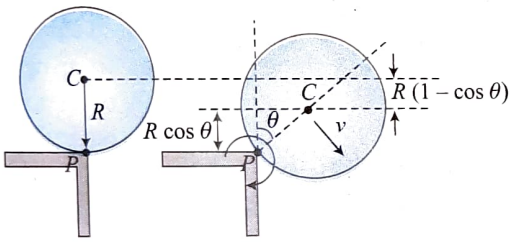
There is sufficient friction present at the edge so that a very small displacement causes the cylinder to roll off the edge without slipping. Determine:

(a) the angle θ through which the cylinder rotates before its leaves contact with the edge,

(b) the speed of the centre of mass of the cylinder before leaving contact with the edge.

Sol.

(a) The cylinder rolls without slipping about an axis passing through CM. Let the vertical diameter of the cylinder has rotated the angle θ on the edge of the table before its leaves contact with the edge,



Applying conservation of mechanical energy for the cylinder

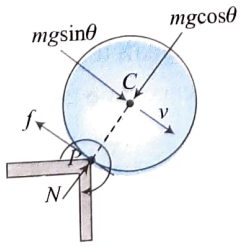
$$\Delta K + \Delta U = 0$$

$$\left((1+c)\frac{1}{2}mv^2 - 0\right) + \left(-mgR(1-\cos\theta)\right) = 0 \qquad \ldots(\text{i})$$

For cylinder $c = \dfrac{k^2}{R^2} = \dfrac{1}{2}$

From (i), we get

$$\frac{v^2}{R} = \frac{4}{3}g(1-\cos\theta) \qquad \ldots(\text{ii})$$



The cylinder is rotating about an axis passing through P. From

$$mg\cos\theta - N = \frac{mv^2}{R} \qquad \ldots(\text{iii})$$

When the cylinder leaves the contact, normal reaction $N = 0$. Hence

$$mg\cos\theta = \frac{mv^2}{R}$$

or $$\frac{v^2}{R} = g\cos\theta \qquad \ldots(\text{iv})$$

From Eqs. (ii) and (iv) we get,

$$\frac{4}{3}g(1-\cos\theta) = g\cos\theta \quad \text{or} \quad \cos\theta = \frac{4}{7}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{7}\right)$$

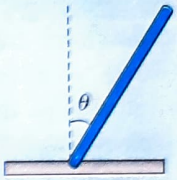
On substituting the value of $\cos\theta$ in Eq. (iv), we get

$$\therefore \quad v = \sqrt{\frac{4}{7}gR}$$

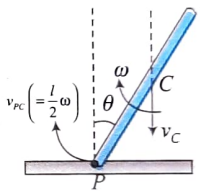
Therefore, speed of CM of cylinder just before it leaves the contact is $\sqrt{\dfrac{4}{7}gR}$.

EXAMPLE 4.11

A uniform rod of mass m and length l is released from rest when it makes a very small angle θ_0 with vertical. Consequently, the rod rotates in the vertical plane, on the smooth horizontal surface. Find the angular speed of the rod when it makes an angle θ with vertical.



Sol. As the horizontal surface is smooth, the. reaction offered by the surface is normal. Since the point of contact P of the rod moves horizontally, work done by normal reaction is zero. Hence, we can conserve the mechanical energy of the rod.



ΔU: While rotating through an angle θ the CM of the rod falls vertically (As no net horizontal force acts on the rod).

Through a distance $\Delta h = \dfrac{l}{2}(1-\cos\theta)$. Then substituting Δh in $\Delta U = -mg\Delta h$, we have

$$\Delta U = -mg\frac{l}{2}(1-\cos\theta)$$

ΔK: the process of combined translation and rotation, the change in kinetic energy

$$\Delta K = \frac{1}{2}I_C\omega^2 + \frac{1}{2}mv_C^2$$

where $I_C = \dfrac{ml^2}{12}$

This gives $\Delta K = \dfrac{ml^2}{24}\omega^2 + \dfrac{1}{2}mv_C^2$

Now substituting ΔU and ΔK in the equation $\Delta U + \Delta K = 0$, we have

$$-\frac{mgl}{2}(1-\cos\theta) + \frac{ml^2\omega^2}{24} + \frac{1}{2}mv_C^2 = 0$$

This gives $$v_C^2 + \frac{l^2\omega^2}{12} = gl(1-\cos\theta) \qquad \ldots(\text{i})$$

Since we have one equation and two unknowns v_C and ω, we need to establish a relation between v_C and ω by using kinematics.

Kinematics: Since the point P touches the surface.

$$v_{P_y} = 0, \text{ where } \vec{v}_{P_y} = \vec{v}_{PC_y} = \vec{v}_{C_y}$$

Substituting $v_{PC_y} = v_{PC}\sin\theta = \frac{l}{2}\omega\sin\theta$ and $v_{C_y} = -v_C$, we have

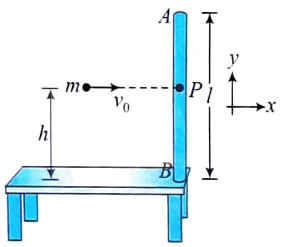
$$v_C = \frac{l\omega}{2}\sin\theta \qquad ...(ii)$$

Finally substituting v_C from Eq. (ii) in Eq. (i), we have

$$\omega = \sqrt{\frac{12g(1-\cos\theta)}{(1+3\sin^2\theta)l}}$$

EXAMPLE 4.12

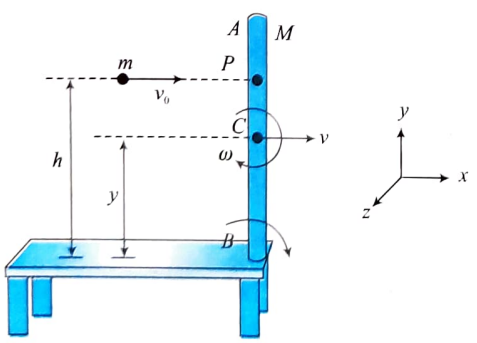
A uniform thin rod AB of mass $M = 0.6$ kg and length $l = 60$ cm stands at the edge of a frictionless table as shown in figure. A particle of mass $m = 0.3$ kg flying horizontally with velocity $v_0 = 24$ m/s strikes the rod at point P at a height $h = 45$ cm from the base and sticks to it. The rod is immediately driven off the table. Find the time after collision when the rod becomes horizontal for the first time ($g = 10$ m/s^2).



Sol. The position of center the combined system with B as origin.

$$y = \frac{0.6\times30 + 0.3\times45}{0.6+0.3} = 35\,\text{cm} \text{ above the table}$$

If we take "rod + particle" as system, we have no external force acting on the system in horizontal direction hence we can conserve the linear momentum of the system in horizontal direction just before and just after collision.



Let v_{CM} be the linear velocity of CM and ω is the angular velocity about an axis passing through CM immediately after collision. From conservation of linear momentum, we get

$$mv_0 = (M+m)v_{CM}$$

$$\text{or} \quad v_{CM} = \frac{mv_0}{M+m} = \frac{(0.3)(24)}{0.6+0.3} = 8 \text{ m/s}$$

The system has no external impulse acting on the system. Hence, we can conserve angular momentum of the system about any point. Let us conserve angular momentum about CM of the system.

Angular momentum of the system just before collision.

$$\vec{L}_{\text{initial}} = mv_0(h-y)(-\hat{k})$$

$$= 0.3\times24\times(0.35-0.30)(-\hat{k})$$

Just after collision $\vec{L}_{\text{initial}} = I_{cm}\vec{\omega}$

$$I_{cm} = (I_{cm})_{\text{rod}} + (I_{cm})_{\text{particle}}$$

$$= \frac{Ml^2}{12} + M\left(y-\frac{l}{2}\right)^2 + m(h-y)^2$$

$$= \frac{(0.6)(0.6)^2}{12} + 0.6(0.35-0.30)^2 + 0.3(0.45-0.35)^2$$

$$= 0.0225\,\text{kg m}^2$$

Hence $\vec{L}_{\text{final}} = 0.0225\,\omega(-\hat{k})$

As $\vec{L}_{\text{initial}} = \vec{L}_{\text{final}}$

$$0.75(-\hat{k}) = 0.0225\,\omega(-\hat{k})$$

$$\text{or} \quad \omega = \frac{0.72}{0.225} = 32 \text{ rad/s}$$

Initially the rod was in vertical orientation then it becomes horizontal. It means angle rotated by the rod is $\pi/2$.

Hence time taken in this process,

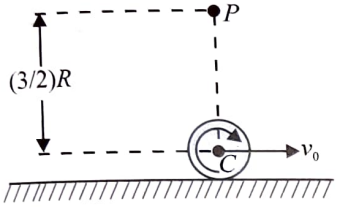
$$t = \frac{\pi/2}{\omega} = \frac{\pi}{2\times32} = \frac{\pi}{64}\,\text{s}$$

Exercises

Single Correct Answer Type

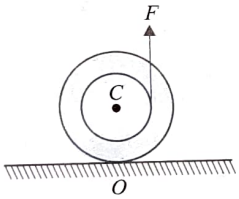
1. Two bodies with moments of inertia I_1 and I_2 $(I_1 > I_2)$ have equal angular momenta. If their kinetic energies of rotation are E_1 and E_2, respectively, then
(1) $E_1 = E_2$ (2) $E_1 < E_2$
(3) $E_1 > E_2$ (4) $E_1 \geq E_2$

2. A disc of mass M and radius R rolls without slipping on a horizontal surface. If the velocity of its centre is v_0, then the total angular momentum of the disc about a fixed point P at a height 3/2 R above the centre C



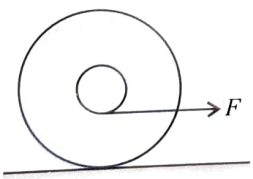
(1) increases continuously as the disc moves away
(2) decreases continuously as the disc moves away
(3) is equal to $2MRv_0$
(4) is equal to MRv_0

3. A yo-yo is placed on a rough horizontal surface and a constant force F, which is less than its weight, pulls it vertically. Due to this



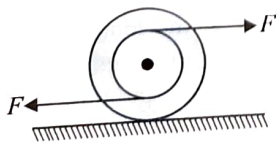
(1) friction force acts towards left, so it will move towards left
(2) friction force acts towards right, so it will move towards right
(3) it will move towards left, so friction force acts towards left
(4) it will move towards right so friction force acts towards right

4. A yo-yo, arranged as shown, rests on a frictionless surface. When a force F is applied to the string, the yo-yo



(1) moves to the left and rotates counterclockwise
(2) moves to the right and rotates counterclockwise
(3) moves to the left and rotates clockwise
(4) moves to the right and rotates clockwise

5. A spool is pulled horizontally by two equal and opposite forces as shown in figure. Which of the following statements are correct?



(1) The centre of mass moves towards left.
(2) The centre of mass moves towards right.
(3) The centre of mass remains stationary.
(4) The net torque about the centre of mass of the spool is zero.

6. A ring, cylinder and solid sphere are placed on the top of a rough incline on which the sphere can just roll without slipping. When all of them are released at the same instant from the same position, then
(1) all of them reach the ground at the same instant
(2) the sphere reaches first and the ring at the last
(3) the sphere reaches first and the cylinder and ring reach together
(4) none of the above

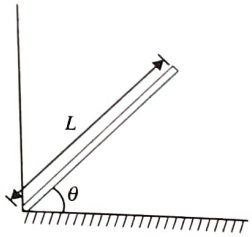
7. Suppose you are standing on the edge of a spinning platform and step off at right angles to the edge (radially outward). Now consider it the other way. You are standing on the ground next to a spinning carousel and you step onto the platform at right angles to the edge (radially inward).
(1) There is no change in rotational speed of the carousel in either situation.
(2) There is a change in rotational speed in the first situation but not the second.
(3) There is a change in rotational speed in the second situation but not the first.
(4) There is a change in rotational speed in both instances.

8. A ball is given a velocity v and angular velocity such that the ball rolls purely on a plank whose upper surface is rough enough to prevent slipping but lower surface in contact with the ground is smooth. No other force is acting on system.
(1) The plank will recoil back.
(2) The plank will also move forward but with a lesser velocity than that of the ball.
(3) The plank will also move forward but with a greater velocity than that of the ball.
(4) The plank will remain at rest.

9. A solid homogenous sphere is moving on a rough horizontal surface, partially rolling and partially sliding. During this kind of motion of the sphere
(1) Total kinetic energy is conserved
(2) Angular momentum of the sphere about the point of contact is conserved
(3) Only the rotational kinetic energy about the centre of mass is conserved
(4) Angular momentum about the centre of mass is conserved

10. A body is rolling without slipping on a horizontal plane. The rotational energy of the body is 40% of the total kinetic energy. Identify the body.
(1) Ring (2) Hollow cylinder
(3) Solid cylinder (4) Hollow sphere

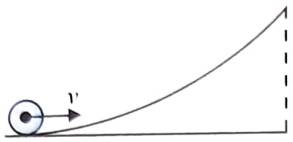
11. A uniform pole of length L and mass M is pivoted on the ground with a frictionless hinge. The pole makes an angle θ with the horizontal. The moment of inertia of the pole about one end is $(1/3)\,ML^2$. If it starts falling from the position shown in the accompanying figure, the linear acceleration of the free end of the pole immediately after release would be



(1) $\left(\frac{2}{3}\right) g \cos\theta$ (2) $\left(\frac{2}{3}\right) g$

(3) g (4) $\left(\frac{3}{2}\right) g \cos\theta$

12. A small object of uniform density rolls up a curved surface with an initial velocity v. It reaches up to a maximum height of $3v^2/4g$ w.r.t. the initial position. The object is



(1) ring (2) solid sphere
(3) hollow sphere (4) disc

13. A solid iron sphere A rolls down an inclined plane, while an identical hollow sphere B of same mass slides down the plane in a frictionless manner. At the bottom of the inclined plane, the total kinetic energy of sphere A is
(1) less than that of B
(2) equal to that of B
(3) more than that of B
(4) sometimes more and sometimes less

14. A string is wrapped around a cylinder of mass M and radius R. The string is pulled vertically upwards to prevent the centre of mass from falling as the cylinder unwinds the string. The tension in the string is

(1) $\frac{Mg}{6}$ (2) $\frac{Mg}{3}$

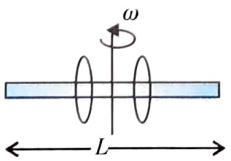
(3) $\frac{Mg}{2}$ (4) $\frac{2Mg}{3}$

15. In problem 14, work done on the cylinder for reaching an angular speed ω is

(1) $\frac{MR^2\omega^2}{4}$ (2) $\frac{MR^2\omega^2}{2}$

(3) $\frac{MR^2\omega^2}{3}$ (4) $\frac{2MR^2\omega^2}{3}$

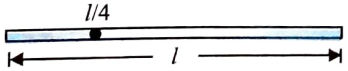
16. A smooth uniform rod of length L and mass M has two identical beads (1 and 2) of negligible size, each of mass m, which can slide freely along the rod. Initially the two beads are at the centre of the rod and the system is rotating with angular velocity ω_0 about an axis perpendicular to the rod and is passing through its midpoint. There are no external forces when the beads reach the ends of the rod, the angular velocity of the system is



(1) $\frac{M\omega_0}{M+6m}$ (2) $\frac{M\omega_0}{m}$

(3) $\frac{M\omega_0}{M+12m}$ (4) ω_0

17. A uniform thin rod of length l and mass m is hinged at a distance $l/4$ from one of the end and released from horizontal position as shown in figure. The angular velocity of the rod as it passes the vertical position is

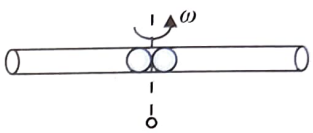


(1) $2\sqrt{\frac{5g}{7l}}$ (2) $2\sqrt{\frac{6g}{7l}}$

(3) $\sqrt{\frac{3g}{7l}}$ (4) $2\sqrt{\frac{g}{l}}$

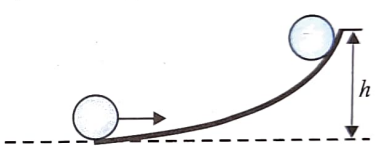
18. A sphere is released on a smooth inclined plane from the top. When it moves down, its angular momentum is
(1) conserved about every point
(2) conserved about the point of contact only
(3) conserved about the centre of the sphere only
(4) conserved about any point on a line parallel to the inclined plane and passing through the centre of the ball

19. A smooth tube of certain mass is rotated in a gravity-free space and released. The two balls shown in figure move towards the ends of the tube. For the whole system, which of the following quantities is not conserved.



(1) Angular momentum (2) Linear momentum
(3) Kinetic energy (4) Angular speed

20. In the figure shown, a ball without sliding on a horizontal surface. It ascends a curved track up to height h and returns. The value of h is h_1 for sufficiently rough curved track to avoid sliding and is h_2 for smooth curved track, then



(1) $h_1 = h_2$ (2) $h_1 < h_2$
(3) $h_1 > h_2$ (4) $h_2 = 2h_1$

21. A solid sphere, a hollow sphere and a disc, all having the same mass and radius, are placed at the top of an incline and released. The friction coefficients between the objects and the incline are same and not sufficient to allow pure rolling. The least time will be taken in reaching the bottom by
(1) the solid sphere (2) the hollow sphere
(3) the disc (4) all will take the same time

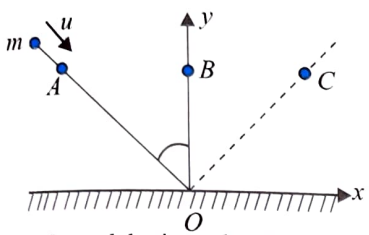
22. In the previous problem, the smallest kinetic energy at the bottom of the incline will be achieved by
(1) the solid sphere (2) the hollow sphere
(3) the disc
(4) all will achieve the same kinetic energy

23. An impulse $J = mv$ at one end of a stationary uniform frictionless rod of mass m and length l which is free to rotate in a gravity-free space. The impact is elastic. Instantaneous axis of rotation of the rod will pass through
(1) its centre of mass
(2) the centre of mass of the rod plus ball
(3) the point of impact of the ball on the rod
(4) the point which is at a distance $2l/3$ from the striking end

24. A particle falls freely near the surface of the earth. Consider a fixed point O (not vertically below the particle) on the ground. Then pick up the incorrect alternative.
(1) Angular momentum of the particle about O is increasing.

(2) The moment of inertia of the particle about O is decreasing.
(3) The moment of inertia of the particle about O is increasing.
(4) The angular velocity of the particle about O is increasing.

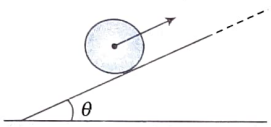
25. A ball of mass moving with constant velocity u collides with a smooth horizontal surface at O as shown in figure. Neglect gravity and friction. The y-axis is drawn normal to the horizontal surface at the point of impact O and x-axis is horizontal as shown. About which point will the angular momentum of ball be conserved?

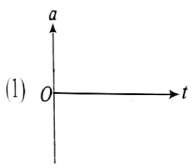


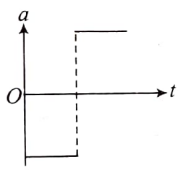
Smooth horizontal surfacce

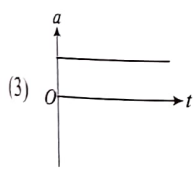
(1) Point A (2) Point B
(3) Point C (4) None of these

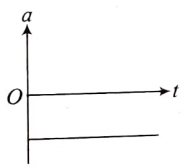
26. A uniform solid sphere rolls up (without slipping) the rough fixed inclined plane, and then back down. Which is the correct graph of acceleration a of centre of mass of solid sphere as function of time t (for the duration sphere is on the incline)? Assume that the sphere rolling up has a positive velocity.



(1) 

(2) 

(3) 

(4) 

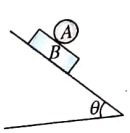
27. A rolling object rolls without slipping down an inclined plane (angle of inclination θ), then the minimum acceleration it can have is

(1) $g \sin\theta$ (2) $\frac{2g\sin\theta}{3}$
(3) $\frac{g\sin\theta}{2}$ (4) zero

28. Three identical solid spheres move down three incline A, B and C are all of the same dimensions. A is without friction, the friction between B and a sphere is sufficient to cause rolling without slipping, the friction between C and a sphere causes rolling with slipping. The kinetic energies of A, B, C at the bottom of the inclines are E_A, E_B, E_C:

(1) $E_A = E_B = E_C$ (2) $E_A = E_B > E_C$
(3) $E_A > E_B > E_C$ (4) $E_A > E_B = E_C$

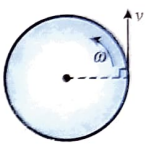
29. A rolling body is kept on a plank B. There is sufficient friction between A and B and no friction between B and the inclined plane. Then body:



(1) A rolls
(2) A does not experience any friction
(3) A and B has equal acceleration and unequal velocities
(4) A rolls depending upon the angle of inclination θ

30. A solid sphere rolls down two different inclined planes of the same height but of different inclinations
(1) the speeds will be same but time of descent will be different
(2) in both cases, the speeds and time of descent will be same
(3) speeds and time of descent both will be different
(4) the speeds will be different but time of descent will be same

31. A child with mass m is standing at the edge of a playground merry-go-round (A large uniform disc which rotates in horizontal plane about a fixed vertical axis in parks) with moment of inertia I, radius R, and initial angular velocity ω as shown in the figure. The child jumps off the edge of the merry-go-round with a velocity v with respect to the ground in direction tangent to periphery of the disc as shown. The new angular velocity of the merry-go-round is



(1) $\sqrt{\frac{I\omega^2 - mv^2}{I}}$ (2) $\sqrt{\frac{(I+mR^2)\omega^2 - mv^2}{I}}$
(3) $\frac{I\omega - mvR}{I}$ (4) $\frac{(I+mR^2)\omega - mvR}{I}$

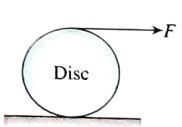
32. A thick walled hollow sphere has outer radius R. It rolls down an inclined plane without slipping and its speed at the bottom is v. If the inclined plane is frictionless and the sphere slides down without rolling, its speed at the bottom is $5v/4$. What is the radius of gyration of the sphere?

(1) $\frac{R}{\sqrt{2}}$ (2) $\frac{R}{2}$
(3) $\frac{3R}{4}$ (4) $\frac{\sqrt{3}R}{4}$

33. If a rigid body rolls on a surface without slipping, then
(1) linear speed is maximum at the highest point but minimum at the point of contact
(2) linear speed is minimum at highest point but maximum at the point of contact
(3) linear speed is same at all points of the rigid body
(4) angular speed is different at different points of a rigid body

34. Two discs, each having moment of inertia 5 kg m^2 about its central axis, rotating with speeds 10 rad s^{-1} and 20 rad s^{-1}, are brought in contact face to face with their axes of rotation coincided. The loss of kinetic energy in the process is
(1) 2 J (2) 5 J
(3) 125 J (4) 0 J

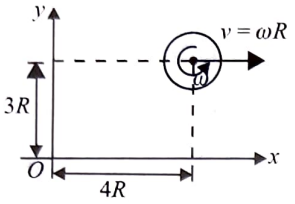
35. A force F acts tangentially at the highest point of a disc of mass m kept on a rough horizontal plane. If the disc rolls without slipping, the acceleration of centre of the disc is



(1) $\frac{2F}{3m}$ (2) $\frac{10F}{7m}$

(3) zero (4) $\frac{4F}{3m}$

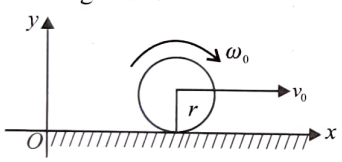
36. A disc of mass m and radius R moves in the x-y plane as shown in figure. The angular momentum of the disc about the origin O at the instant shown is



(1) $-\frac{5}{2}mR^2\omega\hat{k}$ (2) $\frac{7}{3}mR^2\omega\hat{k}$

(3) $-\frac{9}{2}mR^2\omega\hat{k}$ (4) $\frac{5}{2}mR^2\omega\hat{k}$

37. A uniform sphere of mass m, radius r and moment of inertia I about its centre moves along the x-axis as shown in figure. Its centre of mass moves with velocity = v_0, and it rotates about its centre of mass with angular velocity = ω_0. Let $\vec{L} = (I\omega_0 + mv_0 r)(-\hat{k})$. The angular momentum of the body about the origin O is



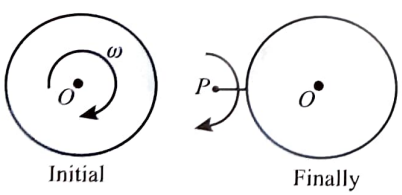
(1) $\vec{L}$, only if $v_0 = \omega_0 r$

(2) greater than $\vec{L}$, if $v_0 > \omega_0 r$

(3) less than $\vec{L}$, if $v_0 > \omega_0 r$

(4) $\vec{L}$, for all values of ω_0 and v_0

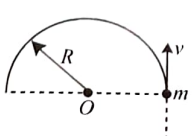
38. A disc is freely rotating with an angular speed ω on a smooth horizontal plane. If it is hooked at a rigid page P and rotates without bouncing about a point on its circumference. Its angular speed after the impact will be equal to



(1) $\frac{2\omega}{3}$ (2) $\frac{\omega}{3}$

(3) $\frac{\omega}{2}$ (4) none of these

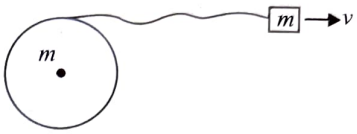
39. A small bead of mass m moving with velocity v gets threaded on a stationary semicircular ring of mass m and radius R kept on a horizontal table. The ring can freely rotate about its centre. The bead comes to rest relative to the ring. What will be the final angular velocity of the system?



(1) $\frac{v}{R}$ (2) $\frac{2v}{R}$

(3) $\frac{v}{2R}$ (4) $\frac{3v}{R}$

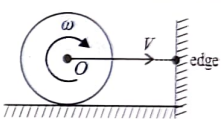
40. A block of mass m is attached to a pulley disc of equal mass m and radius r by means of a slack string as shown. The pulley is hinged about its centre on a horizontal table and the block is projected with an initial velocity of 5 m/s. Its velocity when the string becomes taut will be



(1) 3 m/s (2) 2.5 m/s

(3) $\frac{5}{3}$ m/s (4) $\frac{10}{3}$ m/s

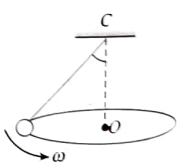
41. A uniform solid sphere of radius r is rolling on a smooth horizontal surface with velocity V and angular velocity $\omega = (V = \omega r)$. The sphere collides with a sharp edge on the wall as shown in figure. The coefficient of friction between the sphere and the edge $\mu = 1/5$. Just after the collision the angular velocity of the sphere becomes equal to zero. The linear velocity of the sphere just after the collision is equal to



(1) V (2) $\frac{V}{5}$

(3) $\frac{3V}{5}$ (4) $\frac{V}{6}$

42. A conical pendulum consists of a simple pendulum moving in a horizontal circle as shown in the figure. C is the pivot. O the centre of the circle in which the pendulum bob moves and ω the constant angular velocity of the bob. If $\vec{L}$ is the angular momentum about point C, then



(1) $\vec{L}$ is constant

(2) only direction of $\vec{L}$ is constant

(3) only magnitude of $\vec{L}$ is constant

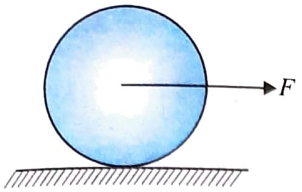
(4) none of the above

43. A ring of radius R is first rotated with an angular velocity ω_0 and then carefully placed on a rough horizontal surface. The coefficient of friction between the surface and the ring is μ. Time after which its angular speed is reduced to half is

(1) $\frac{\omega_0 \mu R}{2g}$ (2) $\frac{\omega_0 g}{2\mu R}$

(3) $\frac{2\omega_0 R}{\mu g}$ (4) $\frac{\omega_0 R}{2\mu g}$

44. A solid sphere of mass m is lying at rest on a rough horizontal surface. The coefficient of friction between the ground and sphere is μ. The maximum value of F, so that the sphere will not slip, is equal to



(1) $\frac{7}{5}\mu mg$ (2) $\frac{4}{7}\mu mg$

(3) $\frac{5}{7}\mu mg$ (4) $\frac{7}{2}\mu mg$

45. A slender rod of mass M and length L rests on a horizontal frictionless surface. The rod is pivoted about one of its ends. The impulse of the force exerted on the rod by the pivot when the rod is struck by a blow of impulse J perpendicular to the rod at other end is

(1) J (2) $\frac{J}{2}$

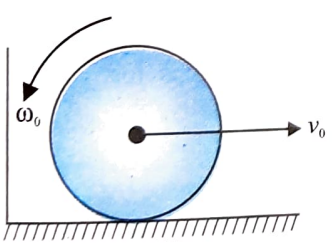
(3) $\frac{J}{3}$ (4) information is insufficient

46. A solid cylinder of mass M and radius R is resting on a horizontal platform (which is parallel to the x–y plane) with its axis fixed along Y-axis and free to rotate about its axis. The platform is given a motion in the X-direction given by $x = A\cos(\omega t)$. There is no slipping between the cylinder and the platform. The maximum torque acting on the cylinder during its motion is

(1) $\frac{M\omega^2 AR}{3}$ (2) $\frac{M\omega^2 AR}{2}$

(3) $\frac{2}{3} \times M\omega^2 AR$ (4) the situation is not possible

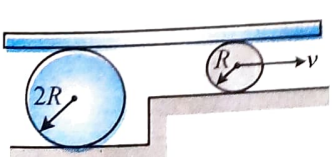
47. A uniform circular disc of radius r is placed on a rough horizontal surface and given a linear velocity v_0 and angular velocity ω_0 as shown. The disc comes to rest after moving some distance to the right. It follows that



(1) $3v_0 = 2\omega_0 r$ (2) $2v_0 = \omega_0 r$

(3) $v_0 = \omega_0 r$ (4) $2v_0 = 3\omega_0 r$

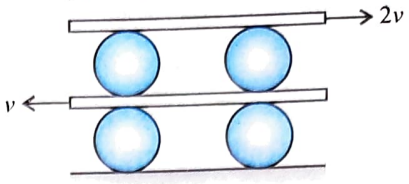
48. Velocity of the centre of a small cylinder is v. There is no slipping anywhere. The velocity of the centre of the larger cylinder is



(1) $2v$ (2) v

(3) $\frac{3v}{2}$ (4) none of these

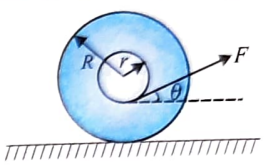
49. A system of identical cylinders and plates is shown in figure. All the cylinders are identical and there is no slipping at any contact. The velocity of lower and upper plates are v and $2v$, respectively, as shown in figure. Then the ratio of angular speeds of the upper cylinders to lower cylinders is



(1) 1/3 (2) 3

(3) 1 (4) none of these

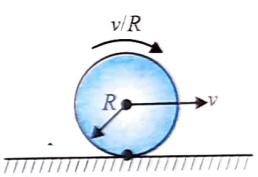
50. The spool shown in the figure is placed on a rough horizontal surface and has inner radius r and outer radius R. The angle θ between the applied force and the horizontal can be varied. The critical angle (θ) for which the spool does not roll and remains stationary is given by



(1) $\theta = \cos^{-1}\left(\frac{r}{R}\right)$ (2) $\theta = \cos^{-1}\left(\frac{2r}{R}\right)$

(3) $\theta = \cos^{-1}\sqrt{\frac{r}{R}}$ (4) $\theta = \sin^{-1}\left(\frac{r}{R}\right)$

51. A disc is performing pure rolling on a smooth stationary surface with constant angular velocity as shown in figure. At any instant, for the lowermost point of the disc,



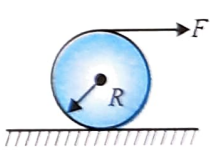
(1) velocity is v, acceleration is zero

(2) velocity is zero, acceleration is zero

(3) velocity is v, acceleration is v^2/R

(4) velocity is zero, acceleration is v^2/R

52. An object of mass M and radius R is performing pure rolling motion on a smooth horizontal surface under the action of a constant force F as shown in figure. The object may be



(1) disk (2) ring

(3) solid cylinder (4) hollow sphere

53. A solid cylinder is placed on the end of an inclined plane. It is found that the plane can be tipped at an angle θ before the cylinder starts to slide. When the cylinder turns on its sides and is allowed to roll, it is found that the steepest angle at which the cylinder performs pure rolling is ϕ. The ratio $\tan\phi/\tan\theta$ is

(1) 3 (2) 1/3

(3) 1 (4) 1/2

54. A uniform box of height 2 m and having a square base of side 1 m, weight 150 kg, is kept on one end on the floor of a truck. The maximum speed with which the truck can round a curve of radius 20 m without causing the block to tip over is (assume that friction is sufficient so that there is no sliding).

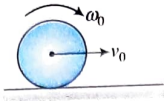
(1) 15 m/s

(2) 10 m/s

(3) 8 m/s

(4) depends on the value of coefficient of friction

55. A sphere of mass M and radius R is moving on a rough fixed surface, having coefficient of friction μ as shown in figure. It will attain a minimum linear velocity after time ($v_0 > \omega_0 R$)



(1) $v_0/\mu g$ (2) $\omega_0 R/\mu g$

(3) $2(v_0 - \omega_0 R)/5\mu g$ (4) $2(v_0 - \omega_0 R)/7\mu g$

56. A solid sphere and a hollow sphere of equal mass and radius are placed over a rough horizontal surface after rotating it about its mass centre with same angular velocity ω_0. Once the pure rolling starts let v_1 and v_2 be the linear speeds of their centres of mass. Then

(1) $v_1 = v_2$ (2) $v_1 > v_2$

(3) $v_1 < v_2$ (4) data is insufficient

57. In the above problem, let t_1 and t_2 be the times when pure rolling of solid sphere and of hollow sphere is started. Then

(1) $t_1 = t_2$ (2) $t_1 < t_2$

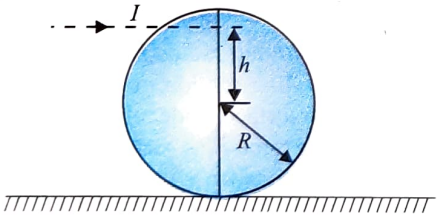
(3) $t_1 > t_2$ (4) none of these

58. A uniform ball of radius r rolls without slipping down from the top of a sphere of radius R. The angular velocity of the ball when it breaks from the sphere is

(1) $\sqrt{\frac{5g(R+r)}{17r^2}}$ (2) $\sqrt{\frac{10g(R+r)}{17r^2}}$

(3) $\sqrt{\frac{5g(R-r)}{10r^2}}$ (4) $\sqrt{\frac{10g(R+r)}{7r^2}}$

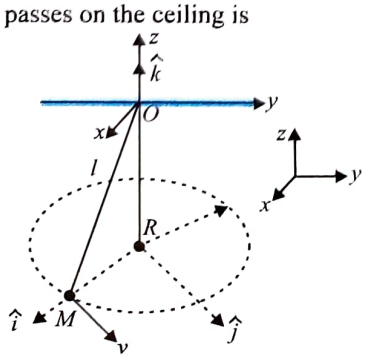
59. A solid sphere rests on a horizontal surface. A horizontal impulse is applied at height h from centre. The sphere starts rolling just after the application of impulse. The ratio h/r will be



(1) $\frac{1}{2}$ (2) $\frac{2}{5}$

(3) $\frac{1}{5}$ (4) $\frac{2}{3}$

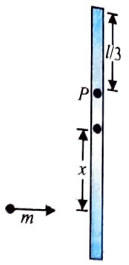
60. A conical pendulum consists of a mass M suspended from a strong sling of length l. The mass executes a circle of radius R in a horizontal plane with speed v. At time t, the mass is at position $R\hat{i}$ and has $v\hat{j}$ velocity. At time t, the angular momentum vector of mass M about the point from which the string passes on the ceiling is



(1) $Mv R\hat{k}$ (2) $Mvl\hat{k}$

(3) $Mvl\left[\frac{\sqrt{l^2-R^2}}{l}\hat{i} + \frac{R}{l}\hat{k}\right]$ (4) $-Mvl\left[\frac{\sqrt{l^2-R^2}}{l}\hat{i} + \frac{R}{l}\hat{k}\right]$

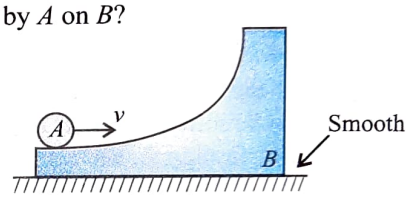
61. A thin uniform rod of mass m and length l is kept on a smooth horizontal surface such that it can move freely. At what distance from centre of rod should a particle of mass m strike on the rod such that the point P at a distance $l/3$ from the end of the rod is instantaneously at rest just after the elastic collision?



(1) $l/2$ (2) $l/3$

(3) $l/6$ (4) $l/4$

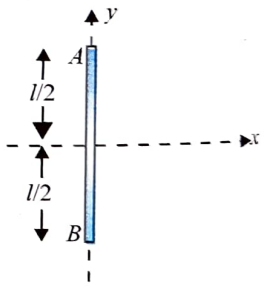
62. In the figure shown, a ring A is initially rolling without sliding with a velocity v on the horizontal surface of the body B (of same mass as A). All surfaces are smooth. B has no initial velocity. What will be the maximum height reached by A on B?



(1) $\frac{3v^2}{4g}$ (2) $\frac{v^2}{4g}$

(3) $\frac{v^2}{2g}$ (4) $\frac{v^2}{3g}$

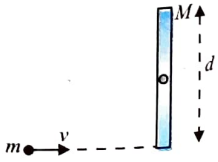
63. A uniform rod of mass m and length l is placed over a smooth horizontal surface along the y-axis and is at rest as shown in figure. An impulsive force F is applied for a small time Δt along x-direction at point A. The x-coordinate of end A of the rod when the rod becomes parallel to x-axis for the first time is [initially, the coordinate of centre of mass of the rod is (0, 0)]



(1) $\frac{\pi l}{12}$ (2) $\frac{l}{2}\left(1 + \frac{\pi}{12}\right)$

(3) $\frac{l}{2}\left(1 - \frac{\pi}{6}\right)$ (4) $\frac{l}{2}\left(1 + \frac{\pi}{6}\right)$

64. A mass m is moving at speed v perpendicular to a rod of length d and mass $M = 6m$ which pivots around a frictionless axle running through its centre. It strikes and sticks to the end of the rod. The moment of inertia of the rod about its centre is $Md^2/12$. Then the angular speed of the system just after the collision is

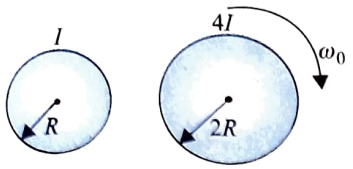


(1) $\frac{2v}{3d}$ (2) $\frac{2v}{d}$

(3) $\frac{v}{d}$ (4) $\frac{3v}{2d}$

65. Two cylinders having radii $2R$ and R and moment of inertia $4I$ and I about their central axes are supported by

axles perpendicular to their planes. The large cylinder is initially rotating clockwise with angular velocity ω_0. The small cylinder is moved to the right until it touches the large cylinder and is caused to rotate by the frictional force between the two. Eventually slipping ceases and the two cylinders rotate at constant rates in opposite directions. During this



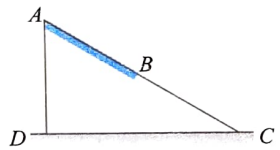
(1) angular momentum of system is conserved
(2) kinetic energy is conserved
(3) neither the angular momentum nor the kinetic energy is conserved
(4) both the angular momentum and kinetic energy are conserved

66. In the above problem, the final angular velocity of the small cylinder is

(1) $\frac{\omega_0}{4}$ (2) ω_0

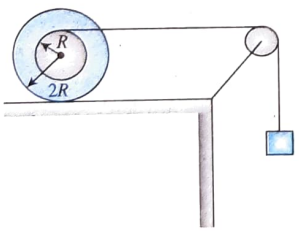
(3) $\frac{\omega_0}{2}$ (4) $\frac{\omega_0}{8}$

67. Portion AB of the wedge shown in the figure is rough and BC is smooth. A solid cylinder rolls without slipping from A to B. If $AB = BC$, then ratio of translational kinetic energy to rotational kinetic energy, when the cylinder reaches point C is



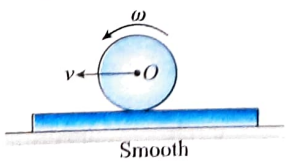
(1) 3/5 (2) 5
(3) 7/5 (4) 8/3

68. In the figure shown mass of both, the spherical body and block is m. Moment of inertia of the spherical body about centre of mass is $2mR^2$. The spherical body rolls on the horizontal surface. There is no slipping at any surfaces in contact. The ratio of kinetic energy of the spherical body to that of block is



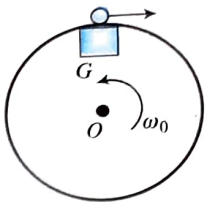
(1) 3/4 (2) 1/3
(3) 2/3 (4) 1/2

69. A cylinder executes pure rolling without slipping with a constant velocity on a plank, whose upper surface is rough enough, but lower surface is smooth. The plank is kept at rest on a smooth horizontal surface by the application of an external force F. Choose the correct alternative.



(1) The direction of F is towards right.
(2) The direction of F is towards left.
(3) The value of F is zero.
(4) The direction of F depends on the ratio of the relative masses of disc and plank.

70. A horizontal turn table in the form of a disc of radius r carries a gun at G and rotates with angular velocity ω_0 about a vertical axis passing through the centre O. The increase in angular velocity of the system if the gun fires a bullet of mass m with a tangential velocity v with respect to the gun is (moment of inertia of gun + table about O is I_0)



(1) $\frac{mvr}{I_0 + mr^2}$ (2) $\frac{2mvr}{I_0}$

(3) $\frac{v}{2r}$ (4) $\frac{mvr}{2I_0}$

71. A circular platform is mounted on a vertical frictionless axle. Its radius is $r = 2$ m and its moment of inertia is $I = 200$ kg m^2. It is initially at rest. A 70 kg man stands on the edge of the platform and begins to walk along the edge at speed $v_0 = 10$ ms^{-1} relative to the ground. The angular velocity of the platform is

(1) 1.2 rad s^{-1} (2) 0.4 rad s^{-1}
(3) 2.0 rad s^{-1} (4) 0.7 rad s^{-1}

72. In the above problem, when the man has walked once around the platform so that he is at his original position on it, what is his angular displacement relative to ground

(1) $\frac{6}{5}\pi$ (2) $\frac{5}{6}\pi$

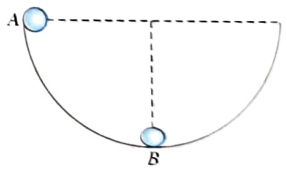
(3) $\frac{4}{5}\pi$ (4) $\frac{5}{4}\pi$

73. A uniform rod AB of mass m and length $2a$ is falling freely without rotation under gravity with AB horizontal. Suddenly the end A is fixed when the speed of the rod is v. The angular speed with which the rod begins to rotate is

(1) $\frac{v}{2a}$ (2) $\frac{4v}{3a}$

(3) $\frac{v}{3a}$ (4) $\frac{3v}{4a}$

74. A ball of mass m and radius r rolls inside a hemispherical shell of radius R. It is released from rest from point A as shown in the figure. The angular velocity of centre of the ball in position B about the centre of the shell is



(1) $2\sqrt{\frac{g}{5(R-r)}}$ (2) $2\sqrt{\frac{g}{7(R-r)}}$

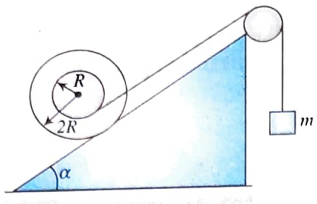
(3) $\sqrt{\frac{2g}{5(R-r)}}$ (4) $\sqrt{\frac{5g}{2(R-r)}}$

75. In the previous problem the normal force between the ball and the shell in position B is

(1) $\frac{12}{7}mg$ (2) $\frac{7}{9}mg$

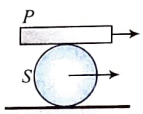
(3) $\frac{17}{7}mg$ (4) $\frac{10}{7}mg$

76. A spool of mass M and radius $2R$ lies on an inclined plane as shown in the figure. A light thread is wound around the connecting tube of the spool and its free end carries a weight of mass m. The value of m so that system is in equilibrium is



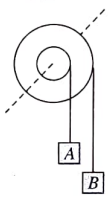
(1) $2M \sin \alpha$ (2) $M \sin \alpha$
(3) $2M \tan \alpha$ (4) $M \tan \alpha$

77. A plank P is placed on a solid cylinder S, which rolls on a horizontal surface. The two are of equal mass. There is no slipping at any of the surfaces in contact. The ratio of kinetic energy of P to the kinetic energy of S is



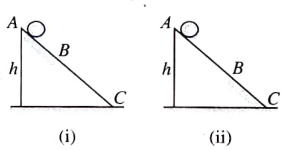
(1) 1 : 1 (2) 2 : 1
(3) 8 : 3 (4) 11 : 8

78. Figure shows a small wheel fixed coaxially on a bigger one of double the radius. The system rotates about the common axis. The strings supporting A and B do not slip on the wheels. If x and y be the distance travelled by A and B in the same time interval, then



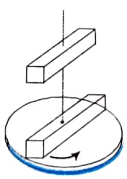
(1) $x = 2y$ (2) $x = y$
(3) $y = 2x$ (4) none of these

79. In both the figures all other factors are same, except that in Fig. (i) AB is rough and BC is smooth while in Fig. (ii) AB is smooth and BC is rough. In Fig. (i), if a sphere is released from rest it starts rolling. Now consider Fig. (ii), if same sphere is released from top of the inclined plane, what will be the kinetic energy of the sphere on reaching the bottom



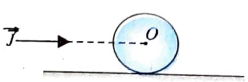
(1) is same in both the cases
(2) is greater in case (i)
(3) is greater in case (ii)
(4) information insufficient

80. A uniform disk turns at 2.4 rev/s around a frictionless axis. A nonrotating rod, of the same mass as the disk and length equal to the disk's diameter is dropped onto the freely spinning disk (see the figure). They then both turn around the axis with their centres superposed. What will be the angular frequency in rev/s of the combination when they start rotating together?



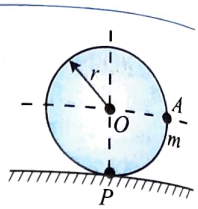
(1) 1.2 rev/s (2) 2.0 rev/s
(3) 1.44 rev/s (4) 0.96 rev/s

81. An impulse J is applied on a ring of mass m along a line passing through its centre O. The ring is placed on a rough horizontal surface. The linear velocity of centre of ring once it starts rolling without slipping is



(1) J/m (2) $J/2m$
(3) $J/4m$ (4) $J/3m$

82. A particle of mass 'm' is rigidly attached at 'A' to a ring of mass '$3m$' and radius 'r'. The system is released from rest and rolls without sliding. The angular acceleration of ring just after release is

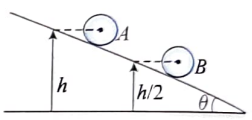


(1) $\frac{g}{4r}$ (2) $\frac{g}{6r}$
(3) $\frac{g}{8r}$ (4) $\frac{g}{2r}$

83. A uniform smooth rod (mass m and length l) placed on a smooth horizontal floor is hit by a particle (mass m) moving on the floor, at a distance $l/4$ from one end elastically ($e = 1$). The distance travelled by the centre of the rod after the collision when it has completed three revolutions will be

(1) $2\pi l$ (2) cannot be determined
(3) πl (4) none of these

84. Two identical uniform solid spherical balls A and B of mass m each are placed on a the fixed wedge as shown in figure. Ball B is kept at rest and it is released just before two balls collides. Ball A rolls down without slipping on inclined plane and collide elastically with ball B. The kinetic energy of ball A just after the collision with ball B is

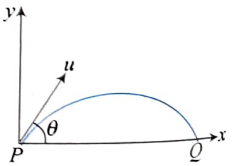


(1) $\frac{mgh}{7}$ (2) $\frac{mgh}{2}$
(3) $\frac{2mgh}{5}$ (4) $\frac{7mgh}{5}$

85. A homogenous rod of length $l = \eta x$ and mass M is lying on a smooth horizontal floor. A bullet of mass m hits the rod at a distance x from the middle of the rod at a velocity v_0 perpendicular to the rod and comes to rest after collision. If the velocity of the farther end of the rod just after the impact is in the opposite direction of v_0, then

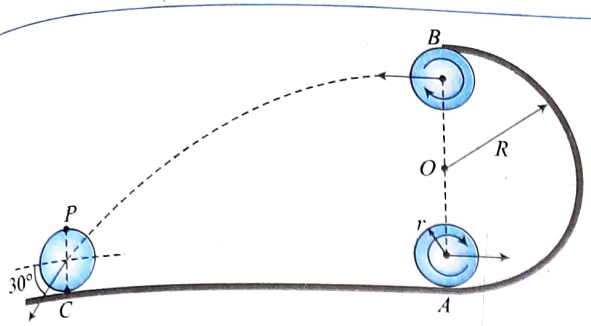
(1) $\eta > 3$ (2) $\eta < 3$
(3) $\eta > 6$ (4) $\eta < 6$

86. Average torque on a projectile of mass m, initial speed u and angle of projection θ between initial and final positions P and Q as shown in the figure about the point of projection is



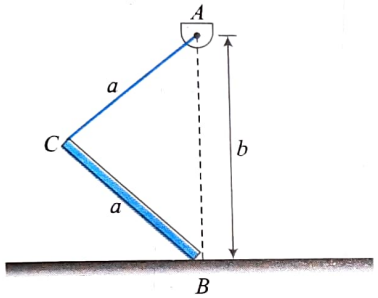
(1) $\frac{mu^2 \sin 2\theta}{2}$ (2) $mu^2 \cos \theta$
(3) $mu^2 \sin \theta$ (4) $\frac{mu^2 \cos \theta}{2}$

87. A disc of radius $r = 0.1$ m is rolled from a point A on a track as shown in the figure. The part AB of the track is a semi-circle of radius R in a vertical plane. The disc rolls without sliding and leaves contact with the track at its highest point B. Flying through the air it strikes the ground at point C. The velocity of the center of mass of the disc makes an angle of 30° below the horizontal at the time of striking the ground. At the same instant, velocity of the topmost point P of the disc is found to be 6 m/s. The value of R is (Take $g = 10$ m/s^2).



(1) 1 m (2) 5 m
(3) 2.5 m (4) 2 m

88. A uniform rod BC with length a is attached to a light string AC. End A of the string is fixed to the ceiling and the end B of the rod is on a smooth horizontal surface. B is exactly below point A and length AB is b $(a < b < 2a)$. The system is released from rest and the rod begins to slide. Find the speed of the centre of the rod when the string becomes vertical.

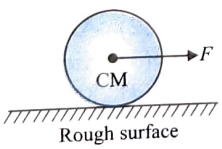


(1) $\sqrt{g\left(a+\frac{b}{2}\right)}$ (2) $\sqrt{g\left(a-\frac{b}{2}\right)}$

(2) $\sqrt{g\left(a+\frac{b}{3}\right)}$ (4) $\sqrt{g\left(a-\frac{b}{3}\right)}$

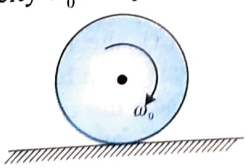
Multiple Correct Answers Type

1. A solid sphere of mass M and radius R is pulled horizontally on a rough surface as shown in figure. Choose the incorrect alternatives.



(1) The magnitude of the frictional force is $F/3$.
(2) The frictional force on the sphere acts forward.
(3) The acceleration of the centre of mass is $2F/3M$.
(4) The acceleration of the centre of mass is F/M.

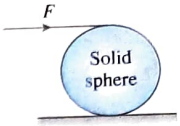
2. A disc is given an initial angular velocity ω_0 and placed on a rough horizontal surface as shown in figure. The quantities which will not depend on the coefficient of friction is/are



(1) the time until rolling begins
(2) the displacement of the disc until rolling begins
(3) the velocity when rolling begins
(4) the work done by the force of friction

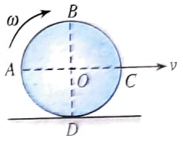
3. Consider three uniform solid spheres, sphere (i) has radius 'r' and mass 'm', sphere (ii) has radius r and mass $3m$, sphere (iii) has radius $3r$ and mass 'm'. All the spheres can be placed at the same point on the same inclined plane where they will roll without slipping to the bottom. If allowed to roll down the incline, then at the bottom of the incline
(1) sphere (i) will have the largest speed
(2) sphere (ii) will have the largest speed
(3) sphere (ii) will have the largest kinetic energy
(4) all the spheres will have equal speed

4. In the situation given, a force F is applied at the top of sphere. Study following statements.



(1) The sphere will move faster if friction is absent.
(2) The sphere will move faster if friction is present.
(3) The sphere will move with same velocity irrespective the friction.
(4) Friction supports the motion.

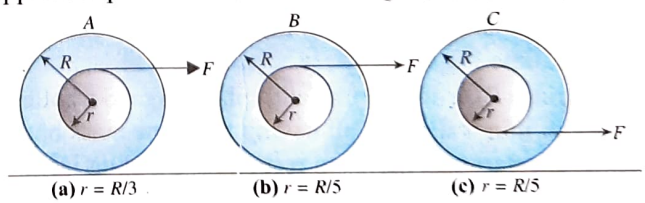
5. A ring rolls without slipping on a horizontal surface. At any instant, its position is as shown in the figure. Then



(1) section ABC has greater kinetic energy than section ADC
(2) section BC has greater kinetic energy than section CD
(3) section BC has the same kinetic energy as section DA
(4) the sections AB, BC, CD and DA have the same kinetic energy

6. A uniform disc is rotating at a constant speed in a vertical plane about a fixed horizontal axis, passing through the centre of the disc. A piece of disc from its rim detaches itself from the disc at the instant when it is on the same horizontal with the centre of the disc and moving upwards. Then about the fixed axis, the angular speed of the
(1) remaining disc remains unchanged
(2) remaining disc decreases
(3) remaining disc increases
(4) broken away piece decreases initially and later increases

7. Three spools A, B and C each having moment of inertia $I = MR^2/4$ are placed on rough ground and equal force F is applied at positions as shown in Figs. (a), (b) and (c). Then



(1) frictional force on spool A acts in forward direction
(2) frictional force on spool A acts in backward direction
(3) frictional force on spool B acts in forward direction
(4) frictional force on spool B and C acts in backward direction

8. A horizontal disc rotates freely about a vertical axis through its centre. A ring, having the same mass and radius as the disc, is now gently placed on the disc. After some time, the two rotate with a common angular velocity, then
(1) some friction exists between the disc and the ring
(2) the angular momentum of the 'disc plus ring' is conserved
(3) the final common angular velocity is (2/3)rd of the initial angular velocity of the disc
(4) (2/3)rd of the initial kinetic energy changes to heat

9. Two horizontal discs of different radii are free to rotate about their central vertical axes. One is given some angular velocity, the other is stationary. Their rims are now brought in contact. There is friction between the rims. Then
(1) the force of friction between the rims will disappear when the discs rotate with equal angular speeds
(2) the force of friction between the rims will disappear when they have equal linear velocities
(3) the angular momentum of the system will be conserved
(4) the rotational kinetic energy of the system will not be conserved

10. A constant external torque τ acts for a very brief period Δt on a rotating system having moment of inertia I, then
(1) the angular momentum of the system will change by $\tau\Delta t$
(2) the angular velocity of the system will change by $\tau\Delta t/I$
(3) if the system was initially at rest, it will acquire rotational kinetic energy $(\tau\Delta t)^2/2I$
(4) the kinetic energy of the system will change by $(\tau\Delta t)^2/2I$

11. Two identical spheres A and B are free to move and to rotate about their centres. They are given the same impulse J. The lines of action of the impulses pass through the centre of A and away from the centre of B, then
(1) A and B will have the same speed
(2) B will have greater kinetic energy than A
(3) they will have the same kinetic energy, but the linear kinetic energy of B will be less than that of A
(4) the kinetic energy of B will depend on the point of impact of the impulse on B

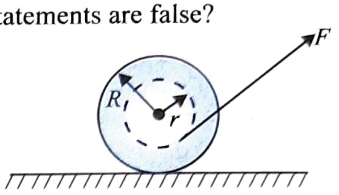
12. A disc of circumference s is at rest at a point A on a horizontal surface when a constant horizontal force begins to act on its centre. Between A and B there is sufficient friction to prevent slipping and the surface is smooth to the right of B, $AB = s$. The disc moves from A to B in time T. To the right of B,
(1) the angular acceleration of the disc will disappear, linear acceleration will remain unchanged
(2) linear acceleration of the disc will increase
(3) the disc will make one rotation in time $T/2$
(4) the disc will cover a distance greater than s in further time T

13. A ring (R), a disc (D), a solid sphere (S) and a hollow sphere with thin walls (H), all having the same mass but different radii, start together from rest at the top of an inclined plane and roll down without slipping. Then
(1) all of them will reach the bottom of the incline together
(2) the body with the maximum radius will reach the bottom first
(3) they will reach the bottom in the order S, D, H and R
(4) all of them will have the same kinetic energy at the bottom of the incline

14. A solid sphere rolls without slipping on a rough horizontal floor, moving with a speed v. It makes an elastic collision with a smooth vertical wall. After impact
(1) it will move with a speed v initially
(2) its motion will be rolling without slipping
(3) its motion will be rolling without slipping initially and its rotational motion will stop momentarily at some instant
(4) its motion will be rolling without slipping only after some time

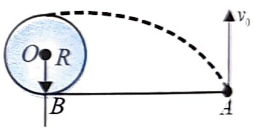
15. A thin uniform rod of mass m and length l is free to rotate about its upper end. When it is at rest, it receives an impulse J at its lowest point, normal to its length. Immediately after impact,
(1) the angular momentum of the rod is Jl
(2) the angular velocity of the rod is $3J/ml$
(3) the kinetic energy of the rod is $3J^2/2m$
(4) the linear velocity of the midpoint of the rod is $3J/2m$

16. Inner and outer radii of a spool are r and R, respectively. A thread is wound over its inner surface and spool is placed over a rough horizontal surface. Thread is pulled by a force F as shown in figure. In case of pure rolling, which of the following statements are false?



(1) Thread unwinds, spool rotates anticlockwise and friction acts leftwards.
(2) Thread winds, spool rotates clockwise and friction acts leftwards.
(3) Thread winds, spool moves to the right and friction acts rightwards.
(4) Thread winds, spool moves to the right and friction does not come into existence.

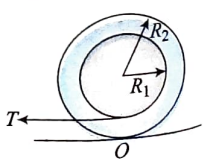
17. A horizontal plane supports a fixed vertical cylinder of radius R and a particle is attached to the cylinder by a horizontal thread AB as shown in figure. The particle initially rest on a horizontal plane. A horizontal velocity v_0 is imparted to the particle, normal to the threading during subsequent motion. Point out the false statements.



(1) Angular momentum of particle about O remains constant.
(2) Angular momentum about B remains constant.
(3) Momentum and kinetic energy both remain constant.
(4) Kinetic energy remains constant.

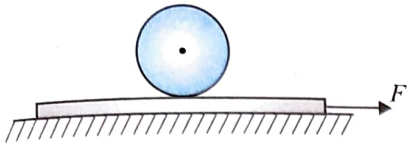
18. A uniform rod is resting freely over a smooth horizontal plane. A particle moving horizontally strikes at one end of the rod normally and gets stuck. Then
(1) the momentum of the particle is shared between the particle and the rod and remains conserved
(2) the angular momentum about the mid-point of the rod before and after the collision is equal
(3) the angular momentum about the centre of mass of the combination before and after the collision is equal
(4) the centre of mass of the rod particle system starts to move translationally with the original momentum of the particle

19. A stepped cylinder having mass 50 kg and a radius of gyration (K) of 0.30 m. The radii R_1 and R_2 are respectively 0.3 m and 0.6 m. A pull T equal to 200 N is exerted on the rope attached to inner cylinder. The coefficient of static and dynamic friction between cylinder and ground are 0.1 and 0.08 respectively.



(1) The angular acceleration is 2.67 rad s^{-2}.
(2) The force of kinetic friction is 40 N.
(3) The acceleration is 3.2 m s^{-2} towards left.
(4) The acceleration is 3.2 m s^{-2} towards right.

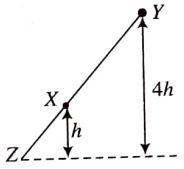
20. A plank with a uniform sphere placed on it rests on a smooth horizontal plane. The plank is pulled to the right by a constant force F. If the sphere does not slip over the plank, then



(1) both have the same acceleration
(2) acceleration of the centre of sphere is less than that of the plank
(3) work done by friction acting on the sphere is equal to its total kinetic energy
(4) total kinetic energy of the system is equal to work done by the force F

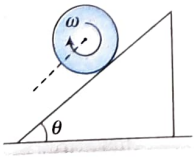
21. A sphere A moving with speed u and rotating with an angular velocity ω makes a head-on elastic collision with an identical stationary sphere B. There is no friction between the surfaces of A and B. Choose the correct alternative(s). Disregard gravity.
(1) A will stop moving but continue to rotate with an angular velocity ω.
(2) A will come to rest and stop rotating.
(3) B will move with speed u without rotating.
(4) B will move with speed u and rotate with an angular velocity ω.

22. Two wheels A and B are released from rest from points X and Y respectively on an inclined plane as shown in the figure. Which of the following statement(s) is/are incorrect?



(1) Wheel B takes twice as much time to roll from Y to Z than that of wheel A from X to Z.
(2) At point Z velocity of wheel A is four times that of wheel B.
(3) Acceleration of the wheel A is twice that of the wheel B.
(4) Both wheel take same time to arrive at point Z.

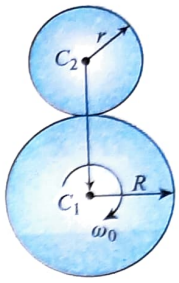
23. A solid sphere is given an angular velocity ω and kept on a rough fixed incline plane. Then choose the correct statement.



(1) If $\mu = \tan\theta$ then sphere will be in linear equilibrium for some time and after that pure rolling down the plane will start.
(2) If $\mu = \tan\theta$ then sphere will move up the plane and frictional force acting all the time will be 2 $mg \sin\theta$.
(3) If $\mu = \dfrac{\tan\theta}{2}$ there will never be pure rolling (consider inclined plane to be long enough).
(4) If incline plane is not fixed and it is on smooth horizontal surface then linear momentum of the system (wedge and sphere) can be conserved in horizontal direction.

24. A wheel A starts rolling up a rough inclined plane and another identical wheel B starts rolling up a smooth plane having same inclination with the horizontal. If initial velocity of both the wheels is same then
(1) A stops ascending earlier than B
(2) kinetic energy of B never becomes zero
(3) maximum height ascended by A is less than that by B
(4) friction acting on A remains constant during the round trip

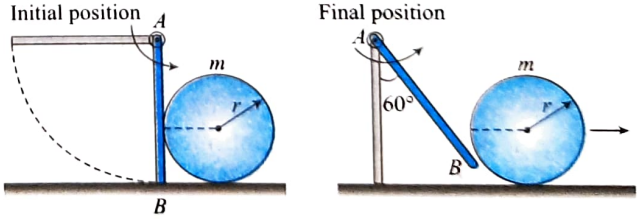
25. Consider a fixed wheel of radius R. A small wheel (in the form, of a uniform solid disc) of radius r is performing pure rolling on periphery of bigger wheel centere of bigger wheel and smaller wheel are joined by rigid rod such that smaller wheel can rotate freely w.r.t. its centre. Rod joining the (centers is rotating with constant angular velocity ω_0. Whole situation is shown in figure. Choose the correct option(s): (Use $R = 4r$)



(1) Angular velocity of smaller wheel is ω_0
(2) Angular velocity of smaller wheel is $5\omega_0$
(3) Kinetic energy of smaller wheel is $\dfrac{75}{4}mr^2\omega_0^2$, where m is the mass of the smaller disc
(4) Radius of curvature of the path of the particle which is lying on the circumference of smaller wheel and at farthest distance from centre of bigger wheel is $\dfrac{10r}{3}$

26. A uniform rod AB of mass M is attached to a hinge at one end A, and released from rest from the horizontal position. The rod rotates about A, and when it reaches the vertical position the rod strikes a sphere of mass m and radius r initially at rest on the smooth horizontal surface as shown in the adjacent figure. The impact is along the horizontal direction and perfectly elastic. If at the moment of impact the lowest end of the rod is very close to the smooth horizontal surface. After the impact, the sphere moves along the horizontal and the rod, subsequently rises to a maximum of 60° with the vertical Choose the correct statement(s) from the following, taking into account the information given above.

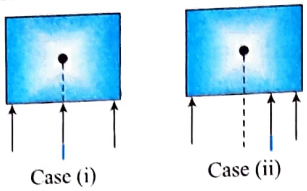
The length of the rod equals $\sqrt{2}\,r$, $\left(r = \dfrac{6\sqrt{2}}{10}\text{ m}\right)$



(1) The ratio $\dfrac{M}{m}$ is $\dfrac{3}{2}$
(2) The ratio $\dfrac{M}{m}$ is $\dfrac{2}{3}$
(3) The speed of the sphere just after collision is 6 m/s
(4) The speed of the sphere just after collision is 3 m/s

27. Two identical blocks are resting on supports as shown. Two identical bullets travelling with the same speed hit the

blocks and get embedded in them in very short time. The blocks leave the contact immediately after bullet strikes them. In case (i) bullet's velocity is toward centre of mass and in case (ii) bullet's velocity is off centre. The kinetic energies after collision are K_1 & K_2 and maximum height reached by the centre of mass are h_1 & h_2 respectively. Then



(1) $K_1 < K_2$ (2) $K_1 = K_2$
(3) $h_1 > h_2$ (4) $h_1 = h_2$

Linked Comprehension Type

For Problems 1–4

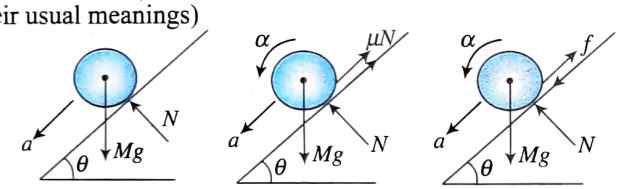
In this passage a brief idea is given of the motion of the rolling bodies on an inclined plane. We will consider three cases: objects are released on an incline plane

Case-A: which is smooth.

Case-B: where friction is insufficient to provide pure rolling.

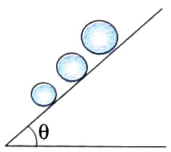
Case-C: where friction is sufficient to provide pure rolling.

Force diagram for three cases are as follows: (where symbols have their usual meanings)



Case (A) $\alpha = 0$ Case (B) $\alpha \neq \alpha R$ Case (C) $\alpha = \alpha R$

1. Three solid uniform spheres are released on an inclined plane as shown in the figure. The distance between the spheres remains constant during motion in



(1) all three masses
(2) case A and B
(3) only case C
(4) depends on the mass of the spheres.

2. We have four objects: a solid sphere, a hollow sphere, a ring and a disc, all of same radius. When these are released on an inclined plane, it may happen that all of them do not perform pure rolling. But from the information of pure rolling, if one object can be confirmed to be purely rolling then it can be said that rest all will perform pure rolling. This object whose pure rolling confirms pure rolling of all other objects is

(1) hollow sphere (2) solid sphere
(3) ring (4) disc

3. If the four objects given in the above question are of same mass, same radius having the same friction coefficient and are released from the same height, then at the bottom the object which will have least kinetic energy for case B will be the

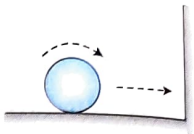
(1) hollow sphere (2) solid sphere
(3) ring (4) disc

4. Two children A and B use bicycles, having wheels of ring type and disc type respectively. During a race, bicycles are given the same velocity from the bottom of the inclined bridge to ascend the bridge without pedalling, then (assuming pure rolling).

(1) both bicycles with reach up to same height
(2) bicycle of child A will reach a greater height
(3) bicycle of child B will reach a greater height
(4) depends on the masses of bicycles and the child

For Problems 5–6

A ball, rolling purely on a horizontal floor with centre's speed v, hits a smooth vertical wall surface elastically. Answer the following questions.



5. Just after the collision is over, the velocity of the lowest point of the ball is

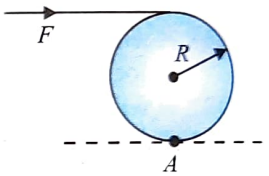
(1) v, leftward (2) $2v$, leftward
(3) v, rightward (4) $2v$, rightward

6. The change in angular momentum of the solid ball (mass m, radius R), about the corner point of floor and wall, due to the collision is

(1) $2mvR$ (2) mvR
(3) $mvR/2$ (4) None

For Problems 7–9

When a body is hinged at a point and a force is acting on the body in such a way that the line of action of force is at some distance from the hinged point, the body will start rotating about the hinged point. The angular acceleration of the body can be calculated by finding the torque of that force about the hinged point. A disc of mass m and radius R is hinged at point A at its bottom and is free to rotate in the vertical plane. A force of magnitude F is acting on the ring at the top most point.



7. Tangential acceleration of the centre of mass is

(1) $\frac{3F}{4m}$ (2) $\frac{F}{m}$
(3) $\frac{2F}{3m}$ (4) $\frac{4F}{3m}$

8. Component of reaction at the hinge in the vertical direction is

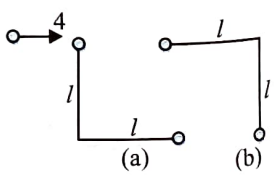
(1) $\frac{4}{3}mg$ (2) mg
(3) $\frac{mg}{2}$ (4) $\frac{2}{3}mg$

9. Component of reaction at the hinge in the horizontal direction is

(1) $\frac{F}{4}$ (2) F
(3) $\frac{F}{3}$ (4) $\frac{F}{2}$

For Problems 10–12

A spherical ball of mass M moving with initial velocity u collides elastically with another ball of mass M, which is fixed at one end of L shaped rigid massless frame as shown in Fig. (a). The L shaped frame contains another mass M connected at the other end.



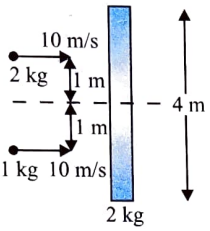
10. The speed of the striking mass after collision is
(1) $u/2$ backwards (2) 0
(3) $u/3$ in same direction (4) $u/7$ backwards

11. The angular speed of L frame immediately after collision is
(1) $\frac{4u}{7l}$ (2) $\frac{u}{3l}$
(3) $\frac{4u}{2l}$ (4) $\frac{u}{7l}$

12. How soon will the frame come to the orientation shown in Fig. (b) after collision?
(1) $\frac{\pi l}{4u}$ (2) $\frac{7\pi l}{8u}$
(3) $\frac{\pi l}{u}$ (4) $\frac{7\pi l}{4u}$

For Problems 13–15

A long slender rod of mass 2 kg and length 4 m is placed on a smooth horizontal table. Two particles of masses 2 kg and 1 kg strike the rod simultaneously and stick to the rod after collision as shown in figure.



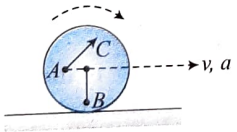
13. Velocity of the centre of mass of the rod after collision is
(1) 12 m/s (2) 9 m/s
(3) 6 m/s (4) 3 m/s

14. Angular velocity of the rod after collision is
(1) $\frac{11}{17}$ rad/s (2) $\frac{24}{17}$ rad/s
(3) $\frac{19}{17}$ rad/s (4) $\frac{30}{17}$ rad/s

15. If the two particles strike the rod in opposite direction, then after collision, as compared to the previous situation the rod will
(1) rotate faster but translate at the same rate
(2) show no change in linear or angular velocity
(3) rotate slower and translate faster
(4) rotate faster and translate slower

For Problems 16–18

A uniform disc of mass 1 kg and radius 20 cm is rolling purely on a flat horizontal surface. Its centre C is moving with acceleration $a = 20\ \text{ms}^{-2}$ and velocity $v = 4$ m/s at a certain instant. At this instant, points A and B are located on the disc as shown in the diagram, with $AC = BC = 10$ cm.



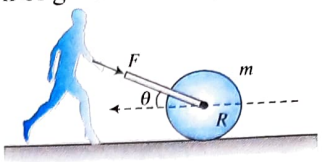
16. What is the kinetic energy of the disc?
(1) 12 J (2) 8 J
(3) 20 J (4) 10 J

17. What is the acceleration magnitude of point A?
(1) $10\sqrt{17}\ \text{ms}^{-2}$ (2) $10\sqrt{37}\ \text{ms}^{-2}$
(3) $10\sqrt{35}\ \text{ms}^{-2}$ (4) $60\sqrt{17}\ \text{ms}^{-2}$

18. What is the acceleration magnitude of point B?
(1) $10\sqrt{17}\ \text{ms}^{-2}$ (2) $10\sqrt{37}\ \text{ms}^{-2}$
(3) $10\sqrt{35}\ \text{ms}^{-2}$ (4) $60\sqrt{17}\ \text{ms}^{-2}$

For Problems 19–21

A gardener presses the grasscutter with a force F at an angle θ. Assume the motion of grasscutter as pure rolling. Find the



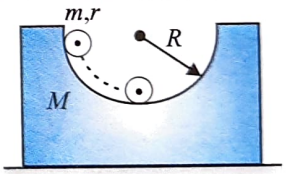
19. Friction between the grasscutter (assumed as a disc) and ground.
(1) $\frac{F\sin\theta}{3}$ (2) $\frac{F\cos\theta}{3}$
(3) $\frac{2F}{3}$ (4) $\frac{2F\cos\theta}{3}$

20. Acceleration of CM of the roller (disc). Assume that the mass of the disc is m and neglect the mass of the connecting rod.
(1) $\frac{2F\cos\theta}{3m}$ (2) $\frac{F\sin\theta}{3m}$
(3) $\frac{2F\sin\theta}{3m}$ (4) $\frac{F\cos\theta}{3m}$

21. Maximum force F for no relative sliding if the coefficient of friction between the roller and ground is μ.
(1) $\frac{\mu mg\cos\theta}{\cos\theta+3\mu\sin\theta}$ (2) $\frac{\mu mg\sin\theta}{\cos\theta+3\mu\sin\theta}$
(3) $\frac{3\mu mg}{\cos\theta-3\mu\sin\theta}$ (4) $\frac{3\mu mg}{\cos\theta+3\mu\sin\theta}$

For Problems 22–24

A uniform solid cylinder of mass 2 kg and radius 0.2 m is released from rest at the top of a semicircular track of radius 0.7 m cut in a block of mass $M = 3$ kg as shown in figure. The block is resting on a smooth horizontal surface and the cylinder rolls down without slipping. Based on the above information, answer the following questions:



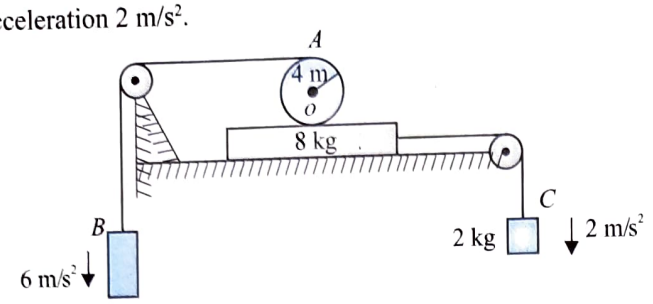
22. The distance moved by the block when the cylinder reaches the bottom of the track is
(1) 0.3 m (2) 0.5 m
(3) 0.7 m (4) 0.2 m

23. The speed of the cylinder when it reaches the bottom of the track is
(1) $\frac{2}{\sqrt{11}}$ m/s (2) $\frac{8}{\sqrt{11}}$ m/s
(3) $\frac{4}{\sqrt{11}}$ m/s (4) $\frac{6}{\sqrt{11}}$ m/s

24. The speed of the point of contact of the cylinder with the block w.r.t. ground, when the cylinder reaches the bottom of the track is
(1) $\frac{2}{\sqrt{11}}$ m/s (2) $\frac{8}{\sqrt{11}}$ m/s
(3) $\frac{4}{\sqrt{11}}$ m/s (4) $\frac{6}{\sqrt{11}}$ m/s

For Problems 25–27

Figure shows a uniform smooth solid cylinder A of radius 4 m rolling without slipping on the 8 kg plank which, in turn, is supported by a fixed smooth surface. Block B is known to accelerate down with 6 m/s² and block C moves down with acceleration 2 m/s².



25. What is the angular acceleration of the cylinder?

(1) $\frac{4}{5}$ rad s^{-2} (2) $\frac{6}{5}$ rad s^{-2}

(3) 2 rad s^{-2} (4) 1 rad s^{-2}

26. What is the ratio of the mass of the cylinder to the mass of block B?

(1) 1 (2) 2

(3) 3 (4) 4

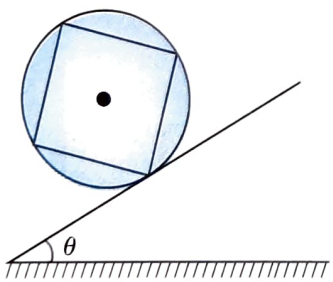
27. If unwrapped length of the thread between the cylinder and block B is 20 m at the beginning, when the system was released from rest, what would it be 2 s later ?

(1) 28 m (2) 30 m

(3) 22 m (4) 32.5 m

For Problems 28–30

Four identical rods of mass M = 6 kg each are welded at their ends to form a square and then welded to a massive ring having mass m = 4 kg and radius R = 1 m. If the system is allowed to roll down the incline of inclination $\theta = 30°$, determine the minimum value of the coefficient of static friction that will prevent slipping.



28. The moment of inertia of the system about the centre of ring will be

(1) 20 kg m² (2) 40 kg m²

(3) 10 kg m² (4) 60 kg m²

29. The acceleration of the system will be

(1) $\frac{g}{2}$ (2) $\frac{g}{4}$

(3) $\frac{7g}{24}$ (4) $\frac{g}{8}$

30. The minimum value of friction coefficient to prevent slipping is

(1) $\frac{5}{7}$ (2) $\frac{5}{12\sqrt{3}}$

(3) $\frac{5\sqrt{3}}{7}$ (4) $\frac{7}{5\sqrt{3}}$

For Problems 31–33

A man of mass 100 kg stands at the rim of a turntable of radius 2 m and moment of inertia 4000 kg m² mounted on a vertical frictionless shaft at its centre. The whole system is initially at rest. The man now walks along the outer edge of the turntable (anticlockwise) with a velocity of 1 m/s relative to the earth.

31. With what angular velocity and in what direction does the turntable rotate?

(1) The table rotates anticlockwise (in the direction of the man motion) with angular velocity 0.05 rad/s.

(2) The table rotates clockwise (opposite to the man) with angular velocity 0.1 rad/s.

(3) The table rotates clockwise (opposite to the man) with angular velocity 0.05 rad/s.

(4) The table rotates anticlockwise (in the direction of the man motion) with angular velocity 0.1 rad/s.

32. Through what angle will it have rotated when the man reaches his initial position on the turntable?

(1) The table rotates through $2\pi/11$ radians anticlockwise

(2) The table rotates through $4\pi/11$ radians clockwise

(3) The table rotates through $4\pi/11$ radians anticlockwise

(4) The table rotates through $2\pi/11$ radians clockwise

33. Through what angle will it have rotated when the man reaches his initial position relative to earth ?

(1) 36° in clockwise direction

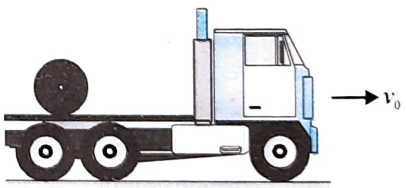
(2) 36° in anticlockwise direction

(3) 72° in clockwise direction

(4) 72° in anticlockwise direction

For Problems 34–35

A solid sphere of mass M and radius R is initially at rest. Solid sphere is gradually lowered onto a truck moving with constant velocity v_0.



34. What is the final speed of the sphere's centre of mass in ground frame when eventually pure rolling sets in

(1) $\frac{5}{7}v_0$ (2) $\frac{2}{7}v_0$

(3) $\frac{7}{5}v_0$ (4) $\frac{7}{2}v_0$

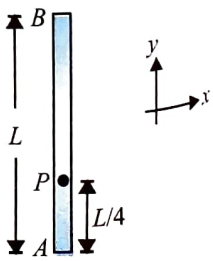
35. If a sphere with twice the radius and four times the mass had been used; what would have been its final speed?

(1) $\frac{5}{7}v_0$ (2) $\frac{2}{7}v_0$

(3) $\frac{7}{5}v_0$ (4) $\frac{7}{2}v_0$

For Problems 36–38

A uniform rod AB hinged about a fixed point P is initially vertical. A rod is released from vertical position. When rod is in horizontal position.



36. The acceleration of the centre of mass of the rod is

(1) $-\frac{6g}{7}\hat{i}-\frac{3g}{7}\hat{j}$ (2) $-\frac{12g}{7}\hat{i}-\frac{6g}{7}\hat{j}$

(3) $-\frac{3g}{7}\hat{i}-\frac{9g}{7}\hat{j}$ (4) $-\frac{9g}{7}\hat{i}-\frac{3g}{7}\hat{j}$

37. The acceleration of end B of the rod is

(1) $-\frac{6g}{7}\hat{i}-\frac{12g}{7}\hat{j}$ (2) $-\frac{5g}{7}\hat{i}-\frac{9g}{7}\hat{j}$

(3) $-\frac{18g}{7}\hat{i}-\frac{5g}{7}\hat{j}$ (4) $-\frac{18g}{7}\hat{i}-\frac{9g}{7}\hat{j}$

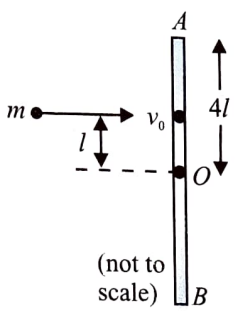
38. The reaction at hinge P is

(1) $-\frac{8mg}{7}\hat{i}-\frac{12mg}{7}\hat{j}$ (2) $-\frac{3mg}{7}\hat{i}-\frac{9mg}{7}\hat{j}$

(3) $-\frac{6mg}{7}\hat{i}+\frac{4mg}{7}\hat{j}$ (4) $-\frac{12mg}{7}\hat{i}-\frac{6mg}{7}\hat{j}$

For Problems 39–42

A rod AB of mass M and length $8l$ lies on a smooth horizontal surface. A particle of mass 'm' and velocity v_0 strikes the rod perpendicular to its length, as shown in figure. As a result of the collision, the centre of mass of rod attains a speed of $v_0/8$ and the particle rebounds back with a speed of $v_0/4$. Find the following.



39. The ratio $\frac{M}{m}$,

(1) $\frac{M}{m}=10$ (2) $\frac{M}{m}=4$

(3) $\frac{M}{m}=8$ (4) $\frac{M}{m}=5$

40. The angular velocity of the rod about O,

(1) $\frac{5v_0}{128l}$ (2) $\frac{3v_0}{128l}$

(3) $\frac{v_0}{128l}$ (4) $\frac{7v_0}{128l}$

41. The coefficient of restitution 'e' for the collision,

(1) $\frac{51}{128}$ (2) $\frac{61}{128}$

(3) $\frac{21}{128}$ (4) $\frac{31}{128}$

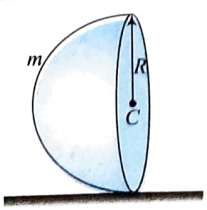
42. The velocities of the ends 'A' and 'B' of the rod, namely v_A and v_B, respectively.

(1) $v_A=\frac{5}{32}v_0;\ v_B=\frac{1}{32}v_0$ (2) $v_A=\frac{7}{32}v_0;\ v_B=\frac{3}{32}v_0$

(3) $v_A=\frac{7}{32}v_0;\ v_B=\frac{1}{32}v_0$ (4) $v_A=\frac{5}{32}v_0;\ v_B=\frac{3}{32}v_0$

For Problems 43–44

Figure shows a hemispherical bowl of uniform mass density placed on a rough horizontal surface. Mass of the bowl is 'm' and its radius is 'R'. Initially the geometrical centre 'C' of the hemispherical bowl is vertically above the point of contact with the ground. Now the bowl is released from rest in the position shown. On the basis of above information, answer the following questions.



43. The minimum coefficient of static friction μ_{min} between hemispherical bowl and surface to prevent initial slipping of the bowl, is

(1) $\frac{3}{17}$ (2) $\frac{6}{17}$

(3) 2 (4) $\frac{1}{2}$

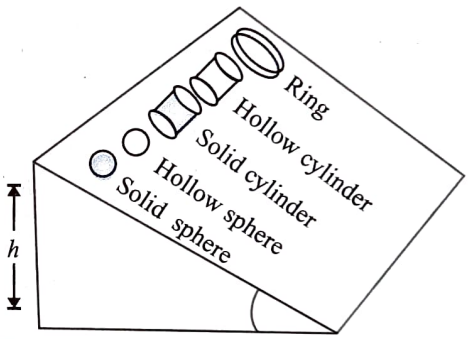
44. If coefficient of static friction $\mu_s=\frac{\mu_{min}}{2}$, then the normal force acting on the bowl, is

(1) $\frac{85}{118}mg$ (2) $\frac{85}{59}mg$

(3) $\frac{17}{59}mg$ (4) $\frac{17}{118}mg$

Matrix Match Type

1. A solid sphere, a thin-walled hollow sphere, a solid cylinder, a thin-walled hollow cylinder and a ring, each of mass m and radius R, are simultaneously released at rest from the top of an inclined plane, as shown in figure. The objects roll down the plane without slipping. Also we may consider the objects and the surface on which they roll to be perfectly rigid. Match Columns I and II.

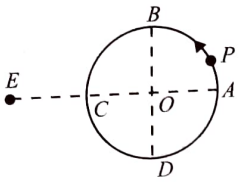


Column I	Column II
i. Time taken to reach the bottom is maximum for	a. Solid sphere
ii. Angular acceleration is maximum for	b. Hollow cylinder
iii. Kinetic energy at the bottom is the same for	c. Hollow sphere
iv. Rotational kinetic energy is maximum for	d. Ring

2. Column I gives the bodies which are released one by one from the top of an inclined plane. Assume each body rolls purely.

Column I	Column II
i. Wooden solid cylinder	a. KE_{rot} is maximum at the bottom
ii. Iron solid cylinder	b. KE_{trans} is minimum at the bottom
iii. Thin iron cylindrical shell	c. Takes minimum time to reach the bottom
iv. Thin wooden cylindrical shell	d. Takes maximum time to reach the bottom

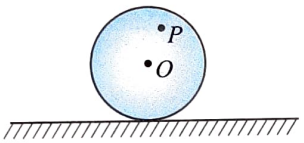
3. A particle P moves with constant speed on a circle in anticlockwise direction as shown in figure.



Match Column I with Column II:

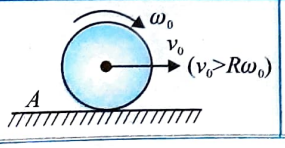
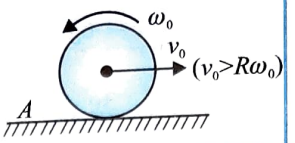
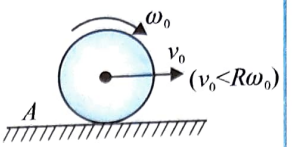
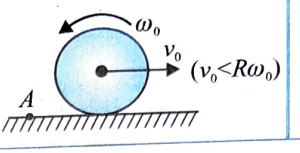
Column I	Column II
i. Angular momentum of the particle about O	a. is minimum when the particle is at A
ii. Angular momentum of the particle about E	b. is maximum when the particle is at A
iii. Angular velocity of the particle about O	c. does not remain constant
iv. Angular velocity of the particle about E	d. remains constant

4. A uniform disc rolls without slipping on a rough horizontal surface with uniform angular velocity. Point O is the centre of disc and P is a point on disc as shown in figure. In each situation of Column I a statement is given and the corresponding results are given in Column II. Match the statements in Column I with the results in Column II.



Column I	Column II
i. The velocity of point P on the disc	a. changes in magnitude with time
ii. The acceleration of point P on the disc	b. always directed from that point (the point on disc given in Column I) towards the centre of disc.
iii. The tangential acceleration of point P on the disc	c. is always zero
iv. The acceleration of the point on the disc which is in contact with rough horizontal surface	d. is non-zero and remains constant in magnitude

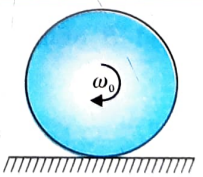
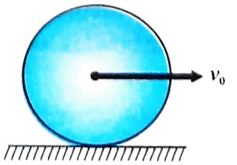
5. In each situation of Column I, a uniform disc of mass m and radius R rolls on a rough fixed horizontal surface as shown. At $t = 0$ (initially) the angular velocity of the disc is ω_0 and velocity of the centre of mass of the disc is v_0 (in horizontal direction). The relation between v_0 and ω_0 for each situation and also initial sense of rotation is given for each situation in Column I. Then match the statements in Column I with the corresponding results in Column II.

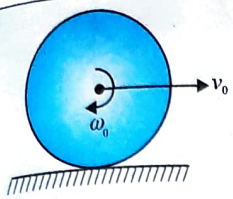
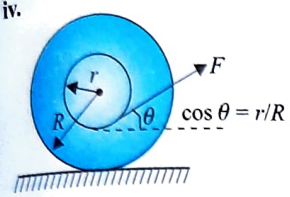
Column I	Column II
i. 	a. The angular momentum of the disc about point A (as shown in the figure) remains conserved.
ii. 	b. The kinetic energy of the disc after it starts rolling without slipping is less than its initial kinetic energy.
iii. 	c. In the duration disc rolls with slipping, the friction acts on the disc towards the left.
iv. 	d. In the duration disc rolls with slipping, the friction acts on the disc for some time to the right and for some time to the left.

6. Match the statements in Column I with those in Column II. One or more matching is possible.

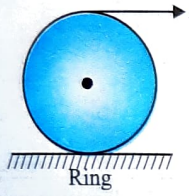
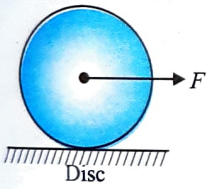
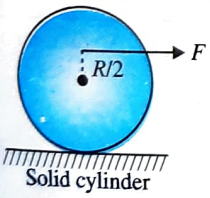
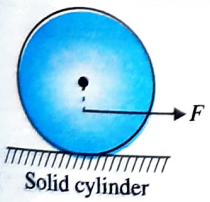
Column I	Column II
i. Solid sphere rolling with slipping on a rough horizontal surface	a. Total kinetic energy is conserved
ii. Solid sphere in pure rolling on a rough horizontal surface and no other force is acting	b. Angular momentum about the CM is conserved
iii. Solid sphere in pure rolling on a smooth horizontal surface	c. Angular momentum about a contact point on the contact surface is conserved
iv. Solid sphere in pure rolling on a rough incline	d. Momentum is conserved
	e. Total mechanical energy is conserved

7. Match the columns.

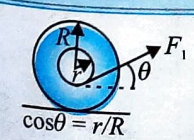
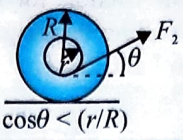
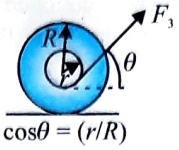
Column I	Column II
i.  A spinning ball lowered on a rough surface	a. Net work done by friction force is negative
ii.  A ball is projected with velocity v_0 on a rough surface	b. Angular momentum about the CM is conserved

iii.  A ball projected with velocity $v_0 = R\omega_0$ projected on a smooth horizontal surface	**c.** Translational work done by friction is positive
iv.  A spool pulled by a force F, rolling without slipping on a rough surface	**d.** Net work done by friction is zero

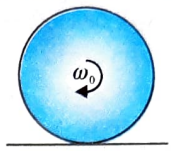
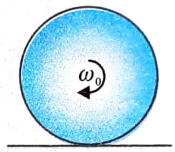
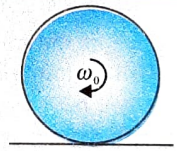
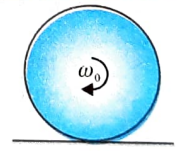
8. Assume sufficient friction to prevent slipping.

Column I	Column II
i. 	**a.** Body accelerates forward
ii. 	**b.** Rotation about the centre of mass is clockwise
iii. 	**c.** Friction force acts backward
iv. 	**d.** No friction acts

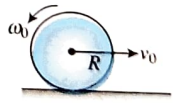
9. A uniform yo-yo (longer radius R, small radius r) is resting on a perfectly rough horizontal table. This equation should be attempted after forces F_1, F_2, F_3 and F_4 are applied separately as shown.

Column I	Column II
i. 	**a.** Centre of mass of the yo-yo accelerates towards the left
ii. 	**b.** Centre of mass of the yo-yo accelerates towards the right
iii. 	**c.** Friction acts on the yo-yo towards the left
	d. Friction acts on the yo-yo towards the right

10. Match the columns

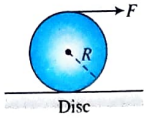
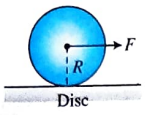
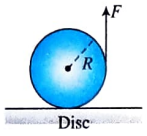
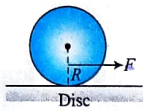
Column I	Column II
i.  Ring	**a.** Rotational work done by the friction is negative till pure rolling begins
ii.  Cylinder	**b.** Translational work done is positive till pure rolling starts
iii.  Solid sphere	**c.** When pure rolling begins velocity of centre of mass is minimum
iv.  Hollow sphere	**d.** Takes maximum time for pure rolling to begin

11. A solid cylinder of mass M and Radius R is given a linear velocity v_0 and angular velocity ω_0 as shown and then it is released gently on a rough horizontal surface. Match the columns appropriately.

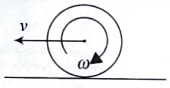
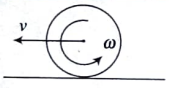
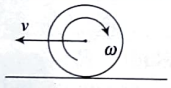


Column I	Column II
i. Finally cylinder moves in forward direction	**a.** may be possible
ii. Finally cylinder moves in backward direction	**b.** not possible at all
iii. Finally cylinder stops	**c.** is conserved about some point
iv. Angular momentum of cylinder	**d.** is not conserved about any point

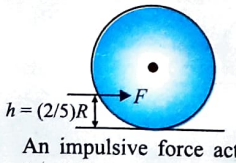
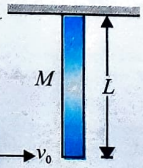
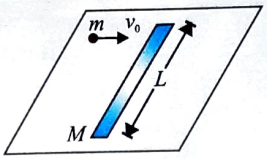
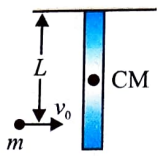
12. In each case, there is sufficient friction for regular rigid uniform body to undergo pure rolling on a rigid horizontal surface. Now match the column I and II

Column I	Column II
i. 	a. The direction of static friction on the disc is towards right.
ii. 	b. The direction of static friction on the disc is towards left.
iii. 	c. The angular acceleration will be clockwise
iv. 	d. Acceleration of the centre mass is in the direction of F.

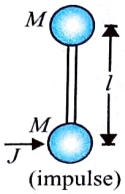
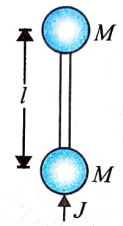
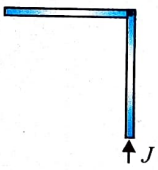
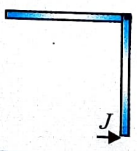
13.

Column I (Initially)	Column II (When rolling without slipping begins)
i. $\omega = \frac{2v}{R}$	a. v_{cm} is towards left in case of uniform ring
ii.  $\omega = \frac{2v}{R}$	b. v_{cm} is towards left in case of solid uniform sphere
iii.  $\omega = \frac{v}{2R}$	c. v_{cm} is towards right in case of uniform ring
iv.  $\omega = \frac{v}{2R}$	d. v_{cm} is towards right in case of solid uniform sphere

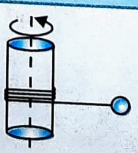
14. Match the entries in Column I with those in Column II.

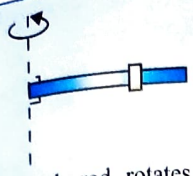
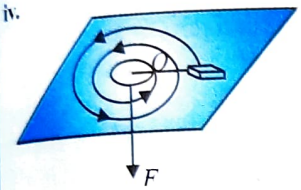
Column I	Column II
i.  An impulsive force acts at a height $h = (2/5)\,R$ on a solid sphere, lying on a rough surface.	a. Momentum is conserved
ii.  A small particle of mass m moving horizontally collides with a vertical hinged rod inelastically; $e = 0$.	b. Angular momentum is conserved
iii.  A small particle of mass m strikes elastically at the end of a horizontal rod kept on a smooth surface	c. Energy is conserved
iv.  A small particle of mass m collides with a vertical rod elastically	d. Momentum increases

15.

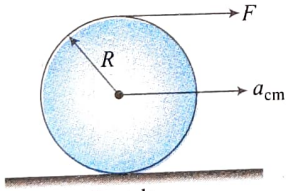
Column I	Column II
i.  (impulse) Rod is massless, dumb-bell is placed on a smooth horizontal surface	a. Translation occurs
ii. 	b. Rotation occurs
iii. 	c. Angular momentum increases
iv. 	d. Linear momentum increases

16.

Column I	Column II
i.  A small particle of mass m is given an initial velocity in horizontal plane and winds its cord around the fixed vertical shaft of radius a.	a. Conservation of angular momentum

ii.  A smooth rod rotates with some angular velocity. A small sleeve starts sliding along the rod.	**b.** Conservation of kinetic energy
iii. Two ice-skaters approach each other at equal speeds along the parallel path separated by some distance. They link hands as they pass by and pull each other to reduce their separation.	**c.** Conservation of total mechanical energy
iv.  A small body tied to a non-stretchable thread of mass m is lying over a smooth horizontal plane. Other end of the thread is being drawn into a hole O.	**d.** Work is done by internal forces.

17. A uniform rigid body of mass m and radius R is placed at rest on a rough horizontal surface. A constant horizontal force F is applied at the topmost point of the body. The body starts rolling without slipping. Different shapes of bodies are given in the column I and based on this problem some physical quantities related to them are given in column II.



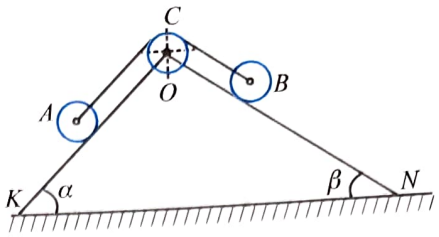
Rough

Column I	Column II
i. Solid sphere	**a.** Friction force is zero
ii. Ring	**b.** Magnitude of friction force is maximum
iii. Hollow sphere	**c.** Acceleration of the centre of mass is $4F/3m$
iv. Disc	**d.** Magnitude of friction force is $F/5$

(1) i – b, ii – a, iii – c, iv – c
(2) i – b, ii – a, iii – d, iv – c
(3) i – d, ii – b, iii – a, iv – c
(4) i – c, ii – a, iii – d, iv – b

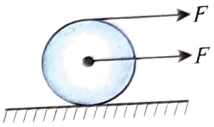
Numerical Value Type

1. A wheel A is connected to a second wheel B by means of inextensible string, passing over a pulley C, which rotates about a fixed horizontal axle O, as shown in the figure. The system is released from rest. The wheel A rolls down the inclined plane OK thus pulling up wheel B which rolls along the inclined plane ON.

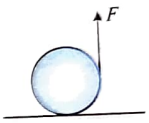


Determine the velocity (in m/s) of the axle of wheel A, when it has travelled a distance $s = 3.5$ m down the slope. Both wheels and the pulley are assumed to be homogeneous disks of identical weight and radius. Neglect the weight of the string. The string does not slip over C.
[Take $\alpha = 53°$ and $\beta = 37°$]

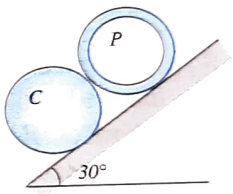
2. Two forces of magnitude F are acting on a uniform disc kept on a horizontal rough surface as shown in the figure. Friction force by the horizontal surface on the disc is nF. Find the value of n.



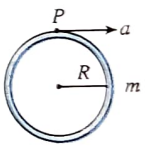
3. We apply a force of 10 N on a cord wrapped around a cylinder of mass 2 kg. The cylinder rolls without slipping on the floor. What is its kinetic energy (in joule) when cylinder has moved by a distance of 3/5 m.



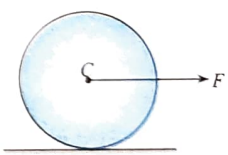
4. A solid cylinder C and a hollow pipe P of same diameter are in contact when they are released from rest as shown in the figure on a long incline plane. Cylinder C and pipe P roll without slipping. Determine the clear gap (in m) between them after $2\sqrt{3}$ seconds.



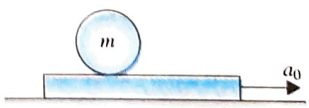
5. A smooth ring of mass m and radius $R = 1$ m is pulled at P with a constant acceleration $a = 4$ m s^{-2} on a horizontal surface such that the plane of the ring lies on the surface. Find the angular acceleration of the ring at the given position. (in rad/s^2)



6. A horizontal force $F = 14$ N acts at the centre of mass of a sphere of mass $m = 1$ kg. If the sphere rolls without sliding, find the frictional force (in N)



7. A rolling body of mass $m = 4$ kg, radius R and radius of gyration $k = R/\sqrt{3}$ is placed as a plank which moves with an acceleration $a_0 = 1$ ms^{-2}. Find the frictional force acting on the body if it rolls without sliding. (in N).



8. A solid sphere rolls on a smooth horizontal surface at 10 m/s and then rolls up a smooth inclined plane of inclination 30° with horizontal. The mass of the sphere is 2 kg. Find the height attained by the sphere before it stops (in m).

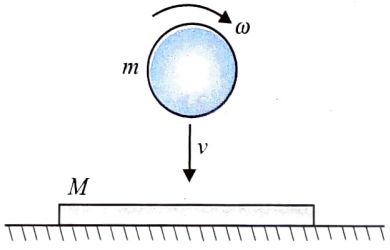
9. A loop and a disc roll without slipping with same linear velocity v. The mass of the loop and the disc is same. If the

total kinetic energy of the loop is 8 J, find the kinetic energy of the disc (in J).

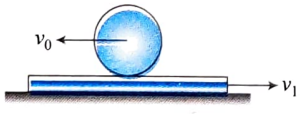
10. A ring of mass 3 kg is rolling without slipping with linear velocity 1 ms^{-1} on a smooth horizontal surface. A rod of same mass is fitted along its one diameter. Find total kinetic energy of the system (in J).

11. A solid sphere of mass 3 kg is kept on a horizontal surface. The coefficient of static friction between the surfaces in contact is 2/7. What maximum force can be applied at the highest point in the horizontal direction so that the sphere does not slip on the surface? (in N)

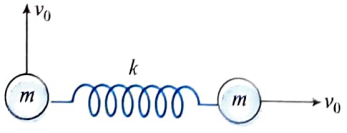
12. A solid ball of mass m and radius r spinning with angular velocity ω falls on a horizontal slab of mass M with rough upper surface (coefficient of friction μ) and smooth lower surface. Immediately after collision the normal component of velocity of the ball remains half of its value just before collision and it stops spinning. Find the velocity of the sphere in horizontal direction immediately after the impact (given: $R\omega = 5$).



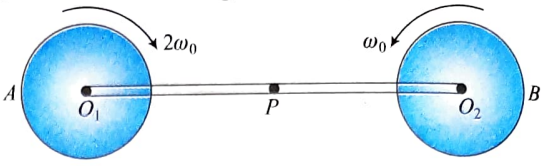
13. A sphere of radius r and mass m has a linear velocity v_0 m/s directed to the left and no angular velocity as it is placed on a horizontal platform moving to the right with a constant velocity 10 m/s. If after sliding on the platform the sphere is to have no linear velocity relative to the ground as it starts rolling on the platform without sliding. The coefficient of kinetic friction between the sphere and the platform is μ_k Determine the required value of v_0 in m/s.



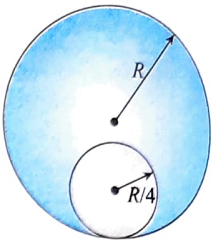
14. Two small spheres of mass 1 kg each are connected to each other by means of an unstretched spring of force constant $k = 100$ N/m and natural length 2.4 m. The system is placed on a smooth horizontal surface and the two spheres are given velocities v_0, as shown in the figure. The maximum elongation in the spring is found to be 0.4 m during the subsequent motion. Calculate v_0 in m/s. Take $\sqrt{31} = 5.6$



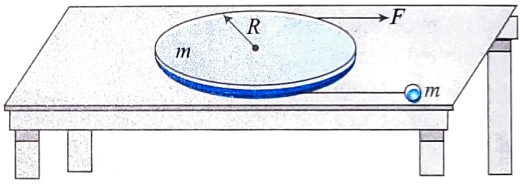
15. A rod of mass 'M' and length $6R$ is pivoted at its centre (point P) to rotate in vertical plane as shown. Two discs of same radius 'R' are joined at the ends O_1 and O_2 of the rod. Mass of the discs A and B are 'M' and '$2M$' respectively. Initial angular velocities of discs are shown in figure. The system is released from rest from horizontal position. Find the angular velocity of the rod (in rad/sec)when it is vertical. (Given $R = 0.5$ m and $g = 10$ m/s²)



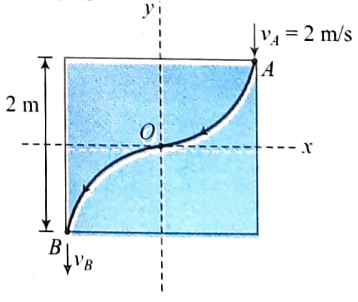
16. A ring of mass m and radius $R/4$ rolls inside a fixed hoop of radius R such that at the highest point of its trajectory the normal reaction becomes zero. The friction on the hoop is sufficient to ensure pure rolling. Find the angular velocity of the ring in rad/s. (Given $R = 0.3$ m and $g = 10$ m/s²)



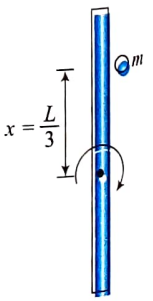
17. A uniform disc of mass $m = 1$ kg and radius $R = 20$ cm lies flat on a smooth horizontal table. A mass less string runs halfway around it as shown in figure. One end of the string is attached to a small block of mass m and the other end is being pulled with a force $F = 20$ N. The circumference of the disc is sufficiently rough so that the string does not slip over it. Find acceleration of the small block (in m/s²).



18. A metal plate which is square in shape of side length $l = 2$ m has a groove made in the shape of two quarter circles joining at the centre of the plate. The plate is free to rotate about vertical axis passing through its centre. The moment of inertia of the plate about this axis is $I_0 = 4$ kg-m². A particle of mass $m = 1$ kg enters the groove at end A travelling with a velocity of $v_A = 2$ m/s parallel to the side of the square plate. The particle move along the frictionless groove of the horizontal plate and comes out at the other end B with speed v. Find the magnitude of v (in m/s) assuming that width of the groove is negligible.



19. A uniform thin stick of mass $M = 24$ kg and length L rotates on a frictionless horizontal plane, with its centre of mass stationary. A particle of mass m is placed on the plane at a distance $x = L/3$ from the centre of the stick. This stick hits the particle elastically. Find the value of m/M so that after the collision, there is no rotational motion of the stick.



Archives

JEE MAIN

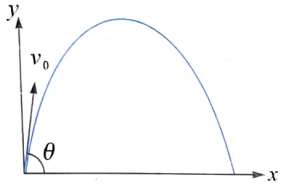
Single Correct Answer Type

1. A thin uniform rod of length l and mass m is swinging freely about a horizontal axis passing through its end. Its maximum angular speed is ω. Its centre of mass rises to a maximum height of

(1) $\frac{1}{3}\frac{l^2\omega^2}{g}$ (2) $\frac{1}{6}\frac{l\omega}{g}$

(3) $\frac{1}{2}\frac{l^2\omega^2}{g}$ (4) $\frac{1}{6}\frac{l^2\omega^2}{g}$ **(AIEEE 2009)**

2. A small particle of mass m is projected at an angle θ with the x-axis with an initial velocity v_0 in the x-y plane as shown in the figure. At a time $t < (v_0 \sin\theta/g)$, the angular momentum of the particle is



(1) $-mgv_0t^2\cos\theta\,\hat{j}$ (2) $mgv_0t\cos\theta\,\hat{k}$

(3) $-\frac{1}{2}mgv_0t^2\cos\theta\,\hat{k}$ (4) $\frac{1}{2}mgv_0t^2\cos\theta\,\hat{i}$

(AIEEE 2010)

3. A thin horizontal circular disc is rotating about a vertical axis passing through its centre. An insect is at rest at a point near the rim of the disc. The insect now moves along a diameter of the disc to reach its other end. During the journey of the insect, the angular speed of the disc

(1) remains unchanged
(2) continuously decreases
(3) continuously increases
(4) first increases and then decreases **(AIEEE 2011)**

4. A hoop of radius r and mass m rotating with an angular velocity ω_0 is placed on a rough horizontal surface. The initial velocity of the centre of the hoop is zero. What will be the velocity of the centre of the hoop when it ceases to slip?

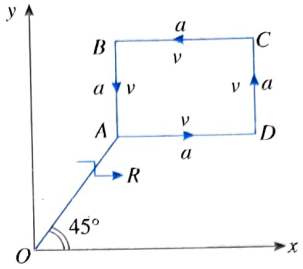
(1) $\frac{r\omega_0}{3}$ (2) $\frac{r\omega_0}{2}$

(3) $r\omega_0$ (4) $\frac{r\omega_0}{4}$ **(JEE Main 2013)**

5. A bob of mass m attached to an inextensible string of length l is suspended from a vertical support. The bob rotates in a horizontal circle with an angular speed ω rad/s about the vertical. About the point of suspension,

(1) angular momentum changes in direction but not in magnitude.
(2) angular momentum changes both in direction and in magnitude.
(3) angular momentum is conserved.
(4) angular momentum changes in magnitude but not in direction. **(JEE Main 2014)**

6. A particle of mass m is moving along side of a square of side 'a', with a uniform speed v in the x-y plane as shown in the figure:



Which of the following statements is false for the angular momentum $\vec{L}$ about the origin?

(1) $\vec{L} = -\frac{mv}{\sqrt{2}}R\hat{k}$ when the particle is moving from A to B

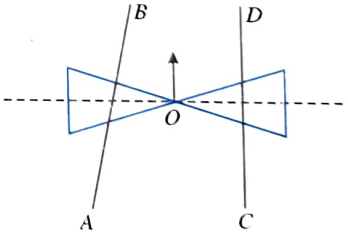
(2) $\vec{L} = mv\left[\frac{R}{\sqrt{2}} - a\right]\hat{k}$ when the particle is moving from C to D.

(3) $\vec{L} = mv\left[\frac{R}{\sqrt{2}} + a\right]\hat{k}$ when the particle is moving from B to C.

(4) $\vec{L} = \frac{mv}{\sqrt{2}}R\hat{k}$ when the particle is moving from D to A.

(JEE Main 2016)

7. A roller is made by joining together two cones at their vertices O. It is kept on two rails AB and CD which are placed asymmetrically (see figure), with its axis perpendicular to CD and its centre O at the centre of line joining AB and CD (see figure). It is given a light push so that it starts rolling with its centre O moving parallel to CD in the direction shown. As it moves, the roller will tend to:



(1) turn left
(2) turn right
(3) go straight
(4) turn left and right alternately

(JEE Main 2016)

JEE ADVANCED

Single Correct Answer Type

1. If the resultant of all the external forces acting on a system of particles is zero, then from an inertial frame, one can surely say that

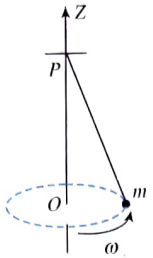
(1) linear momentum of the system does not change in time
(2) kinetic energy of the system does not change in time

(3) angular momentum of the system does not change in time

(4) potential energy of the system does not change in time

(IIT-JEE 2009)

2. A small mass m is attached to a massless string whose other end is fixed at P as shown in the figure. The mass is undergoing circular motion in the x–y plane with centre at O and constant angular speed ω. If the angular momentum of the system, calculated about O and P are denoted by $\vec{L}_O$ and $\vec{L}_P$ respectively, then



(1) $\vec{L}_O$ and $\vec{L}_P$ do not vary with time.

(2) $\vec{L}_O$ varies with time while $\vec{L}_P$ remains constant.

(3) $\vec{L}_O$ remains constant while $\vec{L}_P$ varies with time.

(4) $\vec{L}_O$ and $\vec{L}_P$ both vary with time.

(IIT-JEE 2012)

3. Two solid cylinders P and Q of same mass and same radius start rolling down a fixed inclined plane from the same height at the same time. Cylinder P has most of its mass concentrated near its surface, while Q has most of its mass concentrated near the axis. Which statement(s) is/are correct?

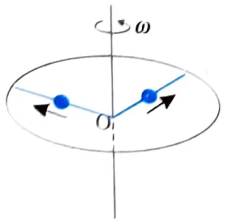
(1) Both cylinders P and Q reach the ground at the same time

(2) Cylinder P has larger linear acceleration than cylinder Q

(3) Both cylinders reach the ground with same translational kinetic energy.

(4) Cylinder Q reaches the ground with larger angular speed. **(IIT-JEE 2012)**

4. A ring of mass M and radius R is rotating with angular speed ω about a fixed vertical axis passing through its centre O with two point masses each of mass $\frac{M}{8}$ at rest at O. These masses can move radially outwards along two massless rods fixed on the ring as shown in the figure. At some instant the angular speed of the system is $\frac{8}{9}\omega$ and one of the masses is at a distance of $\frac{3}{5}R$ from O. At this instant the distance of the other mass from O is



(1) $\frac{2}{3}R$ (2) $\frac{1}{3}R$

(3) $\frac{3}{5}R$ (4) $\frac{4}{5}R$

(JEE Advanced 2015)

5. The potential energy of a particle of mass m at a distance r from a fixed point O is given by $V(r) = kr^2/2$, where k is a positive constant of appropriate dimensions. This particle is moving in a circular orbit of radius R about the point O. If v is the speed of the particle and L is the magnitude of its angular momentum about O, which of the following statements is (are) true?

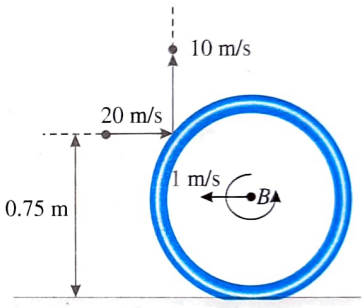
(1) $v = \sqrt{\frac{k}{2m}}R$ (2) $v = \sqrt{\frac{k}{m}}R$

(3) $L = \sqrt{mk}\,R^2$ (4) $L = \sqrt{\frac{mk}{2}}R^2$

(JEE Advanced 2018)

Multiple Correct Answers Type

1. A thin ring of mass 2 kg and radius 0.5 m is rolling without slipping on a horizontal plane with velocity 1 m/s. A small ball of mass 0.1 kg, moving with velocity 20 m/s in the opposite direction, hits the ring at a height of 0.75 m and goes vertically up with velocity 10 m/s. Immediately after the collision



(1) the ring has pure rotation about its stationary

(2) the ring comes to a complete stop

(3) friction between the ring and the ground is to the left

(4) there is no friction between the ring and the ground

(IIT-JEE 2011)

2. The position vector $\vec{r}$ of a particle of mass m is given by the following equation $\vec{r}(t) = \alpha t^3\hat{i} + \beta t^2\hat{j}$, where $\alpha = \frac{10}{3}\text{ms}^{-3}$, $\beta = 5\text{ ms}^{-2}$ and $m = 0.1$ kg. At $t = 1$ s, which of the following statement(s) is(are) true?

(1) The velocity $\vec{v}$ is given by $\vec{v} = (10\hat{i} + 10\hat{j})\text{ms}^{-1}$

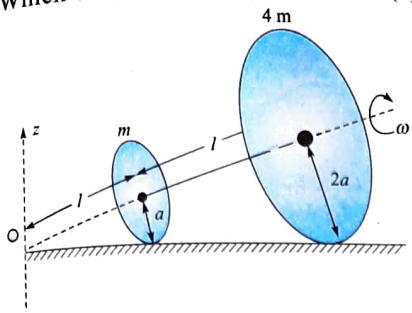
(2) The angular momentum $\vec{L}$ with respect to the origin is given by $\vec{L} = -\left(\frac{5}{3}\right)\hat{k}$ Nms

(3) The force $\vec{F}$ is given by $\vec{F} = (\hat{i} + 2\hat{j})$ N

(4) The torque $\vec{\tau}$ with respect to the origin is given by $\vec{\tau} = -\left(\frac{20}{3}\right)\hat{k}$ Nm **(JEE Advanced 2016)**

3. Two thin circular discs of mass m and $4m$, having radii of a and $2a$, respectively, are rigidly fixed by a massless, rigid rod of length $l = \sqrt{24}a$ through their centers. This assembly is laid on a firm and flat surface, and set rolling without slipping on the surface so that the angular speed

about the axis of the rod is ω. The angular momentum of the entire assembly about the point 'O' is $\vec{L}$ (see the figure). Which of the following statement (s) is (are) true?



(1) The magnitude of the z-component of $\vec{L}$ is $55\ ma^2\omega$.

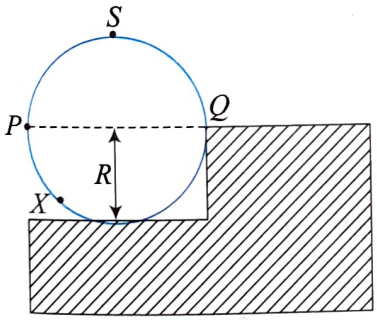
(2) The magnitude of angular momentum of the assembly about its centre of mass is $17\ ma^2\dfrac{\omega}{2}$.

(3) The magnitude of angular momentum of centre of mass of the assembly about the point O is $81\ ma^2\omega$.

(4) The centre of mass of the assembly rotates about the z-axis with an angular speed of $\dfrac{\omega}{5}$.

(JEE Advanced 2016)

4. A wheel of radius R and mass M is placed at the bottom of a fixed step of height R as shown in the figure. A constant force is continuously applied on the surface of the wheel so that it just climbs the step without slipping. Consider the torque τ about an axis normal to the plane of the paper passing through the point Q. Which of the following options is/are correct?



(1) If the force is applied normal to the circumference at point X then τ is constant

(2) If the force is applied tangentially at point S then $\tau \neq 0$ but the wheel never climbs the step

(3) If the force is applied normal to the circumference at point P then τ is zero

(4) If the force is applied at point P tangentially then τ decreases continuously as the wheel climbs

(JEE Advanced 2017)

Linked Comprehension Type

For Problems 1–2

One twirls a circular ring (of mass M and radius R) near the tip of one's finger as shown in Figure 1. In the process the finger never loses contact with the inner rim of the ring. The finger traces out the surface of a cone, shown by the dotted line. The radius of the path traced out by the point where the ring and the finger is in contact is r. The finger rotates with an angular velocity ω_0. The rotating ring rolls without slipping on the outside of a smaller circle described by the point where the ring and the finger is in contact (Figure 2). The coefficient of friction between the ring and the finger is μ and the acceleration due to gravity is g.

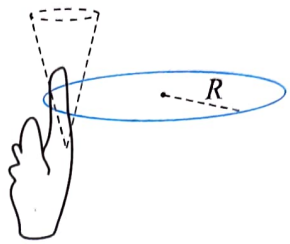


Figure 1

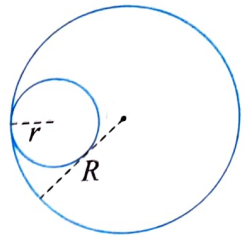


Figure 2

(JEE Advanced 2017)

1. The total kinetic energy of the ring is:

(1) $M\omega_0^2R^2$ (2) $M\omega_0^2(R-r)^2$

(3) $\dfrac{1}{2}M\omega_0^2(R-r)^2$ (4) $\dfrac{3}{2}M\omega_0^2(R-r)^2$

2. The minimum value of ω_0 below which the ring will drop down is:

(1) $\sqrt{\dfrac{3g}{2\mu(R-r)}}$ (2) $\sqrt{\dfrac{g}{\mu(R-r)}}$

(3) $\sqrt{\dfrac{2g}{\mu(R-r)}}$ (4) $\sqrt{\dfrac{2g}{2\mu(R-r)}}$

Matrix Match Type

1. In the Column I below, four different paths of a particle are given as functions of time. In these functions, α and β are positive constants of appropriate dimensions and $\alpha \neq \beta$.

In each case, the force acting on the particle is either zero or conservative. In Column II, five physical quantities of the particle are mentioned; $\vec{p}$ is the linear momentum $\vec{L}$ is the angular momentum about the origin, K is the kinetic energy, U is the potential energy and E is the total energy.

Match each path in Column I with those quantities in Column II, which are conserved for that path.

(JEE Advanced 2018)

Column I		Column II	
i.	$\vec{r}(t)=\alpha t\hat{i}+\beta t\hat{j}$	a.	$\vec{p}$
ii.	$\vec{r}(t)=\alpha\cos\omega t\hat{i}+\beta\sin\omega t\,\hat{j}$	b.	$\vec{L}$
iii.	$\vec{r}(t)=\alpha\left(\cos\omega t\hat{i}+\sin\omega t\,\hat{j}\right)$	c.	K
iv.	$\vec{r}(t)=\alpha t\hat{i}+\dfrac{\beta}{2}t^2\,\hat{j}$	d.	U
		e.	E

(1) i. → a, b, c, d, e; ii. → b, e; iii. → b, c, d, e; iv. → e.

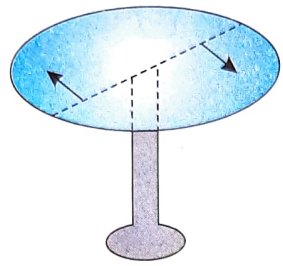
(2) i. → a, b, c, d, e; ii. → c, e; iii. → b, c, d, e; iv. → b, e.

(3) i. → b, c, d; ii. → e; iii. → a, b, d; iv. → b, e.

(4) i. → a, b, c, e; ii. → b, e; iii. → b, c, d, e; iv. → b, e.

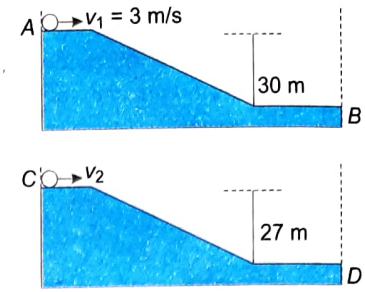
Numerical Value Type

1. A horizontal circular platform of radius 0.5 m and mass 0.45 kg is free to rotate about its axis. Two massless spring toy-guns, each carrying a steel ball of mass 0.05 kg are attached to the platform at a distance 0.25 m from the centre on its either sides along its diameter (see figure). Each gun simultaneously fires the balls horizontally and perpendicular to the diameter in opposite directions. After leaving the platform, the balls have horizontal speed of 9 ms^{-1} with respect to the ground. The rotational speed of the platform in rad s^{-1} after the balls leave the platform is



(JEE Advanced 2014)

2. Two identical uniform discs roll without slipping on two different surfaces AB and CD (see figure) starting at A and C with linear speeds v_1 and v_2, respectively, and always remain in contact with the surfaces. If they reach B and D with the same linear speed and $v_1 = 3$ m/s, then v_2 in m/s is _____. ($g = 10$ m/s^2)



(JEE Advanced 2015)

3. The densities of two solid spheres A and B of the same radii R vary with radial distance r as $\rho_A(r) = k\left(\frac{r}{R}\right)$ and $\rho_B(r) = k\left(\frac{r}{R}\right)^5$, respectively, where k is a constant. The moments of inertia of the individual spheres about axes passing through their centres are I_A and I_B, respectively, If $\frac{I_B}{I_A} = \frac{n}{10}$, the value of n is:

(JEE Advanced 2015)

4. A ring and a disc are initially at rest, side by side, at the top of an inclined plane which makes an angle 60° with the horizontal. They start to roll without slipping at the same instant of time along the shortest path. If the time difference between their reaching the ground is $\frac{(2-\sqrt{3})}{\sqrt{10}}$ s, then the height of the top of the inclined plane, in meters, is _____. Take $g = 10$ ms^{-2}.

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)	**2.** (4)	**3.** (1)	**4.** (2)	**5.** (2)
6. (1)	**7.** (3)	**8.** (4)	**9.** (2)	**10.** (4)
11. (4)	**12.** (4)	**13.** (2)	**14.** (2)	**15.** (1)
16. (1)	**17.** (2)	**18.** (4)	**19.** (4)	**20.** (3)
21. (4)	**22.** (2)	**23.** (4)	**24.** (3)	**25.** (2)
26. (4)	**27.** (3)	**28.** (2)	**29.** (3)	**30.** (1)
31. (4)	**32.** (3)	**33.** (1)	**34.** (3)	**35.** (4)
36. (1)	**37.** (4)	**38.** (2)	**39.** (3)	**40.** (4)
41. (1)	**42.** (3)	**43.** (4)	**44.** (4)	**45.** (2)
46. (2)	**47.** (2)	**48.** (2)	**49.** (2)	**50.** (1)
51. (4)	**52.** (2)	**53.** (1)	**54.** (2)	**55.** (4)
56. (3)	**57.** (2)	**58.** (2)	**59.** (2)	**60.** (3)
61. (1)	**62.** (2)	**63.** (4)	**64.** (1)	**65.** (3)
66. (2)	**67.** (2)	**68.** (3)	**69.** (3)	**70.** (1)
71. (4)	**72.** (2)	**73.** (4)	**74.** (2)	**75.** (3)
76. (1)	**77.** (3)	**78.** (3)	**79.** (1)	**80.** (3)
81. (2)	**82.** (2)	**83.** (1)	**84.** (1)	**85.** (4)
86. (1)	**87.** (1)	**88.** (2)		

Multiple Correct Answers Type

1. (1),(2),(3),(4)	**2.** (3),(4)	**3.** (3),(4)
4. (2),(4)	**5.** (1),(2)	**6.** (1),(4)
7. (1),(4)	**8.** (1),(2),(4)	**9.** (2),(4)
10. (1),(2),(3)	**11.** (1),(2),(4)	**12.** (2),(3),(4)
13. (3),(4)	**14.** (1),(3),(4)	**15.** (1),(2),(3),(4)
16. (1),(3),(4)	**17.** (1),(2),(3)	**18.** (1),(2),(3),(4)
19. (1),(2),(4)	**20.** (2),(3),(4)	**21.** (1),(3)
22. (2),(3),(4)	**23.** (1),(4)	**24.** (2),(4)
25. (2),(3),(4)	**26.** (1),(4)	**27.** (1),(4)

Linked Comprehension Type

1. (1)	**2.** (3)	**3.** (3)	**4.** (2)	**5.** (2)
6. (1)	**7.** (4)	**8.** (2)	**9.** (3)	**10.** (4)
11. (1)	**12.** (4)	**13.** (3)	**14.** (4)	**15.** (4)
16. (1)	**17.** (2)	**18.** (1)	**19.** (2)	**20.** (1)
21. (3)	**22.** (4)	**23.** (4)	**24.** (4)	**25.** (4)
26. (2)	**27.** (1)	**28.** (1)	**29.** (3)	**30.** (2)
31. (3)	**32.** (4)	**33.** (1)	**34.** (2)	**35.** (2)
36. (1)	**37.** (4)	**38.** (3)	**39.** (1)	**40.** (2)
41. (1)	**42.** (3)	**43.** (2)	**44.** (1)	

Matrix Match Type

1. i. → b., d.; ii. → a.; iii → a., b., c., d.; iv. → b., d.
2. i. → c.; ii. → c.; iii, → a., b., c.; iv. → a., b., c.
3. i. → d.; ii. → b., c.; iii. → d.; iv. → a., c.
4. i. → a.; ii. → b., d.; iii. → a.; iv. → b., d.
5. i → a., b., c.; ii → a., b., c.; iii → a., b.; iv. → a., b., c.
6. i. → c.; ii. → a., b., c., d., e.; iii. → a., b., c., d., e.; iv. → e.
7. i. → a., b., c.; ii. → a.; iii. → d.; iv. → d.
8. i. → a., b., d.; ii. → a., b., c.; iii. → a., b., d.; iv. → a., b., c.
9. i → b., c.; ii → a., c.; iii → c.
10. i. → a., b., d.; ii. → a., b.; iii. → a., b., c.; iv. → a., b.
11. i. → a.; ii. → a.; iii. → a.; iv. → c.
12. i. → a., c., d.; ii. → b., c., d.; iii. → b.; iv. → b., c., d.
13. i. → a., b.; ii. → b., c.; iii. → a., b.; iv. → a., b.
14. i. → d.; ii. → b., d.; iii. → a., b., c.; iv. → b., c., d.
15. i. → a., b., c., d.; ii. → a., d.; iii. → a., b., c., d.; iv. → b., c., d.
16. i. → b., c.; ii. → a., b., c.; iii. → a., d.; iv. → a., d.
17. (2)

Numerical Value Type

1. (2) **2.** (0) **3.** (6) **4.** (5) **5.** (2)
6. (4) **7.** (1) **8.** (7) **9.** (6) **10.** (5)
11. (20) **12.** (2) **13.** (4.00) **14.** (5.00) **15.** (2.00)
16. (20.00) **17.** (5.00) **18.** (1.20) **19.** (3.00)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (4) **2.** (3) **3.** (4) **4.** (2) **5.** (1)
6. (2),(4) **7.** (1)

JEE Advanced

Single Correct Answer Type

1. (1) **2.** (3) **3.** (4) **4.** (4) **5.** (3)

Multiple Correct Answers Type

1. (1),(3) **2.** (1),(2),(4) **3.** (2),(4)
4. (2),(3)

Linked Comprehension Type

1. (2) **2.** (2)

Matrix Match Type

1. (1)

Numerical Value Type

1. (4) **2.** (7) **3.** (6) **4.** (0.75)

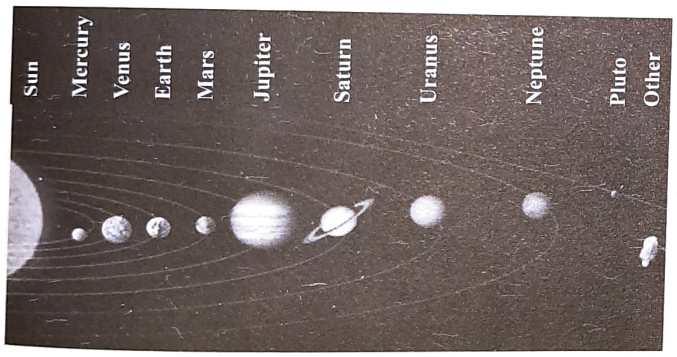
5 Gravitation

INTRODUCTION

Gravitation is the name given to the force of attraction between any two bodies of the universe. It was discovered by Newton in 1665, when he saw an apple falling from the tree. Gravitational force is the weakest force among the four fundamental forces of nature but is the most important force because it played an important role in initiating the birth of stars and controlling the entire structure of the universe. The gravitational force has significant applications in the present advancement of physics.

KEPLER'S LAWS OF PLANETARY MOTION

Johannes Kepler used empirical observations, and sophisticated calculations to arrive at the famous **Kepler's laws of planetary motion**, published in 1609 and 1619. What is particularly significant about Kepler's laws is that they challenged the prevailing world's view, with the Earth in the center of the universe (a geocentric theory) and the Sun and all the planets and stars orbiting around it, just as the Moon does.



Kepler and other pioneers, particularly Nicolaus Copernicus and Galileo Galilei, changed this geocentric view into a heliocentric (Sun-centred) cosmology. The planetary system of the star Sun called solar system consists of nine planets, viz., Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto. Out of these planets Mercury is the smallest and closest to the sun and so hottest. Jupiter is largest and has maximum moons (12). Venus is closest to Earth and brightest. Kepler's complete analysis of planetary motion is summarized in three statements known as **Kepler's laws**.

KEPLER'S FIRST LAW: THE LAW OF ORBITS

Every planet moves around the sun in an elliptical orbit with sun at one of the foci. Kepler's first law says that the circular orbit is a very special case and elliptical orbits are the general situation.

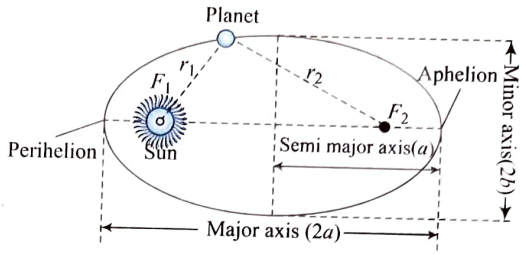


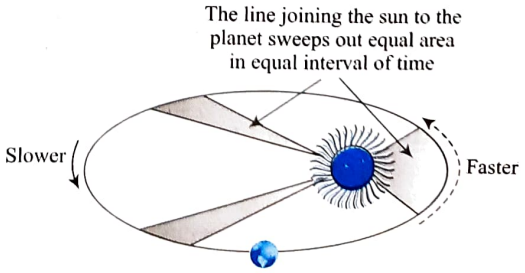
Figure shows the geometry of an ellipse, which serves as our model for the elliptical orbit of a planet. An ellipse is mathematically defined by choosing two points F_1 and F_2, each of which is a called a **focus**, and then drawing a curve through points for which the sum of the distances r_1 and r_2 from F_1 and F_2, respectively, is a constant. The longest distance through the centre between points on the ellipse (and passing through each focus) is called the **major axis**. In figure, the major axis is drawn along the x direction.

Similarly, the shortest distance through the centre between points on the ellipse is called the **minor axis**. In the elliptical orbit of a planet around the Sun, the Sun is at one focus of the ellipse. There is nothing at the other focus.

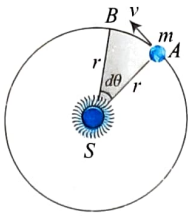
Now imagine a planet in an elliptical orbit as shown in figure, with the Sun at focus F_1. When the planet is at the far right in the diagram, the distance between the planet and the Sun at this point, called the **aphelion**, the planet is at its maximum distance from the Sun. (For an object in orbit around the Earth, this point is called the **apogee**.) Conversely, when the planet is at the left end of the ellipse, the distance between the planet and the Sun at this point, called the **perihelion** (for an Earth orbit, the **perigee**), the planet is at its minimum distance from the Sun.

KEPLER'S SECOND LAW: THE LAW OF AREA

The line joining the sun to the planet sweeps out equal areas in equal interval of time, i.e., areal velocity is constant. According to this law, planet will move slowly when it is farthest from sun and move rapidly when it is nearest to sun. It is similar to law of conservation of angular momentum.



To prove Kepler's second law: Consider that a planet of mass m is revolving around the sun of mass M in a circular orbit of radius r. Let v be its orbital velocity. Suppose that at any instant, the planet is at point A in its orbit and after an infinitesimally small time dt, it reaches point B (see figure). As such, the circular path of the planet between points A and B may be considered as straight.



If dA is small area (shown shaded) swept by the line joining the planet to the sun in time dt, then

$$dA = \text{area of } \Delta ABS = \frac{1}{2}\text{base} \times \text{perpendicular}$$

or $$dA = \frac{1}{2} AB \times AS \quad \ldots(i)$$

If $d\theta$ is the angular displacement of the planet in time dt i.e., when it moves from point A to B, then $AB = r\,d\theta$. Substituting for AB $(= r\,d\theta)$ and AS $(= r)$ in Eq. (i), we have

$$dA = \frac{1}{2} r\,d\theta \times r = \frac{1}{2} r^2\,d\theta \quad \ldots(ii)$$

Dividing both sides of the Eq. (ii) by dt, we have

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

or $$\frac{dA}{dt} = \frac{1}{2} r^2 \omega \quad \ldots(iii)$$

where $\omega = \dfrac{d\theta}{dt}$ is the angular speed of the planet in its orbit and dA/dt is the areal velocity of the planet.

Multiplying and dividing RHS of Eq. (iii) by m (the mass of the planet), we have

$$\frac{dA}{dt} = \frac{1}{2m} mr^2\omega$$

Since $mr^2\omega = L$, the angular momentum of the planet about the axis through the sun, we have

$$\frac{dA}{dt} = \frac{1}{2m} mr^2\omega \quad \ldots(iv)$$

As no external torque acts on the planet during its orbital motion, its angular momentum (L) must remain constant. Since both L and m are constant quantities, the Eq. (iii) becomes

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

Hence, when a planet moves around the sun, its areal velocity remains constant. It proves Kepler's second law of planetary motion.

Kepler's Third Law: The Law of Periods

The square of period of revolution (T) of any planet around sun is directly proportional to the cube of the semi-major axis of the orbit. Because the semimajor axis of a circular orbit is its radius, this equation is valid for both circular and elliptical orbits.

$$T^2 \propto a^3 \text{ or } T^2 = Ka^3$$

where a = semi-major axis.

Notice that the constant of proportionality K is independent of the mass of the planet. Equation (i) is therefore valid for any planet. If we were to consider the orbit of a satellite such as the Moon about the Earth, the constant would have a different value.

Illustration 5.1

The time period of Jupiter is 11.6 years. How far is Jupiter from the Sun? Distance of the Earth from the Sun is 1.5×10^{11} m.

Sol. Here $T_J = 11.6$ years; $r_J = ?$

$T_e = 1$ year; $r_e = 1.5 \times 10^{11}$ m

As $$\frac{T_J^2}{T_e^2} = \frac{r_J^3}{r_e^3} \text{ or } r_J = r_e \left(\frac{T_J}{T_e}\right)^{2/3}$$

$$r_J = 1.5 \times 10^{11} \left(\frac{11.6}{1}\right)^{2/3} = 7.68 \times 10^{11} \text{ m}$$

Illustration 5.2

The mean distance of Mars from the Sun is 1.524 times that of the Earth from the Sun. Find the number of years required for Mars to make one revolution about the Sun.

Sol. For planets revolving around the Sun, $T^2 \propto r^3$

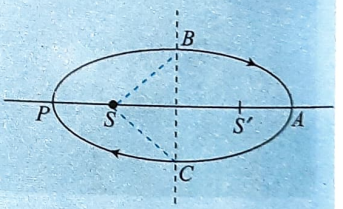
$\Rightarrow$ $\dfrac{T_1^2}{T_2^2} = \dfrac{r_1^3}{r_3^2}$ (T_1 = time period of Mars, T_2 = time period of the Earth)

$\Rightarrow$ $\dfrac{T_1}{T_2} = \left(\dfrac{r_1}{r_2}\right)^{3/2} \Rightarrow T_1 = T_2 \left(\dfrac{r_1}{r_2}\right)^{3/2} = (1 \text{ year})\,(1.524)^{3/2}$

$= 1.88$ years

Illustration 5.3

Let the speed of the planet at the perihelion P in figure be v_P and the Sun–planet distance SP be r_P. Relate r_P, v_P to the corresponding quantities at the aphelion (r_A, v_A). Will the planet take equal times to traverse BAC and CPB?



Sol. Referring figure, we note that $\vec{r}_P$ and $\vec{v}_P$ are perpendicular to each other. Similarly, $\vec{r}_A$ and $\vec{v}_A$ are perpendicular to each other.

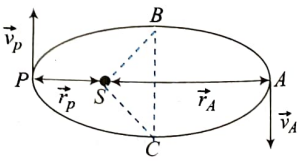
Using the law of conservation of angular momentum,

Angular momentum of the planet at P = Angular momentum of the planet at A

$\Rightarrow$ $m_P v_P r_P = m_A v_A r_A$

or $$\frac{v_P}{v_A} = \frac{r_A}{r_P}$$

Since $r_A > r_P$, $v_P > v_A$

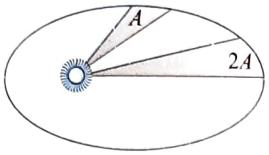


Here area $SBAC$ is greater than area $SCPB$. As the areal velocity of a planet is constant around the Sun, i.e., equal areas are swept

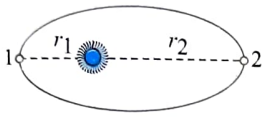
in equal times. Hence the planet will take longer time to traverse *BAC* than *CPB*.

CONCEPT APPLICATION EXECISE 5.1

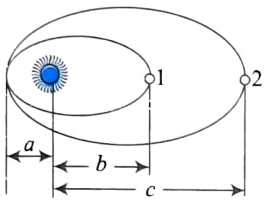
1. A planet sweeps areas A and $2A$ during a small time intervals Δt_1 and Δt_2 respectively. Find $\frac{\Delta t_1}{\Delta t_2}$.



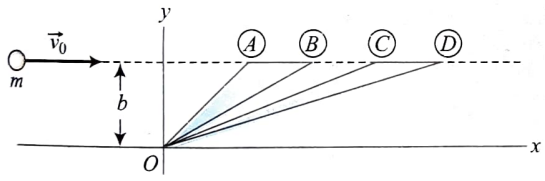
2. Find the ratio of **(a)** velocities **(b)** KE **(c)** areal velocities at 1 and 2.



3. Find the ratio of time period of revolution of the planets 1 and 2.



4. If a planet moving in a circular path having time period of revolution T is suddenly stopped, after what time it will collapse onto the sun?
5. A particle of mass m moves along a straight line with constant velocity $\vec{v}_0$ in the x direction at a distance b from the x axis (figure).



(a) Does the particle possess any angular momentum about the origin?

(b) Explain why the amount of its angular momentum should change or should stay constant.

(c) Show that Kepler's second law is satisfied by showing that the two shaded triangles in the figure have the same area when $t_{(D)} - t_{(C)} = t_{(B)} - t_{(A)}$.

ANSWERS

1. 1/2 **2.** (a) (r_2/r_1) (b) $(r_2/r_1)^2$ (c) 1
3. $((a+b)/(a+c))^{3/2}$ **4.** $T/2^{3/2}$

NEWTON'S LAW OF UNIVERSAL GRAVITATION

Before 1687, a large amount of data had been collected on the motions of the Moon and the planets, but a clear understanding of the forces related to these motions was not available. There was no answer to the questions that people asked about the night sky. Why doesn't the moon fall to earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions. Gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively.

Newton discovered in the 17th century that there must be some force that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of celestial mechanics, the study of the dynamics of objects in space. Newton analyzed astronomical data of the motion of the Moon around the Earth. From that analysis, he made the bold assertion that the force law governing the motion of planets was the same as the force law that attracted a falling apple to the Earth. Newton published his work on the law of gravity in his treatise Mathematical Principles of Natural Philosophy.

Newton's law of universal gravitation states that 'every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.'

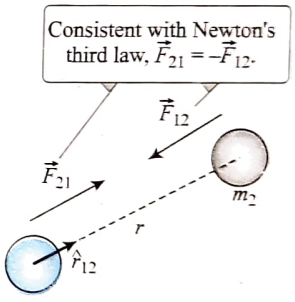
Thus, the magnitude of the gravitational force F that two particles of masses m_1 and m_2, separated by a distance r, exert on each other is given by

$$F_g = G\frac{m_1 m_2}{r^2} \qquad \text{...(i)}$$

where G is a constant, called the *universal gravitational constant.* Its value in SI units is

$$G = 6.674\times10^{-11}\ \text{Nm}^2/\text{kg}^2.$$

Vector form: We can express this force in vector form by defining a unit vector $\hat{r}_{21}$ (figure). Because this unit vector is directed from particle 1 toward particle 2, the force exerted by particle 1 on particle 2 is



$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{21} = \frac{-Gm_1 m_2}{r^3}\vec{r}_{21} = \frac{-Gm_1 m_2}{|\vec{r}_{21}|^3}\vec{r}_{21} \qquad \text{...(ii)}$$

where the negative sign indicates that particle 2 is attracted to particle 1, hence, the force on particle 2 must be directed toward particle 1. By Newton's third law, the force exerted by particle 2 on particle 1, designated $\vec{F}_{21}$, is equal in magnitude to $\vec{F}_{12}$ and acts in the opposite direction. That is, these forces form an action-reaction pair, and $\vec{F}_{21} = -\vec{F}_{12}$.

Important Points:

- The law of universal gravitation was not expressed by Newton in the form of Eq. (i), and Newton did not mention a constant such as G. The universal gravitational constant G was first evaluated in the late nineteenth century. The value of G in the laboratory was first determined by Cavendish using the torsional balance.
- The form of the force law given by Eq. (i) is often referred to as an **inverse-square law** because the magnitude of the

force varies as the inverse square of the separation of the particles.

- The law of gravitation is stated for two point masses, therefore for any two arbitrary finite size bodies, as shown in the figure, It cannot be applied as there is no unique value for the separation.

But if the two bodies are uniform spheres then the separation r may be taken as the distance between their centres because a sphere of uniform mass behave as a point mass for any point lying outside it.

GRAVITY AND GRAVITATION

Gravitation is the force of attraction acting between any two bodies of the universe. Gravity is the earth's gravitational pull on the body lying on or near the surface of earth. It is the force of attraction exerted by earth towards its centre on a body lying on or near the surface of earth. It is merely a special case of gravitation and is also called earth's gravitational pull.

For illustration, the force of attraction between two bodies on the surface of earth is the force of gravitation, but the force of attraction between each body and the earth is the force of gravity.

Force of gravity acting on a body is the measure of weight of the body. Thus, weight of a body is defined as the force of attraction exerted by the earth on the body towards its centre.

If m is the mass of the body placed on the surface of earth, where acceleration due to gravity is g, then

Gravity pull = Weight of body = mg

The unit and dimensions of gravitational pull or weight are the same as those of force.

- **Be clear on g and G:** The symbol g represents the magnitude of the free-fall acceleration near a planet. At the surface of the Earth, g has an average value of 9.8 m/s^2. On the other hand, G is a universal constant that has the same value everywhere in the Universe.
- The value of G does not depend upon the nature and size of the bodies. It also does not depend upon the nature of the medium between the two bodies.
- As G is very small, hence gravitational forces are very small, unless one (or both) of the mass is huge.

ILLUSTRATION 5.4

Gravitational force between two point masses m and M separated by a distance r is F. Now if a point mass $3m$ is placed next to m, what will be the **(a)** force on M due to m, **(b)** total force on M?

Sol.

(a) If r is the distance between two point masses m and M, then the gravitational force on m due to mass M is $F = GMm/r^2$.

Since the gravitational force between two point masses is independent of the presence of other masses, so if a point mass $3m$ is placed next to m, the force on M due to m is

$$F = \frac{GMm}{r^2}.$$

(b) Total force on the body of mass M is F, i.e.,

$$F = \frac{GM \times (m + 3m)}{r^2} = \frac{4GMm}{r^2} = 4F$$

ILLUSTRATION 5.5

The mass of planet Jupiter is 1.9×10^{27} kg and that of the Sun is 1.99×10^{30} kg. The mean distance of Jupiter from the Sun is 7.8×10^{11} m. Calculate the gravitational force which Sun exerts on Jupiter. Assuming that Jupiter moves in circular orbit around the Sun, also calculate the speed of Jupiter. $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}.

Sol. Here, $M_J = 1.9 \times 10^{27}$ kg

$M_S = 1.99 \times 10^{30}$ kg ; $r = 7.8 \times 10^{11}$ m

$G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2} ; $F = ?$

Now $$F = \frac{GM_JM_s}{r^2} = \frac{6.67 \times 10^{-11} \times 1.9 \times 10^{27} \times 1.99 \times 10^{30}}{(7.8 \times 10^{11})^2}$$

$$= 4.15 \times 10^{23} \text{ N}$$

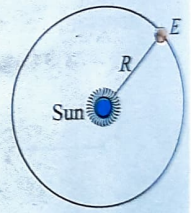
Since the gravitational pull of the Sun on Jupiter provides the required centripetal force to Jupiter, therefore the velocity of Jupiter v can be given by the relation,

$$F = \frac{M_Jv^2}{r}$$

or $$v = \sqrt{\frac{Fr}{M_J}} = \sqrt{\frac{4.15 \times 10^{23} \times 7.8 \times 10^{11}}{1.9 \times 10^{27}}} = 1.3 \times 10^4 \text{ ms}^{-1}$$

ILLUSTRATION 5.6

The orbit of the Earth, mass 6×10^{24} kg, may be assumed to be a circle of radius 1.5×10^{11} m with the Sun at its centre as shown in figure. The time taken for one orbit is 3.2×10^7 s.



(a) Calculate:

(i) the magnitude of the angular velocity of the Earth about the Sun and

(ii) the magnitude of the centripetal force acting on the Earth.

(b) (i) State the origin of the centripetal force calculated in part (a) (ii).

(ii) Determine the mass of the Sun.

Sol. Here, mass of the Earth, $m = 6 \times 10^{24}$ kg,

Radius of the orbit of the Earth, $r = 1.5 \times 10^{11}$ m,

Time period of the earth, $T = 3.2 \times 10^7$ s

(a) (i) Angular velocity of the Earth about the Sun,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.2 \times 10^7} = 1.96 \times 10^{-7} \text{ rad s}^{-1}$$

(ii) The centripetal force acting on the Earth,

$F = m r \omega^2$

$= 6 \times 10^{24} \times 1.5 \times 10^{11} \times (1.96 \times 10^{-7})^2$

$= 3.46 \times 10^{22}$ N

(b) (i) The gravitational pull of the Sun provides the necessary centripetal force to the Earth.

(ii) Let M be the mass of the Sun. Then, $\frac{GMm}{r^2} = F$

$$\text{or } M = \frac{Fr^2}{Gm} = \frac{3.46\times10^{22}\times(1.5\times10^{11})^2}{6.67\times10^{-11}\times6\times10^{24}}$$

$$= 1.95 \times 10^{30} \text{ kg}$$

ILLUSTRATION 5.7

For the sake of simplicity, assume that the planets revolve around the sun in circular orbits. Prove Kepler's third law from Newton laws of gravitation.

Sol. The centripetal force required for making the planet to revolve in the circular orbit is provided by the gravitational pull of the sun i.e.,

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \text{ or } v = \sqrt{\frac{GM}{r}} \quad ...(i)$$

Now, the period of revolution of the planet around the sun,

$$T = \frac{\text{circumference of the orbit}}{\text{orbital velocity}} = \frac{2\pi r}{\sqrt{GM/r}}$$

or $$T^2 = \frac{4\pi^2}{GM} r^3 \quad ...(ii)$$

Since $\frac{4\pi^2}{GM}$ is a constant factor,

$T^2 = \text{constant} \times r^3$ or $T^2 \propto r^3$

Hence, the square of the period of revolution of a planet around the sun is directly proportional to the cube of the radius of is circular orbit. It proves Kepler's third law of planetary motion.

GRAVITATION AND THE PRINCIPLE OF SUPERPOSITION

If a group of particles are given, we find the net (or resultant) gravitational force on any one of them from the others by using the principle of superposition. This is a general principle that says a net effect is the sum of the individual effects. Here, the principle means that we first compute the individual gravitational forces that act on our selected particle due to each of the other particles. We then find the net force by adding these forces vectorially

$$\vec{F}_{1,\text{net}} = \vec{F}_{1,2} + \vec{F}_{1,3} + \vec{F}_{1,4} \cdots + \vec{F}_{1,n} \quad ...(i)$$

Here $\vec{F}_{1,\text{net}}$ is the net force on particle 1 due to the other particles and, for example, $\vec{F}_{1,2}$ is the force on particle 1 from particle 2.We can express this equation more compactly as a vector sum:

$$\vec{F}_{1,\text{net}} = \sum_{i=2}^{i=n} \vec{F}_{1,i} \quad ...(ii)$$

ILLUSTRATION 5.8

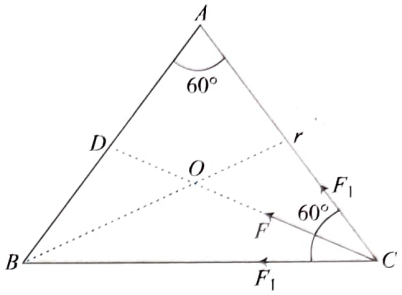
Three identical bodies, each of mass m, are located at the vertices of an equilateral triangle with side r. At what speed must they move if they all revolve under the influence of one another's gravitation in a circular orbit circumscribing the triangle while still preserving the equilateral triangle?

Sol. Refer to figure, force of attraction on body at C due to body at A is

$$F_1 = \frac{Gmm}{r^2} = \frac{Gm^2}{r^2} \text{ along } CA$$

Force of attraction on body at C due to body at B is

$$F_2 = \frac{Gmm}{r^2} = \frac{Gm^2}{r^2} \text{ along } CB$$



These forces, $\vec{F}_1$ and $\vec{F}_2$, are inclined at an angle 60°. The resultant force on the body at C is

$$F = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos 60°}$$

$$= \sqrt{F_1^2 + F_2^2 + 2F_1F_2^{(1/2)}}$$

$$= \sqrt{3}F_1 = \sqrt{3}\frac{Gm^2}{r^2} \text{ acting along } CD$$

$$\left[\because F_1 = F_2 = \frac{Gm^2}{r^2}\right]$$

Here, $OC = \frac{2}{3}CD = \frac{2}{3}AC\sin 60°$

$$= \frac{2}{3}r \times \frac{\sqrt{3}}{2} = \frac{r}{\sqrt{3}}$$

When each body is describing a circular orbit with centre of orbit at O, the force F provides the required centripetal force. The radius of the circular orbit is $OC = r/\sqrt{3}$. If v is the speed of the body in circular orbit, then

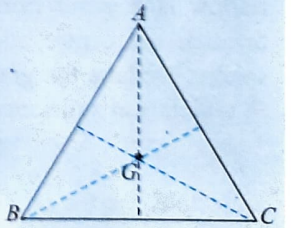
$$\frac{Mv^2}{r/\sqrt{3}} = \frac{\sqrt{3}Gm^2}{r^2} \text{ or } v = \sqrt{\frac{Gm}{r}}$$

ILLUSTRATION 5.9

Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.

(a) What is the force acting on a mass $2m$ placed at the centroid G of the triangle?

(b) What is the force if the mass at the vertex A is doubled? Take $AG = BG = CG = 1$ m.



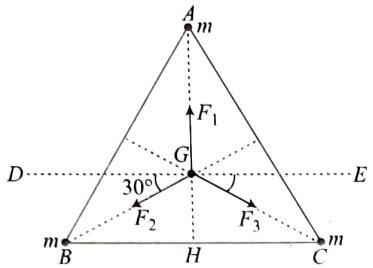
Sol.

(a) From figure, the gravitational force on mass $2m$ at G due to mass at A is

$$F_1 = G\frac{m\times 2m}{1^2} = 2Gm^2 \text{ along } GA$$

Gravitational force on mass $2m$ at G due to mass at B is

$$F_2 = G\frac{m\times 2m}{1^2} = 2Gm^2 \text{ along } BG$$



Gravitational force on mass $2m$ at G due to mass at C is

$$F_3 = G\frac{m \times 2m}{1^2} = 2Gm^2 \text{ along } GC$$

Draw DE parallel to BC passing through point G. Then $\angle EGC = 30° = \angle DGB$.

Resolving $\vec{F}_2$ and $\vec{F}_3$ into two rectangular components, we have

$F_2 \cos 30°$ along GD and $F_2 \sin 30°$ along GH

$F_3 \cos 30°$ along GE and $F_3 \sin 30°$ along GH

Here, $F_2 \cos 30°$ and $F_3 \sin 30°$ are equal in magnitude and acting in opposite directions, and cancel out each other. The resultant force on mass $2m$ at G is

$$F_1 - (F_2 \sin 30° + F_3 \sin 30°)$$
$$= 2Gm^2 - \left(2Gm^2 \times \frac{1}{2} + 2Gm^2 \times \frac{1}{2}\right)$$
$$= 0$$

(b) When mass at A is $2m$, then gravitational force on mass $2m$ at G due to mass $2m$ at A is

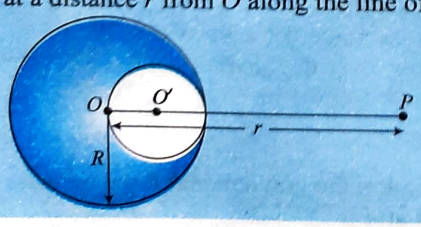
$$F_1' = G\frac{2m \times 2m}{1^2} = 4Gm^2 \text{ along } GA$$

The resultant force on mass $2m$ at G due to masses at A, B and C is

$$F_1' - (F_2 \sin 30° + F_3 \sin 30°)$$
$$= 4Gm^2 - \left(2Gm^2 \times \frac{1}{2} + 2Gm^2 \times \frac{1}{2}\right)$$
$$= 2Gm^2 \text{ along } GA$$

ILLUSTRATION 5.10

A solid sphere of lead has mass M and radius R. A spherical hollow is dug out from it (see figure). Its boundary passing through the centre also touches the boundary of the solid sphere. Deduce the gravitational force on a mass m placed at P, which is at a distance r from O along the line of centres.



Sol. Let O be the centre of the solid sphere and O' that of the hollow. For an external point the sphere behaves as if its entire mass is concentrated at its centre. Therefore, the gravitational force on a mass 'm' at P due to the original sphere (of mass M) is

$$F = G\frac{Mm}{r^2}, \text{ along } PO$$

The diameter of the smaller sphere (which would be cut off) is R, so that its radius OO' is $R/2$. The force on m at P due to this sphere of mass M' (say) would be

$$F' = G\frac{M'm}{(r - \frac{R}{2})^2} \text{ along } PO' \qquad (\because \text{ distance } PO' = r - R/2)$$

As the radius of this sphere is half of that of the original sphere, we have $M' = \frac{m}{8}$

Therefore, $F' = G\frac{Mm}{8(r - R/2)^2}$ along PO'

As both F and F' point along the same direction, the force due to the hollowed sphere is

$$F - F' = \frac{GMm}{r^2} - \frac{GMm}{8r^2(1 - \frac{R}{2r})^2}$$

$$= \frac{GMm}{r^2}\left[1 - \frac{1}{8(1 - \frac{R}{2r})^2}\right]$$

ILLUSTRATION 5.11

Approximate the gravitational force exerted on the moon by the earth and the sun:
(a) during a solar eclipse and
(b) during a lunar eclipse.
Given that mass of the sun $M_s = 2 \times 10^{30}$ kg, mass of the earth $M_e = 6 \times 10^{24}$ kg, mass of the moon $M_m = 7.4 \times 10^{22}$ kg, radius of the earth's orbit around the sun $r_{es} = 1.5 \times 10^{11}$ m and radius of the moon's orbit around the earth $r_{em} = 3.8 \times 10^8$ m and $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

Sol. Force exerted by the earth on the moon,

$$F_{em} = \frac{GM_eM_m}{r_{em}^2} = \frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 7.4 \times 10^{22}}{(3.8 \times 10^8)^2}$$

$$= 2.05 \times 10^{20} \text{ N}$$

Force exerted by the sun on the moon,

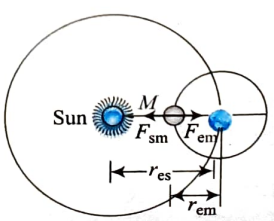
$$F_{sm} = \frac{GM_sM_m}{r_{sm}^2} = \frac{GM_sM_m}{(r_{es} \pm r_{em})^2}$$

The –ve sign is to be taken, when the moon comes between the sun and the earth (during a solar eclipse) and the + ve sign, when the earth is between the sun and the moon (during a lunar eclipse). Since $r_{em} << r_{es}$, $r_{es} \pm r_{em} \approx r_{es}$.

$$\therefore \quad F_{sm} = \frac{GM_sM_m}{r_{es}^2} = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30} \times 7.4 \times 10^{22}}{(1.5 \times 10^{11})^2}$$

$$= 4.39 \times 10^{20} \text{ N}$$

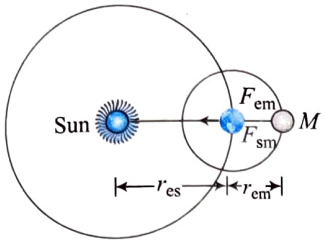
(a) During a solar eclipse: When the solar eclipse takes place (see figure), the forces exerted by the sun and the earth on the moon are in opposite directions.



Therefore, net force exerted on the moon by the earth and the sun during a solar eclipse,

$$F_{solar} = F_{sm} - F_{em} = 4.39\times10^{20} - 2.05\times10^{20}$$
$$= 2.34 \times 10^{20}\ \text{N} \qquad \text{(towards the sun)}$$

(b) **During a lunar eclipse:** When the lunar eclipse takes place (see figure), the forces exerted by the sun and the earth on the moon are in the same direction.



Therefore, net force exerted on the moon by the earth and the sun during a solar eclipse,

$$F_{lunar} = F_{sm} + F_{em} = 4.39\times10^{20} + 2.05\times10^{20}$$
$$= 6.44 \times 10^{20}\ \text{N} \qquad \text{(towards the sun)}$$

GRAVITATIONAL FORCE BETWEEN AN EXTENDED OBJECT AND A PARTICLE

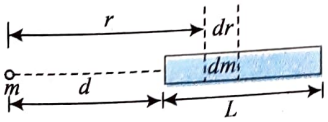
It should be clearly understood that the law of universal gravitation given by equation is valid only if the interacting bodies are considered as particles. What about the gravitational force on a particle from a real (extended) object? This force is found by dividing the object into parts small enough to treat as particles. Here we can divide the extended object into differential parts each of mass dm and each producing a differential force dF on the particle. In this case the vector sum of the forces becomes an integral and the integral is taken over the entire extended object and we drop the subscript "net."

$$\vec{F}_{1,net} = \int dF \qquad ...(i)$$

ILLUSTRATION 5.12

What is the magnitude of the gravitational force on the particle due to the rod?

Sol. Here we need to calculate the gravitational force on the particle due to the rod. We can divide the rod into parts small differential parts each of mass dm. The force between the point mass m and elemental mass dm of the rod at distance r,



$$dF = \frac{G(m).(dm)}{r^2} = \frac{G(m)\left(\frac{M}{L}\right)dr}{r^2}$$

Or $$F = \int_d^{L+d} \frac{GMm}{L}\cdot\frac{1}{r^2}dr$$

$$\Rightarrow \quad F = \frac{GMm}{L}\left(-\frac{1}{r}\right)\Big|_d^{L+d} = \frac{GMm}{d(L+d)}$$

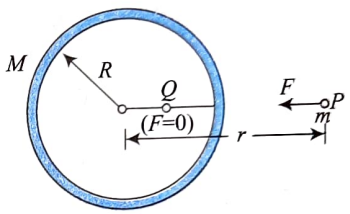
GRAVITATIONAL FORCE BETWEEN A UNIFORM SPHERE OR SPHERICAL SHELL AND A PARTICLE

If the extended object is a uniform sphere or a spherical shell, we can avoid the integration by assuming that the object's mass is concentrated at the object's center and treating it as point mass. The following two results, obtained in this manner, give us the force on a particle when the extended object is (i) a spherical shell and (ii) a solid sphere.

Spherical Shell

Case I: If a particle of mass m is located at P, a point outside a spherical shell of mass M, the shell attracts the particle as though the mass of the shell were concentrated at its centre, i.e.,

$$F = G\frac{Mm}{r^2}$$



Case II: If the particle of mass m is located at Q, a point inside the shell, the force acting on it is zero, i.e., $F = 0$.

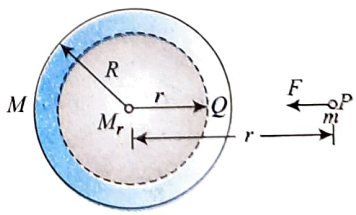
We can express these two important results as follows:

$$F = G\frac{Mm}{r^2} \quad \text{for } r \geq R$$

$$F = 0 \text{ for } r \leq R$$

Solid Sphere

Case I: If a particle of mass m is located at P (figure), a point outside a homogeneous solid sphere of mass M, the sphere attracts the particle as though the mass of the particle were concentrated at its centre, i.e., $F = G\frac{Mm}{r^2}$.



[This result follows from case-I (for a spherical shell), since a solid sphere can be considered as a collection of concentric spherical shells].

Case II: If a particle of mass m is located at Q, a point inside a homogeneous solid sphere of mass M (figure), the force on m is due only to the mass M_r contained within the sphere of radius r (represented by the dashed circle), i.e.,

$$F = G\frac{M_r m}{r^2} = \frac{GMm}{R^3}r$$

Since the sphere is assumed to have uniform density (ρ),

$$\frac{M_r}{M}=\frac{V_r\rho}{V\rho}=\frac{(4\pi/3)r^3\rho}{(4\pi/3)R^3\rho}=\frac{r^3}{R^3} \text{ or } M_r=\frac{Mr^3}{R^3}$$

We can express these two important results as follows:

$$F=G\frac{Mm}{r^2} \text{ for } r\geq R \text{ or } F=\frac{GMm}{R^3}r \text{ for } r\leq R$$

INERTIAL AND GRAVITATIONAL MASSES

The mass of a body is the quantity of matter possessed by a body. There are two different concepts about the mass of body as discussed below.

INERTIAL MASS

Inertial mass of a body is related to its inertia of linear motion and is defined by Newton's second law of motion.

Let a body of mass m_i move with acceleration a under the action of an external force F. According to Newton's second law of motion,

$$F=m_ia \text{ or } m_i=\frac{F}{a} \quad ...(i)$$

The mass m_i of the body in this sense is the inertial mass of the body. If $a=1$, then from Eq. (i) $m_i=F$.

Thus, inertial mass of a body is equal to the magnitude of external force required to produce unit acceleration in the body. In fact, inertial mass of a body is the measure of the ability of the body to oppose the production of acceleration in its motion by an external force.

Properties of Inertial Mass:

- Inertial mass of a body is proportional to the quantity of matter contained in the body.
- Inertial mass of a body is independent of size, shape and state of the body.
- Inertial mass of a body does not depend upon the temperature of the body.
- Inertial mass of a body is not affected by the presence or absence of other bodies near it.
- Inertial mass is conserved when the two bodies combine physically or chemically.
- Inertial mass can be added by simple laws of algebra, irrespective of the materials of the bodies.
- Inertial mass of a body increases with the speed of the body. When a body moves with a velocity v, its inertial mass m is given by $m=\frac{m_0}{\sqrt{1-v^2/c^2}}$ where m_0 is the rest mass of the body, c is the velocity of light in vacuum. The mass m is affected only when the velocity of the body is comparable with the velocity of light.

GRAVITATIONAL MASS

Gravitational mass of a body is related to gravitational pull on the body and is defined by Newton's law of gravitation. If a body of mass m_G is placed on the surface of earth of radius R and mass M, then gravitational pull on the body is given by

$$F=\frac{GMm_G}{R^2}\Rightarrow m_G=\frac{F}{(GM/R^2)}=\frac{F}{I} \quad ...(i)$$

Here I is gravitational field intensity.

The mass m_G of the body in this sense is the gravitational mass of the body. The inertia of the body has no effect on the gravitational mass of the body. If $I=1$ then from Eq. (i), $m_G=F$. Thus, gravitational mass of a body is defined as the magnitude of gravitational pull experienced by the body in a gravitational field of unit intensity.

Comparison of Inertial Mass and Gravitational Mass

1. Both inertial and gravitational masses are measured in the same units.
2. Both are scalar quantities.
3. Both do not depend on the shape and state of the body.
4. Gravitational mass of a body is affected by the presence of other bodies near it whereas the inertial mass of a body remains unaffected by the presence of other bodies near it.
5. The inertial mass of a body is measured by finding its acceleration, produced by the applied force and gravity plays no role in the measurement. On the other hand, the gravitational mass of a body is measured by finding the gravitational pull on it and inertia plays no part in the measurement.
6. The gravitational mass is measured by spring balance whereas inertial mass is measured by inertial balance.
7. Both are equivalent to each other.

EQUIVALENCE OF INERTIAL AND GRAVITATIONAL MASSES

Consider two bodies A and B of gravitational masses m_G and m'_G, respectively. Let these be placed at equal distance R from a third body C (say earth). Let M be the gravitational mass of the body C. According to Newton's law of gravitation, the gravitational force exerted by C on A is given by

$$F=\frac{GMm_G}{R^2}$$

and the gravitational force exerted by C on B is

$$F'=\frac{GMm'_G}{R^2}.$$

Therefore, $$\frac{F}{F'}=\frac{GMm_G/R^2}{GMm'_G/R^2}=\frac{m_G}{m'_G} \quad ...(i)$$

Let m_i and m'_i be the inertial masses of bodies A and B, respectively. If both the bodies are allowed to fall in vacuum from the same place under the action of gravity, where acceleration due to gravity is g, then $F=m_ig$ and $F'=m'_ig$.

Therefore, $$\frac{F}{F'}=\frac{m_ig}{m'_ig}=\frac{m_i}{m'_i} \quad ...(ii)$$

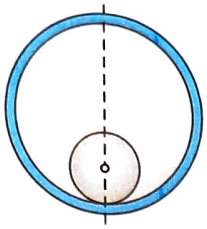
From Eqs. (i) and (ii), we have

$$\frac{m_G}{m'_G}=\frac{m_i}{m'_i}$$

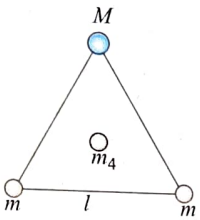
It means that the gravitational mass of a body is proportional to its inertial mass. In fact, inertial and gravitational masses of a body are identical for bodies moving under the action of gravity.

CONCEPT APPLICATION EXERCISE 5.2

1. A solid sphere of mass m and radius r is placed inside a hollow thin spherical shell of mass M and radius R as shown in figure. A particle of mass m' is placed on the line joining the two centres at a distance x from the point of contact of the sphere and the shell. Find the magnitude of the resultant gravitational force on this particle due to the sphere and the shell if **(a)** $r<x<2r$; **(b)** $2r<x<2R$ and **(c)** $x>2R$.



2. As seen in figure, two spheres of mass m and a third sphere of mass M form an equilateral triangle, and a fourth sphere of mass m_4 is at the center of the triangle. The net gravitational force on that central sphere from the three other spheres is zero.

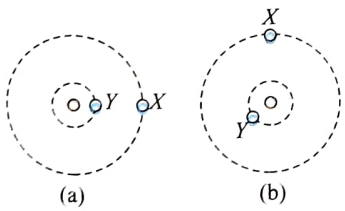


   (a) What is M in terms of m?

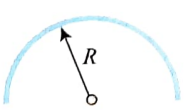
   (b) If we double the value of m_4, what then is the magnitude of the net gravitational force on the central sphere?

3. Three identical particles, each of mass m, are located in space at the vertices of an equilateral triangle of side length a. They are revolving in a circular orbital under mutual gravitational attraction. Find the speed of each particle.

4. Two planets X and Y travel counterclockwise in circular orbits about a star as shown in figure. The radii of their orbits are in the ratio 3:1. At one moment, they are aligned as shown in Fig. (a), making a straight line with the star. During the next five years, the angular displacement of planet X is 90.0° as shown in Fig. (b). What is the angular displacement of planet Y at this moment?



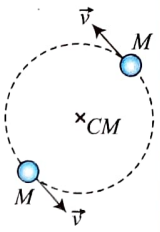
5. A thin rod with mass M is bent in a semicircle of radius R (figure).



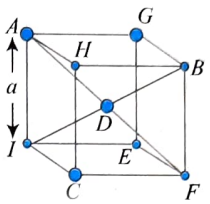
   (a) What is its gravitational force (both magnitude and direction on a particle with mass m at P, the center of curvature?

   (b) What would be the force on the particle if the rod were a complete circle?

6. A binary system consists of two stars that revolve in a circular orbit about a center of mass midway between them. This statement implies that the masses of the two stars are equal (figure). Assume the orbital speed of each star is $|\vec{v}| = 200$ km/s and the orbital period of each is 20 days. Find the mass M of each star.



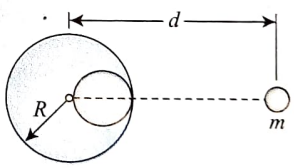
7. Three identical points of mass m each are kept at the vertices A, B, and C of a cube having side length 'a' (see figure). Another identical mass is placed at the center point D of the cube.



   (a) Where will you place a fifth identical mass so that the net gravitational force acting on mass at D becomes zero?

   (b) Calculate the net gravitational force acting on the mass at D.

8. Figure shows a hollow sphere inside a lead sphere of radius R; the surface of the hollow passes through the center of the sphere and "touches" the right side of the sphere. The mass of the sphere before hollowing was M. With what gravitational force does the hollowed-out lead sphere attract a small sphere of mass m that lies at a distance d from the center of the lead sphere, on the straight line connecting the centers of the spheres and of the hollow?



9. Two small balls of masses m and $2m$ are placed at a separation d. A third ball of mass m is placed at an unknown location on the line joining the first two balls such that the net gravitational force experienced by the first ball is $\frac{6Gm^2}{d^2}$. What is the location of the third ball?

ANSWERS

2. (a) $m = M$ (b) zero **3.** $\sqrt{Gm/a}$ **4.** 468°

5. (a) $2GMm/\pi r^2$ (b) 0 **6.** 1.32×10^{32} kg

7. (a) at E (b) $4Gm^2/3a^2$ **9.** $d/2\sqrt{2}$

GRAVITATIONAL FIELD

We know that earth attracts everybody towards its centre. According to Newton's law of gravitation, the force of attraction varies inversely as the square of the distance of the body from the centre of earth. This shows that in the space

all around the earth, its gravitational pull can be experienced by other material bodies. The same is true for material body. Hence, the space around a material body in which its gravitational pull can be experienced is called its gravitational field.

According to the field concept, a material particle placed in space modifies the space around it. This modified space is called gravitational field. When another material particle is brought in this field, it experiences a gravitational force of attraction due to interaction of gravitational fields of the particles.

INTENSITY OF GRAVITATIONAL FIELD

In our daily life, forces are transmitted by direct contact (as either somebody is pushing or pulling some other body). But the gravitational forces act through millions of miles of empty space without direct contact, e.g. Sun attracting earth and Earth in turn attracting Moon. This type of situation where forces are transmitted without direct contact is called **action at a distance**.

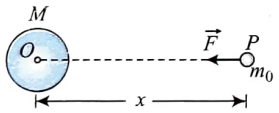
But this old view has now been replaced by the concept of gravitational field. According to this concept:

(i) every mass creates a gravitational field around it and

(ii) any other mass placed in the gravitational field feels a force on it. Thus:

"Gravitational field around a mass is the sphere of influence around it within which its force of attraction can be experienced and the intensity of the gravitational field at any point is equal to the force experienced by a unit mass placed at that point."

Let M be the mass of a body with centre of gravity at O. Let F be the gravitational force of attraction experienced by a test mass m_0 when placed at P in the gravitational field of the body, such that $OP = x$.



According to Newton's law of gravitation,

$$F = MGm_0 / x^2$$

Intensity of gravitational field at P will be

$$I = \frac{F}{m_0} = \frac{GMm_0 / x^2}{m_0} = \frac{GM}{x^2} \quad \text{...(i)}$$

In vector form, $\vec{I} = -\frac{Gm}{x^2}\hat{x}$ ...(ii)

Here the negative sign shows that the gravitational intensity is of attractive force. From Eq. (ii), we note that as x increases, I decreases and when $x = \infty$, $I = 0$. It means that the intensity of gravitational field is zero only at infinite distance from the body.

If the test mass is free to move at P, it will be accelerated due to force of attraction F. Let a be the acceleration produced in the test mass due to force F, then

$$F = m_0 a \text{ or } \frac{F}{m_0} = a \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), $I = a$.

It means that the intensity of gravitational field at a point in the gravitational field is equal to the acceleration of test mass placed at that point.

If the gravitational field is due to earth and point P lies on (near) the surface of the earth, then $x = R$. Now

$$I = \frac{GM}{R^2} = g$$

where g is the acceleration due to gravity at the surface of earth.

Unit of intensity of gravitational field in SI system is N kg^{-1} or m s^{-2} and in CGS system, it is dyne g^{-1} or cm s^{-2}.

Dimensional formula of gravitational intensity

$$I = \frac{F}{m_0} = \frac{MLT^{-2}}{[M]} = [M^0 LT^{-2}]$$

ILLUSTRATION 5.13

Find the relation between the gravitational field on the surface of two planets A and B of masses m_A, m_B and radii R_A and R_B, respectively, if

(a) they have equal mass,

(b) they have equal (uniform) density.

Sol. Let E_A and E_B be the gravitational field intensities on the surface of planets A and B.

Then, $E_A = \frac{GM_A}{R_A^2} = \frac{G\frac{4}{3}\pi R_A^3 \rho_A}{R_A^2} = \frac{4G\pi}{3}\rho_A R_A$

Similarly, $E_B = \frac{Gm_B}{R_B^2} = \frac{4G}{3}\pi\rho_B R_B$

(a) For $m_A = m_B$, $\frac{E_A}{E_B} = \frac{R_B^2}{R_A^2}$

(b) For $\rho_A = \rho_B$, $\frac{E_A}{E_B} = \frac{R_A}{R_B}$

ILLUSTRATION 5.14

Two bodies of masses 100 kg and 10,000 kg are at a distance 1 m apart. At which point on the line joining them will the resultant gravitational field intensity be zero?

Sol. Let x be the distance of point P from the 100 kg body where the resultant gravitational intensity is zero.

Therefore, gravitational intensity at P due to body of mass 100 kg is equal and opposite to that due to body of mass 10,000 kg. Hence,

$$\frac{G \times 100}{x^2} = \frac{G \times 10{,}000}{(1-x)^2}$$

$\Rightarrow \quad 100\,x^2 = (1-x)^2$

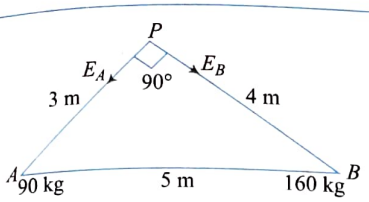
$\Rightarrow \quad 10\,x = 1 - x$

$\Rightarrow \quad 11\,x = 1 \text{ or } x = \frac{1}{11}$ m

ILLUSTRATION 5.15

Two masses 90 kg and 160 kg are at a distance 5 m apart. Compute the magnitude of intensity of the gravitational field at a point distance 3 m from the 90 kg and 4 m from the 160 kg mass. $G = 6.67 \times 10^{-11}$ kg^{-2}.

Sol. Let A and B be the positions of the masses and P be the point at which the gravitational intensity is to be computed.



Gravitational intensity at P due to mass at A will be

$$E_A = G\frac{90}{3^2} = 10G, \text{ along } PA.$$

Gravitational intensity at P due to mass at B will be

$$E_B = G\frac{160}{4^2} = 10G, \text{ along } PB.$$

In ΔAPB, $AB^2 = AP^2 + PB^2$ $(\therefore \angle APB = 90°)$

Therefore, the magnitude of resultant gravitational intensity at P will be

$$E = \sqrt{E_A^2 + E_B^2} = \sqrt{(10\,G)^2 + (10\,G)^2} = 10\sqrt{2}G$$

$$= 10\sqrt{2} \times 6.67 \times 10^{-11} \text{ N/kg} = 9.43 \times 10^{-10} \text{ N kg}^{-1}$$

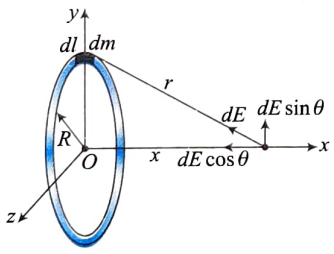
GRAVITATIONAL FIELD INTENSITY DUE TO CONTINOUS MASS DISTRIBUTION

GRAVITATIONAL FIELD DUE TO A UNIFORM CIRCULAR RING AT A POINT ON ITS AXIS

At a Point on the Axis of Ring

Figure shows a ring of mass M and radius R placed in YZ plane with centre at origin. Here we wish to find the gravitational field strength at a point P on its axis at a distance x from its centre.

To find this we consider an element of length dl on ring as shown in figure. The mass dm of this element can be given as



$$dm = \frac{M}{2\pi R} \qquad \ldots(i)$$

Let the gravitational field strength at point P due to the element dm is dM then it is given as

$$dE = \frac{G\,dm}{(x^2 + R^2)} = \frac{GM\,dx}{2\pi R(x^2 + R^2)}$$

This elemental gravitational field strength dE has two rectangular components, one along the axis of ring $dE\cos\theta$ and other perpendicular to the axis of ring, $dE\sin\theta$. Here when we integrate the result for the complete ring, $dE\sin\theta$ component will be cancelled out by symmetry and $dE\cos\theta$ will be summed up to give the net gravitational field strength at P.

Thus here net gravitational field strength at P is given as

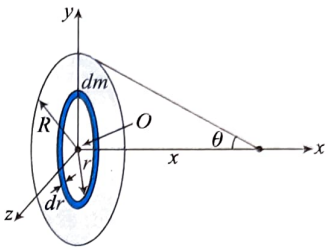
$$E = \int dE\cos\theta = \int_0^{2\pi} \frac{GM\,dx}{2\pi R(x^2 + R^2)} \times \frac{x}{\sqrt{x^2 + R^2}}$$

$$= \frac{GMx}{2\pi R(x^2 + R^2)^{3/2}} \int_0^{2\pi R} dl = \frac{GMx}{2\pi R(x^2 + R^2)^{3/2}} [2\pi R]$$

$$\Rightarrow E = \frac{GMx}{(x^2 + R^2)^{3/2}} \qquad \ldots(ii)$$

GRAVITATIONAL FIELD DUE TO A UNIFORM DISC AT A POINT ON ITS AXIS

The situation is shown in figure. Let the mass of the disc be M and its radius be R. Let O be the centre of the disc and P be a point on its axis at a distance x from the centre. We have to find the gravitational field at P due to the disc.



Let us draw a circle of radius r with the centre at O. We draw another concentric circle of radius $r + dr$. The part of the disc enclosed between these two circles can be treated as a uniform ring of radius x. The point P is on its axis at a distance x from the centre. The area of this ring is $2\pi r\,dr$. The area of the whole disc is πR^2. As the disc is uniform, the mass of this ring is

$$dm = \frac{M}{\pi R^2} 2\pi r dr = \frac{2Mrdr}{R^2}$$

The gravitational field at P due to the ring is given by equation

$$dE = \frac{G\left(\frac{2Mrdr}{R^2}\right)x}{(x^2 + r^2)^{3/2}} = \frac{2GMx}{R^2}\frac{rdr}{(x^2 + r^2)^{3/2}}$$

As r varies from 0 to R, the rings cover up the whole disc. The field due to each of these rings is in the same direction PO. Thus, the net field due to the whole disc is along PO and its magnitude is

$$E = \int_0^R \frac{2GMx}{R^2}\frac{r\,dr}{(x^2 + r^2)^{3/2}} = \frac{2GMx}{R^2}\int_0^a \frac{x\,dx}{(x^2 + r^2)^{3/2}} \qquad \ldots(i)$$

Let $x^2 + r^2 = z^2$

Then, $2r\,dr = 2z\,dz$ and

$$\int \frac{x\,dx}{(x^2 + r^2)^{3/2}} = \int \frac{z\,dz}{z^3}$$

$$\int \frac{1}{z^2}\,dz = -\frac{1}{z} = -\frac{1}{\sqrt{x^2 + r^2}}$$

From (i), $E = \frac{2GMx}{R^2}\left[-\frac{1}{\sqrt{x^2 + r^2}}\right]_0^a$

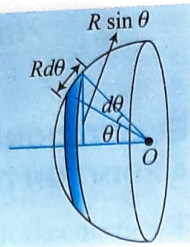
$$E = \frac{2GMx}{R^2}\left[\frac{1}{x} - \frac{1}{\sqrt{x^2 + R^2}}\right] \qquad \ldots(ii)$$

Equation (ii) may be expressed in terms of the angle θ subtended by a radius of the disc at P as,

$$E = \frac{2GM}{R^2}(1 - \cos\theta)$$

ILLUSTRATION 5.16

Calculate the gravitational field intensity at the centre of the base of a hollow hemisphere of mass M and radius R. (Assume the base of hemisphere to be open)



Sol. We consider the shaded elemental ring of mass,

$$dm = \frac{M}{(2\pi R^2)} 2\pi R \sin\theta (Rd\theta)$$

Field due to this ring at O, $dE = \frac{Gdm\, R\cos\theta}{R^3}$

(see formulae for field due to a ring)

or $dE = \frac{GM}{R^2}\sin\theta\cos\theta\, d\theta$

Hence, $E = \int_0^E dE = \int_0^{\pi/2} \frac{GM}{R^2}\sin\theta\cos\theta\, d\theta$

or $E = \frac{GM}{2R^2}$

ACCELERATION DUE TO GRAVITY

If a force F acting on a body of mass m produces an acceleration a then according to Newton's second law of motion $F = ma$. If the force on the body is due to gravity of earth, then acceleration in the body is called acceleration due to gravity, which is denoted by g, i.e., $a = g$. Then,

$$F = mg \text{ or } g = \frac{F}{m}$$

If $m = 1$, then $g = \frac{F}{1} = F$. Thus, acceleration due to gravity is defined as the force of gravity acting on a unit mass of the body placed on or near the surface of the earth.

If the body is falling freely under the effect of gravity, then the acceleration in the body is also called acceleration due to gravity. Thus, acceleration due to gravity is the acceleration set up in the body when it falls freely under the effect of gravity alone. It is denoted by g. The SI unit of acceleration due to gravity is m s^{-2} or N kg^{-1}.

The value of g is independent of the shape, size, mass, etc. of the body but depends upon mass and radius of the earth or planet due to which there is a gravity.

When two bodies of different masses are allowed to fall from same height in vacuum, they reach the earth at the same time. If the two bodies of same mass but of different volumes are allowed to fall from same height in air medium, they will not be reaching simultaneously. The body of smaller volume will reach the earth earlier than the other body, since the upward thrust due to air on the body of smaller volume will be less than that on the body of larger volume.

RELATION BETWEEN g AND G

Consider earth to be a spherical body of mass M, radius R with centre O. Suppose a body of mass m is placed on the surface of earth. The force with which a body is attracted towards the centre of earth is called the weight of the body.

$$F = \frac{GMm}{R^2} \quad \ldots(i)$$

where M = mass of the earth and R = radius of the earth.

If g is the acceleration due to gravity, then the force on the body due to earth is given by

Force = mass × acceleration

or $F = mg$ $\quad \ldots(ii)$

From (i) and (ii), we have $mg = \frac{GMm}{R^2}$

$\therefore \quad g = \frac{GM}{R^2}$

or $g = \frac{G}{R^2}\left(\frac{4}{3}\pi R^3\rho\right)$

$$\left[\text{As mass } (M) = \text{volume}\left(\frac{4}{3}\pi R^3\right) \times \text{density } (\rho)\right]$$

$\therefore \quad g = \frac{4}{3}\pi\rho GR$

Important Points:

- From the expression $g = \frac{GM}{R^2} = \frac{4}{3}\pi\rho GR$, it is clear that its value depends upon the mass, radius and density of planet and it is independent of mass, shape and density of the body placed on the surface of the planet. i.e. a given planet (reference body) produces same acceleration in a light as well as heavy body.
- The greater the value of (M/R^2) or ρR, greater will be value of g for that planet.
- Acceleration due to gravity is a vector quantity and its direction is always towards the centre of the planet.
- Dimensional formula $[g] = [LT^{-2}]$
- Its average value is taken to be 9.8 m/s^2 or 981 cm/s^2 or 32 feet/s^2, on the surface of the earth at mean sea level.

ILLUSTRATION 5.17

Using the known radius of the Earth and $g = 9.80$ m/s^2 at the Earth's surface, find the average density of the Earth.

Sol. The density of material in the Earth varies, but let's adopt a simplified model in which we assume the density to be uniform throughout the Earth. The resulting density is the average density of the Earth. Assume the Earth is a perfect sphere.

We know that $g = \frac{GM_E}{R_E^2}$, the mass of the Earth as $M_E = \frac{gR_E^2}{G}$

Substitute this mass and the volume of a sphere into the definition of density:

$$\rho_E = \frac{M_E}{V_E} = \frac{gR_E^2/G}{\frac{4}{3}\pi R_E^3} = \frac{3}{4}\frac{g}{\pi GR_E}$$

$$= \frac{3}{4}\frac{9.80 \text{ m/s}^2}{\pi(6.674\times10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(6.37\times10^6 \text{ m})}$$

$$= 5.50\times10^3 \text{ kg/m}^3$$

ILLUSTRATION 5.18

The acceleration due to gravity at the Moon's surface is 1.67 m/s². If the radius of the Moon is 1.74×10^6 m, calculate the mass of the Moon. Use the known value of G.

Sol. We know that $g = \frac{GM}{R^2}$ or $M = \frac{gR^2}{G}$

Here, $g = 1.67\ \text{m/s}^2$, $R = 1.74 \times 10^6$ m

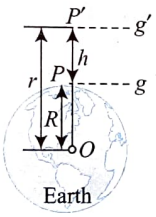
$G = 6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2$

Thus, $M = \frac{(1.67\ \text{m/s}^2)(1.74 \times 10^6\ \text{m})^2}{6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2} = 7.6 \times 10^{22}$ kg

VARIATION OF ACCELERATION DUE TO GRAVITY

The value of acceleration due to gravity changes with height (i.e., altitude), depth, shape of the earth and rotation of earth about its own axis. The effect of each of the above factors on the value of g has been discussed below.

VARIATION IN g WITH HEIGHT



Let us suppose Earth to be a perfect sphere of radius R with centre at O, (figure).

Let M = mass of the Earth (supposed to be concentrated at its centre),

g = acceleration due to gravity at a point P on the surface of the Earth.

h = height of a point from the surface of the Earth

g' = acceleration due to gravity at the point P.

Acceleration due to gravity at the surface of the earth,

$$g = \frac{GM}{R^2} \qquad \ldots(i)$$

Acceleration due to gravity at height h from the surface of the earth,

$$g' = \frac{GM}{(R+h)^2} \qquad \ldots(ii)$$

From (i) and (ii):

$$g' = g\left(\frac{R}{R+h}\right)^2 = g\frac{R^2}{r^2} \qquad \ldots(iii)$$

$$g' = \frac{GM}{r^2} \quad [\text{As } r = R + h] \qquad \ldots(iv)$$

Important Points:

- As we go above the surface of the earth, the value of g decreases because $g' \propto \frac{1}{r^2}$.
- If $r = \infty$ then $g' = 0$, i.e., at infinite distance from the earth, the value of g becomes zero.
- If $h \ll R$ i.e., height is negligible in comparison to the radius then from Eq. (iii) we get

$$g' = g\left(\frac{R}{R+h}\right)^2 = g\left(1+\frac{h}{R}\right)^{-2} = g\left[1-\frac{2h}{R}\right] \quad [\text{As } h \ll R]$$

Absolute decrease $\Delta g = g - g' = \frac{2hg}{R}$

Fractional decrease $\frac{\Delta g}{g} = \frac{g-g'}{g} = \frac{2h}{R}$

Percentage decrease $\frac{\Delta g}{g} \times 100\% = \frac{2h}{R} \times 100\%$

ILLUSTRATION 5.19

How much above the surface of earth does the acceleration due to gravity reduce by 36% of its value on the surface of earth? Radius of earth = 6400 km.

Sol. Since the acceleration due to gravity reduces by 36%, the value of acceleration due to gravity there is 100 – 36 = 64%. It means, $g' = \frac{64}{100}g$. If h is the height of location above the surface of earth, then

$$g' = g\frac{R^2}{(R+h)^2} \quad \text{or} \quad \frac{64}{100}g = g\frac{R^2}{(R+h)^2}$$

$$\frac{8}{10} = \frac{R}{R+h} \quad \text{or} \quad 8R + 8h = 10R$$

$$\Rightarrow \quad h = \frac{2R}{8} = \frac{R}{4} = \frac{6.4 \times 10^6}{4} = 1.6 \times 10^6\ \text{m}$$

ILLUSTRATION 5.20

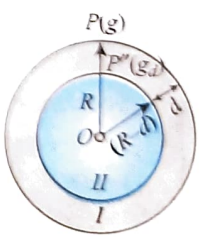
What is the fractional decrease in the value of free-fall acceleration g for a particle when it is lifted from the surface to an elevation h? ($h \ll R$).

Sol. $g = \frac{GM}{R^2} \Rightarrow \frac{dg}{dR} = -\frac{2GM}{R^3}$

$$\Rightarrow \quad \frac{dg}{g} = -\frac{2GM}{R^2}\frac{1}{R}$$

$$\Rightarrow \quad \frac{dg}{g} = -2\left(\frac{h}{R}\right)$$

VARIATION IN g WITH DEPTH



Suppose Earth to be a homogenous sphere (i.e., a sphere of uniform density). Let ρ = mean density of the Earth; M = mass of the Earth (supposed concentrated at its centre); R = radius of the earth; g = acceleration due to gravity at the point P on the surface of the Earth; d = depth of a point P'' inside the Earth; and g_d = acceleration due to gravity at the point P

Clearly, $g = \frac{GM}{R^2} = \frac{G \times (4\pi/3)R^3\rho}{R^2} = \frac{4\pi}{3}GR\rho \qquad \ldots(i)$

With O as centre and $(R - d)$ as radius, let us draw an imaginary sphere so that Earth is divided into two parts (refer figure).

(i) a shell of thickness d denoted by I.

(ii) a sphere of radius $(R - d)$ denoted by II.

The force of attraction on a body placed at the point P'' is due to: (i) shell I and (ii) sphere II.

As discussed earlier, a body lying inside a shell does not experience any force. Hence, the only force with which the body at P'' is attracted is due to the sphere II.

Thus, g_d = acceleration due to gravity at P'' due to sphere II (as the shell I does not contribute in this respect).

Assuming the density ρ to be constant throughout the Earth, we can replace R by $(R-d)$ in Eq. (i) and get

$$g_d = \frac{4\pi}{3}G(R-d)\rho \qquad ...(ii)$$

Dividing Eq. (ii) by Eq. (i), we get

$$\frac{g_d}{g} = \frac{(4\pi/3)G(R-d)\rho}{(4\pi/3)GR\rho} = \frac{R-d}{R} = \left(1-\frac{d}{R}\right)$$

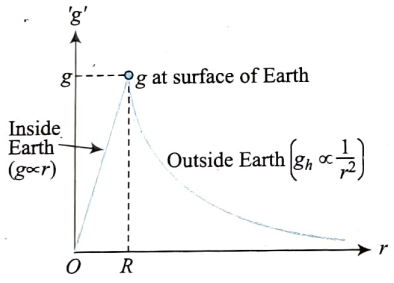
or $$g_d = g\left(1-\frac{d}{R}\right)$$

Clearly, the acceleration due to gravity decreases as we move down into the surface of the Earth, i.e., acceleration due to gravity in the mines is less than that on the surface of the Earth.

At the centre O of the Earth, $d = R$

Thus, $g_d = g\left(1-\frac{d}{R}\right) = \text{zero}$

Hence, acceleration due to gravity at the centre of the Earth is zero.



Variation of g with height and depth

Important Points:

- The value of g decreases on going below the surface of the earth. From Eq. (ii) we get $g' \propto (R-d)$.
  So it is clear that if d increase, the value of g decreases.
- At the centre of earth $d = R$ $\therefore$ $g' = 0$, i.e., the acceleration due to gravity at the centre of earth becomes zero.
- Decrease in the value of g with depth

  Absolute decrease $\Delta g = g - g' = \frac{dg}{R}$

  Fractional decrease $\frac{\Delta g}{g} = \frac{g-g'}{g} = \frac{d}{R}$

  Percentage decrease $\frac{\Delta g}{g}\times 100\% = \frac{d}{R}\times 100\%$
- The rate of decrease of gravity outside the earth ($h << R$) is double to that of inside the earth.

ILLUSTRATION 5.21

Find the percentage decrease in weight of a body, when taken 16 km below the surface of the earth. Take radius of the earth as 6,400 km. What happens to the weight of the body, when it is taken to the centre of the earth?

Sol. Here, $R = 6{,}400$ km; $d = 16$ km

The value of acceleration due to gravity at a depth d,

$$g' = g\left(1-\frac{d}{R}\right) = g\left(1-\frac{16}{6{,}400}\right) = \frac{399}{400}g$$

$$g - g' = g - \frac{399}{400}g = \frac{1}{400}g = 2.5\times 10^{-3}\, g$$

If m is mass of the body, then mg and mg' will be respectively weight of the body on the surface of earth and at a depth d below the surface of the earth. Then, % decrease in the weight of the body

$$= \frac{mg - mg'}{mg}\times 100 = \frac{g-g'}{g}\times 100$$

$$= \frac{2.5\times 10^{-3}\, g}{g}\times 100 = 0.25\%$$

Now, at the centre of the earth,

$$mg' = mg\left(1-\frac{d}{R}\right) = mg\left(1-\frac{R}{R}\right) = 0$$

ILLUSTRATION 5.22

(a) Find the height from the earth's surface where g will be 25% of its value on the surface of earth ($R = 6400$ km).

(b) Find the percentage decrease in the value of g at a depth h from the surface of earth.

Sol.

(a) At a height h above the earth's surface, we have

$$g' = g\left(\frac{R}{R+h}\right)^2 \Rightarrow \frac{g}{4} = g\left(\frac{R}{R+h}\right)^2 \Rightarrow \frac{R}{R+h} = \frac{1}{2}$$

$$\Rightarrow \quad h = R = 6400 \text{ km}$$

(b) At a depth h below earth's surface, we have

$$g' = g\left(1-\frac{h}{R}\right) \Rightarrow \frac{g'}{g} = 1 - \frac{4}{6400}$$

$$\frac{g-g'}{g} = \frac{4}{6400} \Rightarrow \frac{\Delta g}{g} = \frac{1}{1600}$$

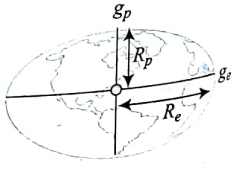
$$\therefore \quad \%\text{age decrease} = \frac{\Delta g}{g}\times 100 = \frac{1}{16}\% \text{ decrease}$$

EFFECT OF SHAPE OF EARTH

Earth is not a perfect sphere. It is flattened at the poles and bulges out at the equator. Equatorial radius R_e of the earth is about 21 km greater than the polar radius R_p.

Now, $g = \frac{GM}{R^2}$

At equator: $g_e = \frac{GM}{R_e^2}$ ...(i)



At poles: $g_p = \dfrac{GM}{R_p^2}$ ...(ii)

From (i) and (ii) $\dfrac{g_e}{g_p} = \dfrac{R_p^2}{R_e^2}$

Since $R_{equator} > R_{pole}$

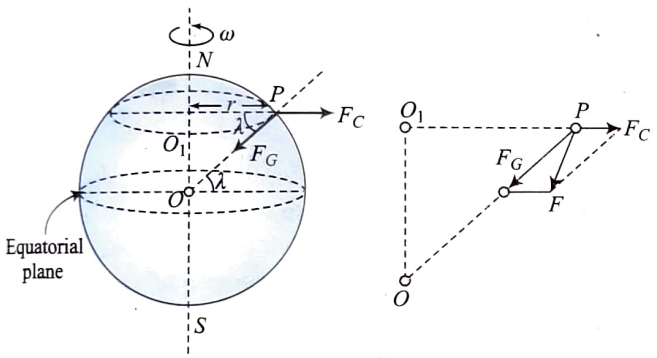
$\therefore \quad g_{pole} > g_{equator}$ and $g_p = g_e + 0.018\ ms^{-2}$

Thus, we conclude that the value of g is least at the equator and maximum at the pole. It means, the value of acceleration due to gravity increases as we go from equator to the pole.

At sea level, the value of g at pole is greater than its value at equator by 1.80 cm s^{-2}.

EFFECT OF ROTATION OF EARTH ABOUT ITS AXIS

Figure below shows the situation in which earth spins about its axis passing through north and south pole, with an angular speed ω. Let P be any point on the earth's surface, with a latitude λ. Let PO_1 be the perpendicular dropped from the point P on the spin axis. Evidently, the point P rotates in a circle of radius $PO_1 = r$ (say) as the earth spins about its axis.



Consider a small mass m kept at P. Earth attracts the body of mass m with a force of gravity given by $F_G = \dfrac{GMm}{R^2}$ (along $\overrightarrow{PO}$). If we attach a reference frame to the earth (which is a non-inertial reference frame) and would like to apply Newton's laws, then we have to incorporate an additional pseudo force F_C (centrifugal force) on the mass m given by

$$F_C = m\omega^2 r \quad (\text{along } \vec{O}_1P)$$

Thus, the apparent force on body of mass m, is the vector sum of $\vec{F}_G$ and $\vec{F}_C$ given by (figure).

$$|\vec{F}| = \sqrt{F_G^2 + F_C^2 - 2F_GF_C\cos\lambda}$$

i.e., $$F = \sqrt{\left(\frac{GMm}{R^2}\right)^2 + (m\omega^2 r)^2 - 2\left(\frac{GMm}{R^2}\right)(m\omega^2 r)\cos\lambda}$$

Now, putting $r = R\cos\lambda$,

acceleration due to gravity (apparent); $g' = \dfrac{F}{m}$ will be

$$g' = \sqrt{\left(\frac{GM}{R^2}\right)^2 + (\omega^2 R\cos\lambda)^2 - 2\left(\frac{GM}{R^2}\right)R\omega^2\cos^2\lambda}$$

Now putting $\dfrac{GM}{R^2} = g$ and remembering that

$$\omega^2 R = \left(\frac{2\pi}{86400}\right)^2 \times 6{\cdot}4\times10^6\ m/s^2 = 0.034\ m/s^2$$

which is so small that its higher powers can be neglected, we have

$$g' = \sqrt{g^2 - 2gR\omega^2\cos^2\lambda}$$

$$g' = g\left(1 - \frac{2R\omega^2\cos^2\lambda}{g}\right)^{1/2}$$

Expanding by binomial theorem.

$$g' = g\left(1 - \frac{R\omega^2\cos^2\lambda}{g} + \text{higher power of}\left(\frac{2R\omega^2\cos^2\lambda}{g}\right)\cdots\right)$$

$$\Rightarrow \quad g' = g - R\omega^2\cos^2\lambda$$

Alternatively, the component of centripetal force along and normal to the radius OP is $m\omega^2(R\cos\lambda)\cos\lambda$ and $m\omega^2(R\cos\lambda)\sin\lambda$.

$\therefore$ Net force along $\overrightarrow{PO} = \dfrac{GMm}{R^2} - m\omega^2 R\cos^2\lambda$ and normal to $\overrightarrow{PO} = m\omega^2 R\cos\lambda\sin\lambda$ which is too small and hence can be neglected.

$\therefore$ Net force $F \approx \dfrac{GMm}{R^2} - m\omega^2 R\cos^2\lambda$

and effective acceleration due to gravity

$$g' = \frac{F}{m} = g - R\omega^2\cos^2\lambda$$

Important Points:

- At poles, latitude $\lambda = 90° \Rightarrow \cos\lambda = 0$

  $\therefore \quad g_P = g - 0 = g \left[\text{where } g = \dfrac{GM}{R^2}\right]$ ...(i)
- At equator; $\lambda = 0°$; $\cos\lambda = 1$

  $\therefore \quad g_e = g - R\omega^2$ ...(ii)
- Thus, from Eqs. (i) and (ii) the acceleration due to gravity at equator is less than that poles.
- The direction of apparent 'g' is not along the centre of earth, but slightly away from it.
- All the above equations are equally applicable for any other planet, where M and R corresponds to the mass and radius of the planet.

ILLUSTRATION 5.23

A body is taken from pole to equator. Ignoring the effect of shape of Earth, find the percentage change in the acceleration due to gravity.

Sol. As we know, $g_P = g$ and $g_e = g - R\omega^2$

$\therefore \quad g_e = g_P - R\omega^2$

$$\therefore \quad \text{Required \% change} = \left(\frac{g_e - g_P}{g_P}\right)100 = -\frac{R\omega^2 \times 100}{g_P}$$

which can be evaluated as $= -\frac{64\times10^5}{9.8}\times\left(\frac{2\pi}{86400}\right)^2 \times 100$

$= -0.34\%$

ILLUSTRATION 5.24

Find the percentage increase in the earth's angular velocity, so that, all bodies lying on the equator, just leave the earth's surface. Assume the earth to be a uniform sphere.

Sol. Let us assume that the acceleration due to gravity, at present, at the equator as

$$9.8 = \frac{GM}{R^2} - R\omega_0^2 \qquad ...(i)$$

where ω_0 is the present value of angular velocity of earth.

Now, if ω be the new angular velocity when, the bodies at the equator leave the earth's surface then $g_e = 0$

[i.e., the effective value of g_e should be zero]

$$\therefore \quad 0 = \frac{GM}{R^2} - R\omega^2$$

$$\Rightarrow \quad R\omega^2 = \frac{GM}{R^2} \qquad ...(ii)$$

From (i) and (ii), $9.8 = R\omega^2 - R\omega_0^2$

$$\Rightarrow \quad \omega^2 - \omega_0^2 = \left(\frac{9.8}{R}\right)$$

$$\Rightarrow \quad \omega^2 = \frac{9.8}{R} + \omega_0^2$$

Now, % increase in $\omega = \left(\frac{\omega - \omega_0}{\omega_0}\right)\times 100$

$$\Rightarrow \quad \left(\frac{\omega}{\omega_0} - 1\right)\times 100 = \left(\sqrt{\frac{9.8}{\omega_0^2 R} + 1} - 1\right)\times 100$$

Substituting $\omega_0 = \left(\frac{2\pi}{86400}\right)$ rad/sec and $R = 64\times10^5$ m

$\therefore$ The value comes out to be $\approx 1700\%$

i.e., the angular velocity of earth should increase about 17 times.

ILLUSTRATION 5.25

What will be the change in the value of 'g' at a place whose latitude is 45°, if the earth stops spinning about its axis? Assume the earth to be a sphere of radius 6.4×10^6 m.

Sol. The acceleration due to gravity at a place with latitude λ is given by $g' = g - R\omega^2\cos^2\lambda$. If the earth stops spinning, then $\omega = 0$ and hence the new acceleration due to gravity $g'' = g$

$\therefore$ Change in the value of $g = g'' - g' = R\omega^2\cos^2\lambda$

$$= 6.4\times10^6 \times \left(\frac{2\pi}{86400}\right)^2 \times \cos^2(45°)$$

$$\approx 0.0168 \text{ m/s}^2$$

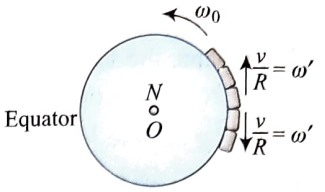
Thus, the value of g will increase by about 0.0168 m/s^2.

ILLUSTRATION 5.26

If a train of mass m moves along the equator with a speed v, find the difference in thrust exerted by it on earth, when it moves (i) from east to west and (ii) from west to east. Take ω_0 as the angular velocity of earth.

Sol. Since the train is on the equator, latitude $\lambda = 0$.

Now, the effective value of g is given by $g' = g - R\omega^2$ where ω is the angular velocity of the reference frame.



If we attach a reference frame to the train. Then, when train moves from west to east (i.e., along the same direction in which earth rotates) (refer figure).

Then, $\omega = \omega_0 + \frac{v}{R}$ (where ω_0 is the angular velocity of earth)

$\therefore$ Force exerted by train on earth will be

$$F_1 = mg' = m\left(g - R\left(\omega_0 + \frac{v}{R}\right)^2\right) \qquad ...(i)$$

Similarly, when the train moves from east to west.

Then, $\omega = \omega_0 - \frac{v}{R} \quad \left[\text{assuming } \omega_0 > \frac{v}{R}\right]$

and hence, the corresponding force will be

$$F_2 = m\left(g - R\left(\omega_0 - \frac{v}{R}\right)^2\right) \qquad ...(ii)$$

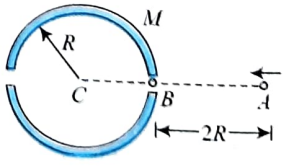
Subtracting (i) and (ii),

$$F_2 - F_1 = mR\left[\left(\omega_0 + \frac{v}{R}\right)^2 - \left(\omega_0 - \frac{v}{R}\right)^2\right]$$

$$= mR\left[\frac{4\omega_0 v}{R}\right] = 4m\omega_0 v$$

CONCEPT APPLICATION EXERCISE 5.3

1. A particle is released from rest from the point A. Find its accelerations, velocities at B and C, if the particle moves under the gravity of thin spherical shell of mass M and radius R.



2. Two spheres of masses 64 kg and 289 kg are placed a distance 10 m apart. Find the position of a point on the line joining the centres of the two spheres, where the gravitational field is zero.
3. If the radius of Earth is 6000 km, what will be the weight of 120 kg body if taken to a height of 2000 km above sea level (i.e., surface of the Earth)?

4. Where will you weigh more and why: **(a)** 2 km above the surface of earth or **(b)** 2 km below the surface of earth?
5. Find the maximum change in reading of a spring balance if a block of mass m is brought from pole to equator.
6. Find the height above the earth's surface, where the value of acceleration due to gravity is $(3/4)^{th}$ the value on the surface. (take the radius of earth as R).
7. Find the percentage change in the acceleration due to gravity when a small body is taken to an altitude $\frac{R}{100}$.
8. The value of 'g' at a depth d, is two-third the value that on the earth's surface. Find d in terms of radius of earth R.
9. The distance between earth and moon is 3.8×10^5 km and the mass of earth is 81 times the mass of the moon. Find the position of a point on the line joining the centres of earth and moon, where the gravitational field is zero. What would be the value of gravitational field there due to the earth and moon separately?
   Given, radius of the earth = 6.4×10^6 m.
10. A uniform solid sphere of mass M and radius a is surrounded symmetrically by a uniform thin spherical shell of equal mass and radius $2a$. Find the gravitational field at a distance **(a)** $3a/2$ from the centre, **(b)** $\frac{5}{2}a$ from the centre.
11. At a depth $h_1 = \frac{R}{2}$ from the surface of the earth acceleration due to gravity is g_1. Its value changes by Δg_1 when one moves down further by 1 km. At a height h_2 above the surface of the earth acceleration due to gravity is g_2. Its value changes by Δg_2 when one moves up further by 1 km. If $\Delta g_1 = \Delta g_2$, find h_2. Assume the earth to be a uniform sphere of radius R.

ANSWERS

1. $a_A = \frac{GM}{9R^2}, a_B = \frac{GM}{R^2}, a_C = 0$ **2.** 3.2 m from 64 kg mass
3. 67.5 kg-wt **5.** 0.033 N **6.** $h = (2\sqrt{3} - 3)\frac{R}{3}$
7. 2% **8.** $d = R/3$ **11.** $h_2 = R\left(2^{\frac{1}{3}} - 1\right)$

GRAVITATIONAL POTENTIAL

Gravitational potential at a point in a gravitational field of a body is defined as the amount of work done in bringing a body of unit mass from infinity to that point without acceleration.

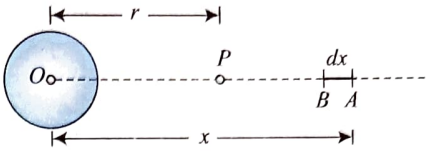
If W is the amount of work done in bringing a body of mass m_0 from infinity to a point P in the gravitational field without acceleration, then gravitational potential at P is $V = W/m_0$.

Gravitational potential is a scalar quantity, since work done is a scalar. Unit of gravitational potential at a point in SI system is J kg^{-1} and in cgs system it is erg g^{-1}. Dimensional formula for gravitational potential is given by

$$\frac{W}{m_0} = \frac{[ML^2T^{-2}]}{[M]} = [M^0L^2T^{-2}]$$

EXPRESSION FOR GRAVITATIONAL POTENTIAL AT A POINT

Let earth to be a perfect sphere of radius R and mass M. The mass of the earth can be supposed to be concentrated at its centre O. Let us calculate the gravitational potential at any point P, where $OP = r$ and $r > R$. Join OP and produce it onwards. Take two points A and B on this line such that $OA = x$ and $AB = dx$.



The gravitational force of attraction on a body of unit mass at point A will be

$$F = \frac{GM \times 1}{x^2} = \frac{GM}{x^2}$$

Small amount of work done in bringing the unit mass of the body without acceleration through a small distance AB $(= dx)$ is

$$dW = Fdx = \frac{GM}{x^2}dx$$

Total work done in bringing the body of unit mass from infinity to point P can be obtained by integrating the above expression within the limits $x = \infty$ to $x = r$, we get

$$\therefore \quad W = \int_{\infty}^{r} \frac{GM}{x^2}dx = -\left(\frac{GM}{x}\right)_{\infty}^{r} = -GM\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

This work done is the measure of gravitational potential at P (i.e., V_P).

$$V_P = W = -\frac{GM}{r} \qquad \ldots(i)$$

From Eq. (i), we note the following:

1. The gravitational potential at a point is always negative.
2. When $r = \infty$, from (i), $V_P = 0$, hence gravitational potential is maximum (= zero) at infinity.
3. At the surface of earth, $r = R$, therefore $V_P = -GM/R$ at the surface of earth.

GRAVITATIONAL POTENTIAL AT A POINT DUE TO THE EARTH

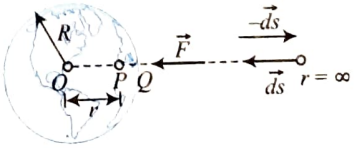
Outside the earth: The work done in bringing a body of mass m from infinity to a point at a distance r from the centre of earth is

$$W = -\frac{GMm}{r} \quad \text{for } r > R$$

Hence, gravitational potential,

$$V = \frac{W}{m} = \frac{-GMm/r}{m} = \frac{-GM}{r}, r > R$$

Inside the earth:



Potential at point P is

$$V_P = \frac{[W_{\infty \to P}]}{m} = -\left[\frac{W_{\infty \to Q} + W_{Q \to P}}{m}\right]$$

$$= -\frac{1}{m}\left[\int_{\infty}^{R} \vec{F} \cdot d\vec{s} + \int_{R}^{r} \vec{F} \cdot d\vec{s}\right] = -\frac{1}{m}\left[\int_{\infty}^{R} F ds + \int_{R}^{r} F ds\right]$$

$$[\vec{F} d\vec{s} = F ds \cos\theta = F ds]$$

$$= -\frac{1}{m}\left[\int_{\infty}^{R} \frac{GMm}{r^2}(-dr) + \int_{R}^{r} \frac{GMm}{r^2}(-dr)\right]$$

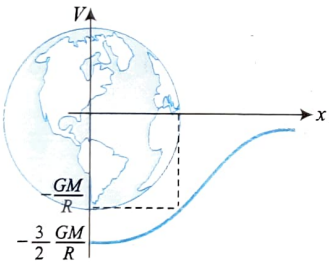
After solving, we get

$$V_P = -GM\left[\frac{3R^2 - r^2}{2R^3}\right] = -\frac{GM}{2R}\left[3 - \frac{r^2}{R^2}\right]$$

We can write gravitational potential at different points

- $V = \dfrac{-GM}{r}, r > R$
- $V = -\dfrac{GM}{R}, r = R$
- $V = -GM\left[\dfrac{3R^2 - r^2}{2R^3}\right], r < R$
- $V = \dfrac{3}{2}\left(\dfrac{-GM}{R}\right), r = 0$

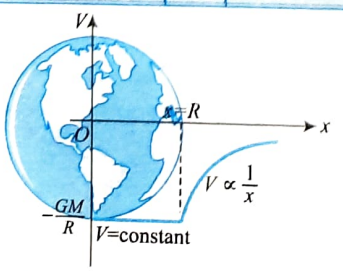
or $V_{\text{centre}} = \dfrac{3}{2} V_{\text{surface}}$



Variation of gravitational potential in case of solid sphere

Comparison of gravitational field and potential of a spherical shell of mass M and radius R

Intensity of gravitational field and potential		Gravitational potential	
i.	$E = -\dfrac{GM}{r^2}, r > R$	i.	$V = -\dfrac{GM}{r}, r > R$
ii.	$E = -\dfrac{GM}{R^2}, r = R$	ii.	$V = -\dfrac{GM}{R}, r = R$
iii.	$E = 0, r < R$	iii.	$V = -GM\left[\dfrac{3R^2 - r^2}{2R^3}\right]$



Variation of gravitational potential of spherical shell

Relation Between Gravitational Field and Potential

The work done by an external agent to move unit mass from a point to another point in the direction of the field E, slowly through an infinitesimal distance dr = Force by external agent × distance moved = $-Edr$.

Thus, $dV = -Edr \Rightarrow E = \dfrac{-dV}{dr}$

Therefore, gravitational field at any point is equal to the negative gradient of potential at that point.

Illustration 5.27

The gravitational field in a region is given by $\vec{E} = -(20 \text{ N kg}^{-1})(\hat{i} + \hat{j})$. Find the gravitational potential at the origin (0, 0) in J kg^{-1}.

Sol. We know the relation between field and potential

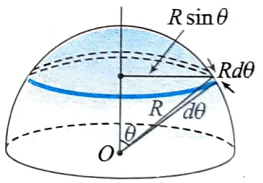
$$V = -\int E.dr = \left[\int E_x.dx + \int E_y.dy\right] = 20x + 20y$$

$\Rightarrow$ $V = 0$ at the origin (0, 0)

Illustration 5.28

Calculate the gravitational field intensity and potential at the centre of the base of a solid hemisphere of mass m, radius R.

Sol. We consider the shaded elemental disc of radius $R \sin\theta$ and thickness $Rd\theta$



Its mass, $dM = \dfrac{M}{\frac{2}{3}\pi R^3} \pi (R \sin\theta)^2 (Rd\theta \sin\theta)$

or $dM = \dfrac{3M}{2} \sin^3\theta \, d\theta$

Field due to this plate at O,

$$dE = \frac{2GdM(1 - \cos\theta)}{(R \sin\theta)^2}$$ (see field due to a uniform disc)

or $$dE = \frac{3GM \sin\theta (1 - \cos\theta) d\theta}{R^2}$$

Therefore, $E = \int_0^{\frac{\pi}{2}} dE = \int_0^{\frac{\pi}{2}} \frac{3GM \sin\theta(1 - \cos\theta)}{R^2} d\theta$

$$= \frac{3GM}{R^2}\left[-\cos\theta + \frac{\cos^2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

or $E = \dfrac{3GM}{2R^2}$

Now, potential due to the element under consideration at the centre of the base of the hemisphere,

$dV = -2GdM/r$ (cosec θ – cot θ)

(see potential due to a circular plate)

$$dV = \frac{-3GM\sin^3\theta\,(\text{cosec}\,\theta - \cot\theta)\,d\theta}{(R\sin\theta)}$$

or

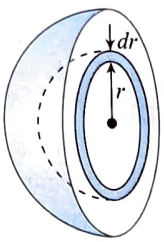
Therefore, $$V = -\frac{3GM}{R}\int_0^{\frac{\pi}{2}} (\sin\theta - \cos\theta\sin\theta)\,d\theta$$

$$= -\frac{3GM}{R}\left[-\cos\theta + \frac{\cos^2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

or $$V = \frac{-3GM}{2R}$$

Alternative method: Consider a hemispherical shell of radius r and thickness dr

Its mass, $dm = \frac{M}{\frac{2}{3}\pi R^3}(2\pi r^2 dr)$ or $dm = \frac{3Mr^2dr}{R^3}$



Since all points of this hemispherical shell are at the same distance r from O, hence potential at O is

$$dV = \frac{-Gdm}{r} = \frac{-3GMrdr}{R^3}$$

$$\therefore\quad V = \int_0^R dV = \frac{-3GM}{2R}$$

GRAVITATIONAL POTENTIAL ENERGY

We know that potential energy of a body at a given position is defined as the energy stored in the body at that position. If the position of a body changes on account of the forces acting on it, then change in its potential energy is equal to the amount of work done on the body by the forces acting on it, for the given change in position. As this work done is independent of the path followed, so the gravitational forces are conservative forces.

When the body is at infinity with respect to another body, the gravitational attraction on the body is zero. Therefore, its potential energy is zero, which is called zero level of potential energy.

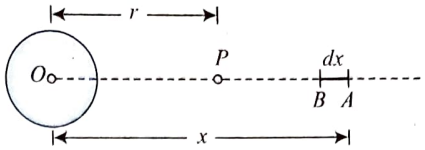
If W_0 is the zero level of potential energy of the body and W is the work done in raising the body from the position of zero level potential energy to present location, then PE of the body $= W + W_0$.

If initially the body is at infinity, then the zero level of PE is zero, i.e., $W_0 = 0$. Now, PE of the body $= W$.

Gravitational potential energy of a body at a point in a gravitational field of another body is defined as the amount of work done in bringing the given body from infinity to that point without acceleration.

EXPRESSION FOR GRAVITATIONAL POTENTIAL ENERGY

Consider a gravitational field due to earth of mass M and radius R. The mass of the earth can be supposed to be concentrated at its centre O. Let us calculate the gravitational potential energy of the body of mass m placed at point P in the gravitational field, where $OP = r$ and $r > R$. Join OP and produce it outwards. Take two points A and B on this line such that $OA = x$ and $AB = dx$.



The gravitational force of attraction on the body at A will be

$$F = GMm/x^2$$

Small amount of work done in bringing the body without acceleration through a very small distance AB $(= dx)$ is given by

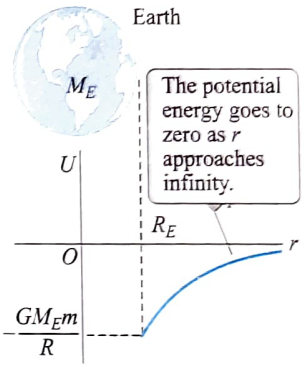
$$dW = Fdx = \frac{GMm}{x^2}dx$$

Total work done in bringing the body from infinity to point P is given by

$$W = \int_\infty^r \frac{GMm}{x^2}dx = -GMm\int_\infty^r x^{-2}dx$$

$$= -GMm\left[\frac{1}{x}\right]_\infty^r = -GMm\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$= \frac{-GMm}{r} \qquad \ldots(\text{i})$$



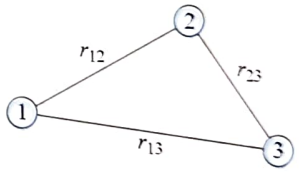
Since this work done is stored in the body as its gravitational potential energy U, therefore gravitational potential energy of the body of mass m placed at point P in the gravitational field of earth of mass M at distance r will be

$$U = \frac{-GMm}{r} \qquad \ldots(\text{ii})$$

Note that $U = V_p \times m$.

That is, gravitational potential energy (GPE) = gravitational potential × mass of the body.

We can extend this concept to three or more particles. In this case, the total potential energy of the system is the sum over all pairs of particles. Each pair contributes a term of the form given by Eq. (iii). For example, if the system contains three particles as in figure,



$$U_{\text{total}} = U_{12} + U_{13} + U_{23} = -\left(\frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}\right) \qquad \ldots(\text{iii})$$

The absolute value of U_{total} represents the work needed to separate the particles by an infinite distance.

Discussion of Result

1. In Eq. (ii), negative sign shows that the potential energy is due to attractive gravitational force exerted by earth on the body. It means to bring a body from infinity to point P, the work has been performed by the gravitational field of earth, i.e., by attractive force.
2. From Eq. (ii), we note that if the value of r increases, the value of gravitational potential energy increases as it becomes less negative. When $r = \infty$, the value of gravitational potential energy becomes zero (i.e., maximum).

3. If a body of mass m is moved from a point at a distance r_1 to a point at a distance r_2, then change in potential energy of the body is given by

$$\Delta U = \int_{r_1}^{r_2} \frac{GMm}{x^2}\,dx = GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \qquad ...(iv)$$

If $r_1 > r_2$, then ΔU is negative. It means as the body is brought closer to the earth, its potential energy decreases (i.e., becomes more negative).

4. If a body is moved from the surface of earth to a point at a height h above the surface of earth, then $r_1 = R$ and $r = R + h$. The change in potential energy will be

$$\Delta U = GMm\left[\frac{1}{R} - \frac{1}{R+h}\right]$$

$$= \frac{GMm}{R}\left[1 - \frac{R}{R+h}\right] = \frac{GMm}{R}\left[1 - \left(1+\frac{h}{R}\right)^{-1}\right]$$

$$= \frac{GMm}{R}\left[1 - \left(1 - \frac{h}{R} + \cdots\right)\right]$$

(Expanding by binomial theorem)

$$= \frac{GMmh}{R^2} = \left(\frac{GM}{R^2}\right)mh = gmh \qquad \left(\because g = \frac{GM}{R^2}\right)$$

$$\therefore \quad \Delta U = mgh$$

Important Points:

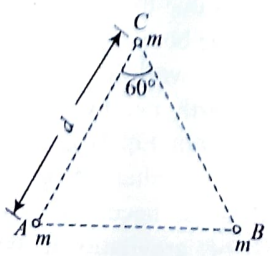
- The gravitational potential energy of a point mass m at a distance r ($>R$) from the centre of earth is
$$= \frac{GMmh}{R^2} = \left(\frac{GM}{R^2}\right)mh = gmh$$
- Gravitational potential energy of a point mass m relative to infinity at the surface of earth will be
$$U = -\frac{GMm}{R} \qquad (\because r = R)$$
- The potential energy of a point mass m at the centre of earth relative to infinity will be given by
$$U = mV_C = -\frac{3}{2}\frac{GMm}{R} \qquad \left[\because V_C = \frac{3}{2}V_S = -\frac{3GM}{2R}\right]$$
- The potential energy of a body at the centre of earth is minimum but not zero.

ILLUSTRATION 5.29

Three particles each of mass m are placed at the corners of an equilateral triangle of side d as shown in figure. Calculate:

(a) the potential energy of the system,

(b) work done on this system if the side of the triangle is changed from d to $2d$.



Sol.

(a) As in case of two-particle system potential energy is given by $(-Gm_1m_2/r)$, so

$$U_1 = U_{12} + U_{23} + U_{31}$$

or $$U_1 = -3\frac{Gmm}{d} = -\frac{3Gm^2}{d}$$

(b) When d is changed to $2d$, $U_f = -\frac{3Gm^2}{2d}$

Thus work done in changing in potential energy is given as

$$W = U_f - U_i = \frac{3Gm^2}{2d}$$

ILLUSTRATION 5.30

If g is the acceleration due to gravity on the surface of earth, find the gain in potential energy of an object of mass m raised from the surface of earth to a height equal to the radius R of the earth.

Sol. Let M, R be the mass and radius of the earth. Then

$$g = GM/R^2$$

or $$GM = gR^2 \qquad ...(i)$$

Potential energy of the object on the surface of earth $U_1 = \frac{-GMm}{R}$

The potential energy of the object at a height equal to radius of the earth is $U_2 = \frac{-GMm}{2R}$

Gain in PE is

$$U_2 - U_1 = -\frac{GMm}{2R} + \frac{GMm}{R} = \frac{GMm}{2R} = \frac{(gR^2)m}{2R} = \frac{1}{2}mgR$$

ILLUSTRATION 5.31

Two particles of masses m and M are initially at rest at an infinite distance apart. They move towards each other and gain speeds due to gravitational attraction. Find their speeds when the separation between the masses becomes equal to d.

Sol. Let v_1 and v_2 be the speeds of two masses m and M respectively, when they are at a separation d.

As they approach each other, the kinetic energy increases and GPE decreases. Hence, for the system,

Loss in GPE = Gain in KE

$$\Rightarrow \quad (\text{GPE})_i - (\text{GPE})_f = \text{KE}_f - \text{KE}_i$$

$$\Rightarrow \quad 0 - \left(-\frac{GMm}{d}\right) = \left(\frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2\right) - 0$$

$$\frac{GMm}{d} = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2$$

As there is no external force on this system, its total momentum remains conserved.

$$P_i = P_f \Rightarrow 0 = mv_1 - Mv_2$$

Combining the two equations, we have

$$v_1 = \sqrt{\frac{2GM^2}{d(m+M)}} \text{ and } v_2 = \sqrt{\frac{2Gm^2}{d(m+M)}}$$

ILLUSTRATION 5.32

A body is released at a distance far away from the surface of the earth. Calculate its speed when it is near the surface of earth. Given $g = 9.8\ \text{m s}^{-2}$, radius of earth $R = 6.37 \times 10^6$ m.

Sol. Conservation of energy implies

$$0 + 0 = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2(R^2g)}{R}}$$

$$= \sqrt{2Rg} \approx 11.2 \times 10^3\ \text{m s}^{-1}$$

ILLUSTRATION 5.33

Calculate the velocity with which a body must be thrown vertically upward from the surface of the earth so that it may reach a height of $10R$, where R is the radius of the earth and is equal to 6.4×10^8 m. (earth's mass = 6×10^{24} kg, gravitational constant G = $6.7 \times 10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}$)

Sol. The gravitational potential energy of a body of mass m on earth's surface is

$$U(R) = -\frac{GMm}{11R}$$

where M is the mass of the earth (supposed to be concentrated at its centre) and R is the radius of the earth (distance of the particle from the centre of the earth). The gravitational energy of the same body at a height $10R$ from earth's surface, i.e., at a distance $11R$ from earth's centre is

$$U(11R) = -\frac{GMm}{11R}$$

Therefore, change in potential energy,

$$U(11R) - U(R) = -\frac{GMm}{11R} - \left(-\frac{GMm}{R}\right) = \frac{10}{11}\frac{GMm}{R}$$

This difference must come from the initial kinetic energy given to the body in sending it to that height. Now, suppose the body is thrown up with a vertical speed v, so that its initial kinetic energy is $1/2\ mv^2$. Then

$$\frac{1}{2}mv^2 = \frac{10}{11}\frac{GMm}{R} \quad \text{or} \quad v = \sqrt{\frac{20}{11}\frac{GM}{R}}$$

Putting the given values:

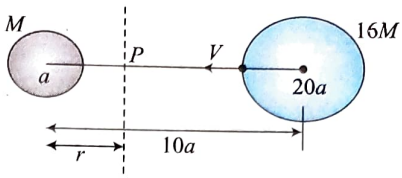
$$v = \sqrt{\left(\frac{20 \times (6.7 \times 10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}) \times (6 \times 10^{24}\ \text{kg})}{11(6.4 \times 10^6\ \text{m})}\right)}$$

$$= 1.07 \times 10^4\ \text{m s}^{-1}$$

ILLUSTRATION 5.34

Distance between centres of two stars is $10a$. The masses of these stars are M and $16M$ and their radii are a and $2a$, respectively. A body is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star?

Sol. Let P be the point on the line joining the centres of the two planets such that the net field at that point is zero.



Then, $\frac{GM}{r^2} - \frac{G(16M)}{(10a - r)^2} = 0$

$\Rightarrow \quad (10a - r)^2 = 16r^2$

$\Rightarrow \quad 10a - r = 4r$

$\Rightarrow \quad r = 2a$

Potential at point P,

$$V_P = \frac{-GM}{r} - \frac{G(16M)}{(10a - r)} = \frac{-GM}{2a} - \frac{2GM}{a} = \frac{-5GM}{2a}$$

Now if the particle projected from the larger planet has enough energy to cross this point, it will reach the smaller planet.

For this, the KE imparted to the body must be just enough to raise its total mechanical energy to a value which is equal to PE at point P, i.e.

$$\frac{1}{2}mv^2 - \frac{G(16M)m}{2a} - \frac{GMm}{8a} = mv_P$$

or $$\frac{v^2}{2} - \frac{8GM}{a} - \frac{GM}{8a} = -\frac{5GM}{2a}$$

or $$v^2 = \frac{45GM}{4a}$$

or $$v_{\min} = \frac{3}{2}\sqrt{\frac{5GM}{a}}$$

ILLUSTRATION 5.35

Estimate whether it takes more energy to get a satellite upto 1600 km above the earth than to put in orbit there (earth's radius is 6400 km). Does your answer remain same for height 3200 km or for height 4800 km?

Sol. Energy required to raise a satellite at height h,

$$E_1 = -\frac{GMm}{(R + h)} - \left(-\frac{GMm}{R}\right)$$

$$= GMm\left(\frac{1}{R} - \frac{1}{R + h}\right) = \frac{GMmh}{R(R + h)}$$

Energy required to put a satellite in orbit at that height = KE of satellite

$$E_2 = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{GM}{r}\right) = \frac{GMm}{2r} = \frac{GMm}{2(R + h)}$$

$$\frac{E_1}{E_2} = \frac{2(R + h)h}{R(R + h)} = \frac{2h}{R} = \frac{2 \times (1600\ \text{km})}{(6400\ \text{km})}$$

$$= \frac{1}{2} = 0.5$$

Clearly, the answer depends on height, if h = 3200 km

$$\frac{E_1}{E_2} = \frac{2h}{R} = \frac{2 \times (3200\ \text{km})}{(6400\ \text{km})} = 1.10$$

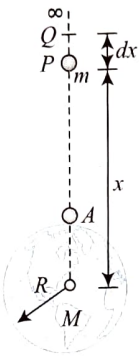
For height h = 4800 km

$$\frac{E_1}{E_2} = \frac{2 \times (4800\ \text{km})}{(6400\ \text{km})} = \frac{3}{2} = 1.5$$

ESCAPE SPEED

If we throw an object vertically upwards, it falls back to the surface of the earth after attaining a height depending upon the velocity of projection. If the object is given a sufficiently large velocity to overcome the gravitational pull and the resistance of earth's atmosphere, it may escape the gravitational pull of the earth and may never return to the earth. The minimum velocity with which a body must be projected up so as to enable it to just overcome the gravitational pull, is known as **escape velocity**.

Consider that the earth is a sphere of mass M and radius R. Suppose that the body of mass m is at the point P at a distance x from the centre of the earth [figure]. Then, the force F exerted by the earth on the body is given by



$$F = \frac{GMm}{x^2}$$

Therefore, work done in moving the body against gravitational attraction through an infinitesimally small distance dx is given by

$$dW = F\,dx$$

or $$dW = \frac{GMm}{x^2}dx \qquad ...(i)$$

Total work done to move the body from the surface of the earth to a point at infinity (where the gravitational pull would reduce to zero) can be obtained by integrating both the sides of Eq. (i) within the proper limits i.e., $x = R$ to $x = \infty$.

$$W = \int_R^\infty \frac{GMm}{x^2}dx = GMm\int_R^\infty x^{-2}dx = GMm\,x^{-1}\Big|_R^\infty$$

$$= -GMm\left|\frac{1}{\infty} - \frac{1}{R}\right| = \frac{GMm}{R}$$

The work required to project the body so as to escape the gravitational pull is performed on the body by providing an equal amount of kinetic energy to it at the surface of the earth. The kinetic energy will be just enough to provide it the necessary escape velocity.

If v_e is the required escape velocity, then kinetic energy which should be given to the body on the surface of earth is $\frac{1}{2}mv_e^2$.

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R} \text{ or } v_e = \sqrt{\frac{2GM}{R}} \qquad ...(ii)$$

The value of acceleration due to gravity at the surface of earth, $g = GM/R^2$. Therefore, setting $GM = gR^2$ in the Eq. (ii), we have

$$v_e = \sqrt{\frac{2gR^2}{R}}$$

or $$v_e = \sqrt{2gR} \qquad ...(iii)$$

Note: In deriving the expression for escape speed, we have neglected the resistance to the body by earth's atmosphere. Therefore in reality, the value of escape speed is slightly greater than the value calculated from these relations.

Important Points

- Escape speed depends on the mass and size of the planet. That is why escape velocity on the Jupiter is more than that on the earth.
- Escape speed is independent of the mass of the body.
- The value of escape speed of a body does not depend upon the mass (m) of the body and its angle of projection from the surface of earth or any other planet.
- The value of escape speed depends upon the mass and radius of the planet from the surface of which the body is to be projected. Clearly, the value of escape speed of a body will be different for different planets.

  For earth, $g = 9.8\,ms^{-2}$, $R = 6.4\times10^6$ m.

  $\therefore\ v_e = \sqrt{2gR_e} = \sqrt{2\times9.8\times6.4\times10^6}$

  $= 11.2\times10^3\ ms^{-1} = 11.2\ kms^{-1}$
- If a body is projected from a planet with a speed v which is smaller than the escape speed v_e (i.e., $v < v_e$) then the body will reach a certain height, may either move in an orbit around the planet or may fall back to the planet.
- If speed of projection (v) of the body from the surface of a planet is greater than the escape speed (v_e) of that planet, the body will escape out from the gravitational field of that planet and will move in the interstaller field of that planet with speed v' which can be obtained using law of conservation of energy, according to which

  $$\frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = \frac{1}{2}mv'^2 + 0$$

  $$v'^2 = v^2 - \frac{2GM}{R} = v^2 - v_e^2 \qquad \left[\because v_e = \sqrt{\frac{2GM}{R}}\right]$$

  $$v' = \sqrt{v^2 - v_e^2}$$

ILLUSTRATION 5.36

Calculate the escape speed on the surface of a planet of mass 7.5×10^{25} g and radius 1.6×10^6 m.
$G = 6.67 \times 10^{-8}$ dyne cm^2 g^{-2}.

Sol. Here, $G = 7.5 \times 10^{25}$ g, $R = 1.6 \times 10^8$ cm

Escape speed, $v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \times 6.67 \times 10^{-8} \times 7.5 \times 10^{25}}{1.6 \times 10^8}}$

$= 2.5006 \times 10^5$ cm s^{-1}

ILLUSTRATION 5.37

The escape speed from earth's surface is 11 kms^{-1}. A certain planet has a radius twice that of earth but its mean density is the same as that of the earth. Find the value of the escape speed from the planet.

Sol. Let M, R, ρ be the mass, radius and density of earth and M, R, ρ be the corresponding values for the planet. Then escape speed from earth's surface is

$$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2G}{R} \times \frac{4}{3}\pi R^3\rho} = \sqrt{\frac{8\pi G\rho R^2}{3}}$$

Escape speed from planet's surface is

$$v_e' = \sqrt{\frac{8\pi G\rho(2R)^2}{3}} = 2\sqrt{\frac{8\pi G\rho R^2}{3}} = 2v_e = 2 \times 11 = 22 \text{ km/s}$$

ILLUSTRATION 5.38

A spaceship goes into a circular orbit close to the earth's surface. What additional velocity must be imparted to the ship so that it is able to escape the gravitational pull of the earth? ($R = 6400$ km, $g = 9.8$ m s^{-2})

Sol. The orbital velocity in a circular orbit close to the earth is $v = \sqrt{gR}$.

The velocity required to escape $v_e = \sqrt{2gR}$.

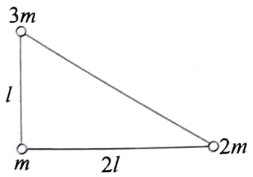
Hence additional velocity required is $v_e - v = (\sqrt{2} - 1)\sqrt{gR}$.

Therefore,

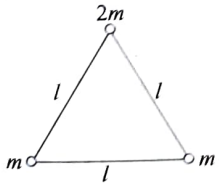
$$v_e - v = 0.414 \times \sqrt{9.8 \times 6400 \times 10^3}$$
$$= 3278.71 \text{ m/s} = 3.278 \text{ km s}^{-1}$$

CONCEPT APPLICATION EXERCISE 5.4

1. Does a planet revolving around the sun in elliptical orbit have a constant **(a)** linear speed, **(b)** angular momentum, (c) kinetic energy and **(d)** potential energy throughout its orbit?
2. Three mass points each of mass m are placed at the vertices of an equilateral triangle of side l. What is the gravitational field and potential due to three masses at the centroid of the triangle?
3. If g is the acceleration due to gravity on the earth's surface, what will be the gain in the potential energy of an object of mass m raised from the surface of the earth to a height equal to twice the radius of the earth?
4. Find the potential energy of the system of three particles.



5. Find the minimum energy required to disassemble the system of three particles.



6. With what velocity should an object be projected from the surface of the earth, so that its height above the surface becomes equal to the radius of the earth?
7. The masses and radii of the earth and moon are M_1, R_1 and M_2, R_2 respectively. Their centres are at a distance r apart. Find the minimum speed with which a body of mass m should be projected from a point mid-way between the centres of the earth and moon so as to escape their gravitational pull.
8. Two masses M_1 and M_2 are initially at rest at infinite distance apart. They approach each other due to gravitational interaction. Find their speed of approach at the instant, when they are distance r apart.
9. A space-ship is stationed on the planet venus. How much energy must be expanded on the space-ship to rocket it out of the solar system? Given that mass of the space-ship $m = 800$ kg, mass of the sun $M_s = 2 \times 10^{30}$ kg; mass of the venus $M_v = 5.0 \times 10^{24}$ kg, radius of the venus $R_v = 6{,}000$ km, radius of the orbit of the venus $r = 1.00 \times 10^{11}$ m and gravitational constant, $G = 6.6 \times 10^{-11}$ N m^2 kg^{-2}.

ANSWERS

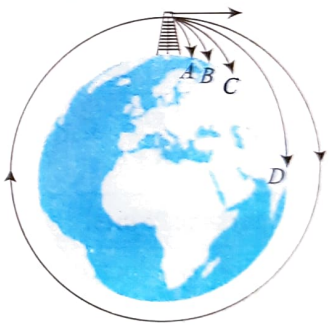
2. 0, $-3\sqrt{3}\,Gm/l$ **3.** $2mgR/3$ **5.** $5Gm^2/l$ **6.** $\sqrt{gR}$

7. $\sqrt{4G(M_1 + M_2)/r}$ **8.** $\sqrt{2G(M_1 + M_2)/r}$

9. 1.1×10^{12} J

ARTIFICIAL SATELLITE

Let us consider a several kilometers high tower on the surface of earth. A stone is thrown from the top of the tower with a horizontal direction with certain velocity, it may hit the surface of the earth at point A after following a parabolic path. Now the stone is thrown with a greater horizontal velocity, the stone will meet the earth at the point say B, a little away from the foot of the tower. If we keep on increasing the velocity of the projection, it will hit the surface of the earth always at a point a little further away, till at a certain horizontal velocity of the projection of the stone, it will be in a state of free fall under gravity i.e., it will be attempting to fall to the earth but missing it all the times. The result will be that the stone will follow a circular path around the earth. As the stone is moving around the earth, it may be called an artificial satellite of the earth.



In practice, an artificial satellite is launched into space with the help of at least a two stage rocket. The satellite is placed at the tip of the rocket. The first stage of the rocket takes the satellite vertically up against the gravity to the desired height of its orbit above the surface of the earth. Then the first stage of the rocket detaches, and the second stage tilts it through 90°. The second stage is used to impart required horizontal velocity to the rocket to make the satellite revolve in a circular orbit around the earth. This horizontal velocity is called **critical orbital velocity** or simply **orbital velocity** and depends on the height of the orbit. In fact, the magnitude of the horizontal velocity imparted to the satellite determines the successful launching of the satellite as explained below:

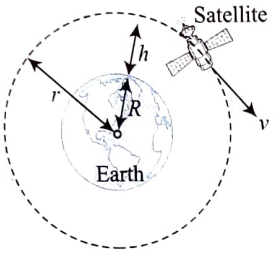
1. In case the horizontal velocity imparted is less than the critical orbital velocity, the satellite will follow a parabolic path and fall to the surface of the earth.
2. In case the horizontal velocity imparted is precisely equal to the critical orbital velocity, the satellite will move in the desired circular orbit around the earth.

3. In case the horizontal velocity imparted is greater than the critical orbital velocity but less than the escape velocity, the satellite will move in an elliptical orbit.
4. In case the horizontal velocity imparted is greater than the escape velocity, the satellite will leave the gravitational field of the earth and escape into the space.

A satellite is a body which is constantly revolving in an orbit around a comparatively much larger body.

Orbital Velocity of Satellite

Satellites are natural or artificial bodies describing orbit around a planet under its gravitational attraction. Moon is a natural satellite while INSAT-1B is an artificial satellite of earth. Condition for establishment of artificial satellite is that the centre of orbit of satellite must coincide with centre of earth or satellite must move around great circle of earth.



Orbital velocity of a satellite is the velocity required to put the satellite into its orbit around the earth.

For revolution of satellite around the earth, the gravitational pull provides the required centripetal force.

$$\frac{mv^2}{r}=\frac{GMm}{r^2}\Rightarrow v=\sqrt{\frac{GM}{r}}$$

$$v=\sqrt{\frac{gR^2}{R+h}}=R\sqrt{\frac{g}{R+h}}\quad[\text{As } GM=gR^2 \text{ and } r=R+h]$$

Important Points:

- Orbital velocity is independent of the mass of the orbiting body and is always along the tangent of the orbit i.e., satellites of different masses have same orbital velocity, if they are in the same orbit.
- Orbital velocity depends on the mass of central body and radius of orbit.
- For a given planet, greater the radius of orbit, lesser will be the orbital velocity of the satellite $\left(v\propto 1/\sqrt{r}\right)$.
- Orbital velocity of the satellite when it revolves very close to the surface of the planet

$$v=\sqrt{\frac{GM}{r}}=\sqrt{\frac{GM}{R+h}}$$

$$\therefore\quad v=\sqrt{\frac{GM}{R}}=\sqrt{gR}\qquad[\text{As } h\approx 0 \text{ and } GM=gR^2]$$

For the earth,

$$v=\sqrt{9.8\times 6.4\times 10^6}=7.9\text{ km/s}\approx 8\text{ km/s}$$

- Close to the surface of planet $v=\sqrt{\frac{GM}{R}}$

$$\therefore\quad v=\frac{v_e}{\sqrt{2}}\text{ i.e., }v_{escape}=\sqrt{2}\,v_{orbital}\qquad\left[\text{As } v_e=\sqrt{\frac{2GM}{R}}\right]$$

It means that if the speed of a satellite orbiting close to the earth is made $\sqrt{2}$ times (or increased by 41%) then it will escape from the gravitational field.

Time Period of Satellite

It is the time taken by satellite to go once around the earth.

$$\therefore\quad T=\frac{\text{Circumference of the orbit}}{\text{orbital velocity}}$$

$$\Rightarrow\quad T=\frac{2\pi r}{v}=2\pi r\sqrt{\frac{r}{GM}}\qquad\left[\text{As } v=\sqrt{\frac{GM}{r}}\right]$$

$$\Rightarrow\quad T=2\pi\sqrt{\frac{r^3}{GM}}=2\pi\sqrt{\frac{r^3}{gR^2}}\qquad[\text{As } GM=gR^2]$$

$$\Rightarrow\quad T=2\pi\sqrt{\frac{(R+h)^3}{gR^2}}=2\pi\sqrt{\frac{R}{g}}\left(1+\frac{h}{R}\right)^{3/2}\qquad[\text{As } r=R+h]$$

Important Points:

- From $T=2\pi\sqrt{\frac{r^3}{GM}}$, it is clear that time period is independent of the mass of orbiting body and depends on the mass of central body and radius of the orbit

  Also, $T=2\pi\sqrt{\frac{r^3}{GM}}\Rightarrow T^2=\frac{4\pi^2}{GM}r^3$ i.e., $T^2\propto r^3$

  This is in accordance with Kepler's third law of planetary motion, r becomes a (semi major axis) if the orbit is elliptic.
- Time period of nearby satellite:

  From $T=2\pi\sqrt{\frac{r^3}{GM}}=2\pi\sqrt{\frac{R^3}{gR^2}}=2\pi\sqrt{\frac{R}{g}}$

  $[\text{As } h\approx 0 \text{ and } GM=gR^2]$

  For earth $R=6400\text{ km}$ and $g=9.8\text{ m/s}^2$

  $T=84.6\text{ minute}\approx 1.4\text{ h}$
- Time period of nearby satellite in terms of density of planet can be given as

$$T=2\pi\sqrt{\frac{r^3}{GM}}=2\pi\sqrt{\frac{R^3}{GM}}=\frac{2\pi(R^3)^{1/2}}{\left[G.\frac{4}{3}\pi R^3\rho\right]^{1/2}}=\sqrt{\frac{3\pi}{G\rho}}$$

Illustration 5.39

If T is the period of a satellite revolving just above the surface of a planet, whose average density is ρ, show that ρT^2 is a universal constant.

Sol. The time period of a satellite revolving around a planet of mass M and radius R at a distance x above its surface is given by

$$T = 2\pi\sqrt{\frac{(R+x)^3}{GM}}$$

When the satellite is revolving just above the surface of the planet, $(R+x) \approx R$. Then, $T = 2\pi\sqrt{\frac{R^3}{GM}}$

Now, $M = \frac{4}{3}\pi R^3\rho$

$$T = 2\pi\sqrt{\frac{R^3}{G\times\frac{4}{3}\pi R^3\rho}} = \sqrt{\frac{3\pi}{G\rho}}$$

or $\rho T^2 = \frac{3\pi}{G}$ = a universal constant

ILLUSTRATION 5.40

Assume the radius of the earth to be 6.4×10^6 m.

(a) Calculate the time period T of a satellite on equatorial orbit at 1.4×10^6 m above the surface of the earth.

(b) What is the speed of the satellite in this orbit?

(c) If the satellite is travelling in the same direction as the rotation of the earth, i.e., west to east, what is the interval between two successive times at which it will appear vertically overhead to an observer at a fixed point on the equator?

Sol.

(a) $r = R_{earth} + h = 6.4 \times 10^6 + 1.4 \times 10^6 = 7.8 \times 10^6$ m

$$\Rightarrow r = 7.8 \times 10^6 \text{ m} \Rightarrow T = \left(\frac{4\pi^2 r^3}{GM_{earth}}\right) = 6831 \text{ s}$$

(b) Speed of satellite, $v = \sqrt{\frac{GM_{earth}}{r}} = 7174$ m s^{-1}

(c) Relative angular velocity $= \omega_{satellite} - \omega_{earth}$

$$\text{Thus } T = \frac{2\pi}{(\omega_s - \omega_e)} = \frac{2\pi}{\left[\frac{2\pi}{T_s} - \frac{2\pi}{T_e}\right]}$$

$$\Rightarrow T = \frac{T_s T_e}{T_e - T_s} = 7417 \text{ s}$$

ILLUSTRATION 5.41

Two satellites A and B of equal mass move in the equatorial plane of the earth, close to earth's surface. Satellite A moves in the same direction as that of the rotation of the earth while satellite B moves in the opposite direction. Calculate the ratio of the kinetic energy of B to that of A in the reference frame fixed to the earth. ($g = 9.8$ m s^{-2} and radius of the earth $= 6.37 \times 10^6$ km)

Sol. Let ω_A and ω_B be the absolute angular speeds of A and B. Since they are in the same orbit, their time periods must be the same, i.e., $\omega_A = \omega_B$. Considering the dynamics of circular motion in the cases,

$$m\omega_A^2 R = \frac{GMm}{R^2} \Rightarrow \omega_A = \sqrt{\frac{g}{R}} \qquad (\because GM = gR^2)$$

Similarly, $\omega_B = \sqrt{\frac{g}{R}}$

And $\omega_A = \omega_B = \sqrt{\frac{9.8}{6.37 \times 10^6}} = 124 \times 10^{-5}$ rad s^{-1}

Now, $\omega_{AE} = \omega_A - \omega_E = 124 \times 10^{-5} - 7.3 \times 10^{-5}$

$= 116.7 \times 10^{-5}$ rad s^{-1}

ω_{BE} (velocity of B relative to E) $= \omega_B - (-\omega_E) = \omega_B + \omega_E$

$= 131.3 \times 10^{-5}$ rad s^{-1}

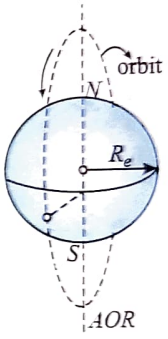
Therefore, $\frac{K_B}{K_A} = \frac{\frac{1}{2} m\,\omega_{BE}^2 r^2}{\frac{1}{2} m\,\omega_{AE}^2 r^2} = \frac{131.3^2}{116.7^2} = 1.27$

ILLUSTRATION 5.42

If a satellite is revolving around the earth in a circular orbit in a plane containing earth's axis of rotation. If the angular speed of satellite is equal to that of earth, find the time it takes to move from a point above north pole to a point above the equator.

Sol. A satellite which rotates with angular speed equal to earth's rotation has an orbit radius $7R_e$ and the angular speed of revolution is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86400} = 7.27\times10^{-5} \text{ rad/s}$$



When satellite moves from a point above north pole to a point above equator, it traces an angle $\frac{\pi}{2}$, this time taken is

$$t = \frac{\pi/2}{\omega} = 21600\,s = 6 \text{ h}$$

HEIGHT OF SATELLITE

As we know, time period of satellite,

$$T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}$$

By squaring and rearranging both sides

$$\frac{gR^2T^2}{4\pi^2} = (R+h)^3 \Rightarrow h = \left(\frac{T^2gR^2}{4\pi^2}\right)^{1/3} - R$$

By knowing the value of time period we can calculate the height of satellite from the surface of the earth.

ANGULAR MOMENTUM, L

When a satellite of mass m is orbiting with linear speed v on the orbital path of radius r around the earth, its angular momentum is given by

$$L = mvr = mr\sqrt{GM/r} = [m^2GMr]^{1/2}$$

From the above equation, it is clear that angular momentum of a satellite depends on both the mass of the satellite (m) and

the mass of the planet (M). It also depends upon the radius of the orbit (r) of the satellite.

Energy of an Orbiting Satellite

The total mechanical energy of a satellite revolving around the earth is the sum of its potential energy (U) and kinetic energy (K).

The potential energy of a satellite is due to its position with respect to earth. It appears because of the gravitational pull acting on the satellite due to earth. If a satellite of mass m is revolving around the earth of mass M, radius R, with orbital velocity v, in an orbit of radius r, then potential energy of the satellite is

$$U = -\frac{GMm}{r}$$

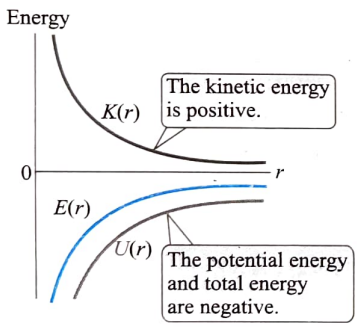
The kinetic energy of a satellite,

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r} \qquad \left[\because v = \sqrt{\frac{GM}{r}}\right]$$

Thus, total mechanical energy of a satellite is

$$E = U + K = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r} = -\frac{GMm}{2r}$$

$$\Rightarrow \quad E = -\frac{GMm}{2r} = -\frac{GMm}{2(R+h)}$$



If the radius r of the orbit of the satellite decreases, then it follows that

(i) its kinetic energy will increase i.e., its orbital velocity will increase,

(ii) its potential energy will decrease (as it becomes more negative) and

(iii) its total energy will decrease (as it also becomes more negative).

If the satellite is orbiting close to earth, then $r \approx R$. Now, total energy of the satellite is $E = -GMm/2R$.

The above expression has been obtained considering the circular orbit of the satellite. When the orbit of satellite becomes elliptical, both the PE and KE of the satellite vary from point to point but the total energy remains constant and is negative, as in the case of circular orbit.

The total energy of a circularly orbiting satellite which is negative, the potential energy is negative and magnitude of potential energy is twice the magnitude of the positive kinetic energy. We know that the total energy of a body is positive or zero, when it is at infinity from the earth. Therefore, the satellite will escape to infinity if its total energy becomes zero or positive.

Since a satellite is always at finite distance from the earth, its total energy can never be positive or zero.

Binding Energy of a Satellite

The energy required to remove the satellite from its orbit around the earth to infinity is called binding energy of the satellite. Binding energy is equal to negative value of total mechanical energy of a satellite in its orbit. Thus, binding energy is,

$$-E = \frac{GMm}{2r}$$

Illustration 5.43

A satellite orbits the earth at a height of 3.6×10^6 m from its surface. Compute its **(a)** kinetic energy, **(b)** potential energy, **(c)** total energy. Mass of the satellite = 500 kg, mass of the earth = 6×10^{24} kg, radius of the earth = 6.4×10^6 m, $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

Sol. Here, $r = R + h = 6.4 \times 10^6 + 3.6 \times 10^6 = 10^7$ m.

Orbital velocity of the satellite around the earth is given by

$$v_0 = \sqrt{\frac{GM}{(R+h)}} = \sqrt{\frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{10^7}}$$

$$= \sqrt{6.67 \times 6 \times 10^6} \text{ m s}^{-1}$$

(a) KE of the satellite $= \frac{1}{2}mv_0^2$

$$= \frac{1}{2} \times 500 \times 6.67 \times 6 \times 10^6 = 10^{10} \text{ J}$$

(b) PE of the satellite $= -\frac{GMm}{R+h}$

$$= -\frac{6.67 \times 10^{-11} \times (6 \times 10^{24}) \times 500}{10^7} = -2 \times 10^{10} \text{ J}$$

(c) Total energy = KE + PE = $10^{10} - 2 \times 10^{10} = -10^{10}$ J

Illustration 5.44

A satellite of mass 1000 kg is rotating around the earth in a circular orbit of radius $3R$. What extra energy should be given to this satellite if it is to be lifted into an orbit of radius $4R$?

Sol. Energy required = $(TE)_f - (TE)_i$

$$= \left(\frac{-GMm}{2(4R)}\right) - \left(\frac{-GMm}{2(3R)}\right)$$

$$= \frac{-GMm}{8R} + \frac{GMm}{6R} = \frac{GMm}{24R}(4-3) = \frac{GMm}{24R}$$

$$= \left(\frac{9.8 \times (6400 \times 10^3)^2 \times 1000}{24(6400 \times 10^3)}\right) = 2.614 \times 10^9 \text{ J}$$

Illustration 5.45

The Earth may be considered to be a uniform sphere with its mass M concentrated at its centre. A satellite of mass m orbits the Earth, such that the radius of the circular orbit is r. The total energy of the satellite gradually decreases. State and explain the effect of this decrease on **(i)** the radius r of the orbit and **(ii)** the linear speed v of the satellite.

Sol. The total energy of the satellite, $E = -\frac{GMm}{2r}$

(i) The total energy of the satellite is negative. If the total energy of the satellite gradually decreases, it implies that it becomes more negative i.e. $\frac{GMm}{2r}$ increases. It will happen so, if r decreases.

(ii) Also, the kinetic energy of the satellite, $E_K = \frac{GMm}{2r}$

When r decreases, it follows that E_K will increase. Since $E_K = \frac{1}{2}mv^2$, it follows that the linear speed v of the satellite will increase.

GEOSTATIONARY AND POLAR SATELLITES

Artificial satellites are of two types: Geostationary satellites and Polar satellites.

(a) A satellite which revolves around the Earth in equatorial plane in the same direction (anticlockwise) with the same angular speed as that of Earth about its own axis is called a **geostationary satellite** or **geosynchronous satellite**.

The speed of such a satellite relative to Earth is zero and as such it appears stationary with respect to any point on the surface of the Earth. The orbit in which stationary satellite revolves is called the **geostationary orbit** or **geosynchronous orbit**. Further, being in the equatorial plane, it is also called an **equatorial orbit**. Obviously, the time period of revolution of such a satellite is the same as the time period of rotation of Earth about its own axis, i.e., approximately 24 hours (exactly, 23 h, 56 min, 4.1 s). We know that the altitude of a satellite is given by

$$h = \left(\frac{T^2R^2g}{4\pi^2}\right)^{\frac{1}{3}} - R \qquad ...(i)$$

Here, $T = 24\text{ h} = 24\times60\times60 = 86400\text{ s}$

R = radius of Earth = 6400 km

$g = 9.8\text{ m/s}^2 = 9.8\times10^{-3}\text{ km/s}^2$

Substituting these values in Eq. (i), we get

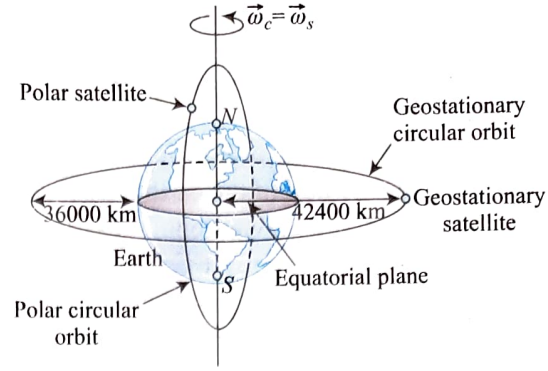
$h = (42400 - 6400)\text{ km} = 36000\text{ km}$

Obviously, **a geostationary satellite has always to be at an altitude of 36000 km, as shown in figure.** The **orbital velocity** of the geostationary satellite can easily be found out from the consideration that it covers a distance equal to $2\pi \times 42400$ km in 24 hours. This comes out to be 3.1 km per second.

Thus, **for a satellite to be geostationary**:

(i) it should rotate in the **equatorial plane** and

(ii) $\vec{\omega}_e$ (angular velocity of Earth) = $\vec{\omega}_s$ (angular velocity of the satellite), i.e., the satellite should have a time period of 24 h and should rotate in the direction of Earth rotation (i.e., from west to east), as shown in figure.



Geostationary satellites are used for the purpose of communication where they act as reflectors of suitable waves carrying the messages. Such satellites are called **communication satellites**. Three geostationary communication satellites placed at the vertices of an equilateral triangle can hold the entire world in view. Television programs and telephone conversations can thus be transmitted across the world.

(b) **Polar satellites** are low to medium altitude satellites (about 1000 km) from the Earth's surface.

In a polar orbit, the artificial satellite moves around the Earth in such a way that it passes over the north and south poles of the Earth repeatedly as the time period of such a satellite is less than 2 hours. Since a polar satellite is close to the Earth, it observes large areas of the Earth around the point over which it passes.

"A polar satellite orbits in a north–south direction as the Earth spins below it in the east–west direction. As a result, the satellite scans the entire surface of the Earth."

For example, a satellite in low polar orbit which flies around the Earth every 90 minutes, completes 16 orbits in one day and crosses the equator at an angle of 90°. During this period, the Earth will have rotated just once. Thus, "the satellite has covered every part of the world in one day. Weather monitoring satellites and spy satellites are almost always in low flying polar orbits (500–800 km)."

WEIGHTLESSNESS

The weight of a body is the force with which it is attracted towards the centre of earth. When a body is stationary with respect to the earth, its weight equals the gravity. This weight of the body is known as its static or true weight.

We become conscious of our weight, only when our weight (which is gravity) is opposed by some other object. Actually, the secret of measuring the weight of a body with a weighing machine lies in the fact that as we place the body on the machine, the weighing machine opposes the weight of the body. The reaction of the weighing machine to the body gives the measure of the weight of the body.

The state of weightlessness can be observed in the following situations.

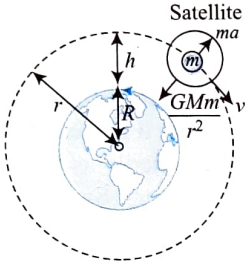
1. **When objects fall freely under gravity:** For example, a lift falling freely, or an airship showing a feat in which it falls freely for a few seconds during its flight, are in state of weightlessness.
2. **When a satellite revolves in its orbit around the earth:** Weightlessness poses many serious problems to the

astronauts. It becomes quite difficult for them to control their movements. Everything in the satellite has to be kept tied down. Creation of artificial gravity is the answer to this problem.

3. **When bodies are at null points in outer space:** On a body projected up, the pull of the earth goes on decreasing, but at the same time the gravitational pull of the moon on the body goes on increasing. At one particular position, the two gravitational pulls may be equal and opposite and the net pull on the body becomes zero. This is zero gravity region or the null point and the body in question is said to appear weightless.

WEIGHTLESSNESS IN A SATELLITE

A satellite, which does not produce its own gravity moves around the earth in a circular orbit under the action of gravity. The acceleration of satellite is $\frac{GM}{r^2}$ towards the centre of earth.



If a body of mass m placed on a surface inside a satellite moving around the earth. Then force on the body are

(i) The gravitational pull of earth $= \frac{GMm}{r^2}$

(ii) The reaction by the surface $= R$

By Newton's law $\frac{GmM}{r^2} - R = ma$

$$\frac{GmM}{r^2} - R = m\left(\frac{GM}{r^2}\right)$$

$\therefore \quad R = 0$

Thus, the surface does not exert any force on the body and hence its apparent weight is zero.

A body needs no support to stay at rest in the satellite and hence all position are equally comfortable. Such a state is called weightlessness.

Condition of weightlessness can be experienced only when the mass of satellite is negligible so that it does not produce its own gravity.

For example, Moon is a satellite of earth but due to its own weight it applies gravitational force of attraction on the body placed on its surface and hence weight of the body will not be equal to zero at the surface of the moon.

ILLUSTRATION 5.46

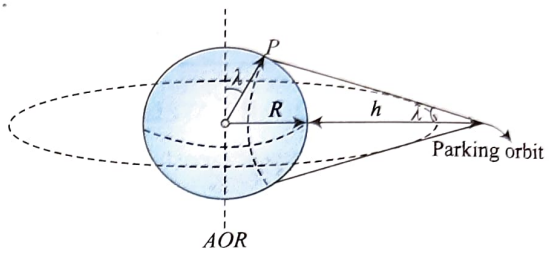
The astronauts in the satellite orbiting the earth feel weightlessness. Does the weightlessness depend upon the distance of the satellite from the earth? If so, how? Explain you answer.

Sol. A satellite revolving in an orbit of any radius experiences weightlessness. It is because, when a satellite revolves in an orbit, its weight is used up in providing the necessary centripetal force. Thus, the weightlessness does not depend upon the distance of the satellite from the earth.

ILLUSTRATION 5.47

Find the minimum co-latitude which can directly receive a signal from a geostationary satellite.

Sol. The farthest point on earth, which can receive signals from the parking orbit is the point where a length is drawn on earth surface from satellite as shown in figure.



The colatitude λ of point P can be obtained from figure as

$$\sin\lambda = \frac{R_e}{R_e + h} \simeq \frac{1}{7}$$

We know for a parking orbit $h \simeq 6R_e$

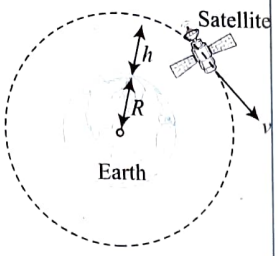
Thus, we have $\lambda = \sin^{-1}\left(\frac{1}{7}\right)$

CONCEPT APPLICATION EXERCISE 5.5

1. Air friction increases the velocity of the satellite. Explain.
2. Two satellites move along a circular orbit in the same direction at a small distance from each other. A container has to be thrown from the first satellite on the second one. When will the container reach the second satellite earlier, if it is thrown in the direction of motion of the first satellite or in the opposite direction? The velocity of the container is small in comparison to that of the satellite.
3. Two satellites of the same mass are launched in the same orbit around the earth so as to rotate opposite to each other. They collide inelastically and stick together as wreckage. Obtain the total energy of the system before and just after the collision. Describe the subsequent motion of the wreckage.
4. If a satellite is revolving around a planet of mass M in an elliptical orbit of semi-major axis a. Show that the orbital speed of the satellite when it is at a distance r from the planet will be given by $v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$.
5. An artificial satellite is in a circular orbit at 500 km above the earth's surface. **(i)** What is the acceleration of the satellite? **(ii)** What is the acceleration due to gravity at any point along the satellite's path? Take radius of the earth as 6.5×10^6 m.
6. A satellite of mass m moves in a circular orbit of radius r around the earth. Calculate the total energy required to

place the satellite in the orbit from the earth's surface. Assuming initially it to be at rest.

7. A satellite is lifted from earth to a height h. Let the work done by the external agent be W_1. Then it is projected with a speed v so as to move in a circular orbit doing an extra work W_2. Compare these works done.



8. A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit to overcome the gravitational pull?
[Radius of the earth = 6,400 km, $g = 9.8$ m s^{-2}]

9. A geostationary satellite is orbiting the earth at a height $6R$ above the surface of the earth; where R is radius of the earth. Find the time period of another satellite orbiting at a height $2.5R$ from the surface of the earth.

10. An artificial satellite is moving in a circular orbit around the Earth with a speed equal to half the magnitude of escape velocity from the Earth.
(a) Determine the height of the satellite above the Earth's surface.
(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely on the Earth, find the speed with which it hits and surface of Earth. Given M = mass of Earth & R = radius of earth.

ANSWERS

5. (i) 8.45 ms^{-2} (ii) 8.45 ms^{-2} **6.** $mg[R-(R^2/2r)]$
7. $W_1/W_2 = 2h/R$ **8.** 3.28 km s^{-1}
9. $6(2)^{1/2}$ h **10.** (a) 6400 km (b) 7.92 km/s

Solved Examples

EXAMPLE 5.1

A body suspended on a spring balance in a ship weighs W_0 when the ship is at rest. When the ship begins to move along the equator with a speed v, show that the scale reading is very close to $W_0(1 \pm 2\omega v/g)$, where ω is the angular speed of the earth.

Sol. Let W_0 be weight of the body when ship is at rest then

$$W_0 = F - \frac{mv_e^2}{R}$$

where F = gravitational pull of the earth when it is at rest and v_e = velocity of the earth at the equator when both move due east, the net speed of the ship

$$v' = v_e + v$$

Therefore, the apparent weight $W = F - \dfrac{m(v_e+v)^2}{R}$

$$W - W_0 = \frac{mv_e^2}{R} - \frac{m(v_e+v)^2}{R} = \frac{mv_e^2}{R}\left[1-\left(1+\frac{v}{v_e}\right)^2\right]$$

As $v \ll v_e$, $W - W_0 = \dfrac{mv_e^2}{R}\left[1-\left(1+\dfrac{2v}{v_e}\right)\right] = -\dfrac{2mv_e v}{R}$

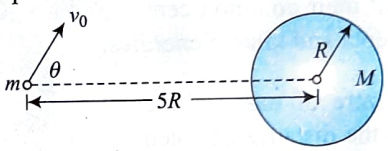
As $m = \dfrac{W_0}{g}$ and $\omega = \dfrac{v_e}{R} \Rightarrow W = W_0\left(1-\dfrac{2\omega v}{g}\right)$

Similarly, if the ship moves due west, then the net velocity of the ship is $v' = v_e - v$

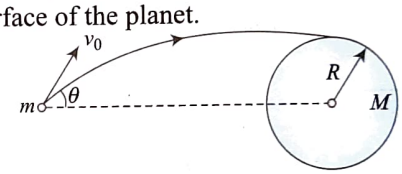
$$W = F - \frac{m(v_e-v)^2}{R} \Rightarrow W = W_0\left(1+\frac{2\omega v}{g}\right)$$

EXAMPLE 5.2

A spaceship is sent to investigate a planet of mass M and radius R. While hanging motionless in space at a distance $5R$ from the centre of the planet, the spaceship fires an instrument package of mass m, which is much smaller than the mass of the spaceship. For what angle θ will the package just graze the surface of the planet?



Sol. Let the speed of the instrument package be v when it grazes the surface of the planet.



Conserving angular momentum of the package about the centre of the planet

$$mv_0 \times 5R\sin\theta = mvR$$

$$\Rightarrow v = 5v_0\sin\theta \quad \text{...(i)}$$

Conserving mechanical energy, we get

$$-\frac{GMm}{5R} + \frac{1}{2}mv_0^2 = -\frac{GMm}{R} + \frac{1}{2}mv^2$$

$$\Rightarrow \frac{1}{2}m(v^2 - v_0^2) = \frac{4GMm}{5R} \Rightarrow v^2 - v_0^2 = \frac{8GM}{5R} \quad \text{...(ii)}$$

Substituting the value of v from Eq. (i) in Eq. (ii), we have

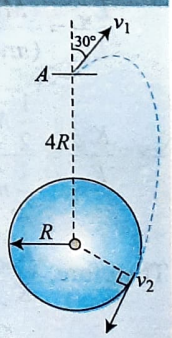
$$25v_0^2\sin^2\theta - v_0^2 = \frac{8GM}{5R}$$

$$\Rightarrow \theta = \sin^{-1}\left[\frac{1}{5}\sqrt{1+\frac{8GM}{5v_0^2R}}\right]$$

EXAMPLE 5.3

A particle is projected from point A, that is at a distance $4R$ from the centre of the Earth, with speed v_1 in a direction making 30° with the line joining the centre of the Earth and point A, as shown. Find the speed v_1 of particle if particle passes grazing the surface of the earth. Consider gravitational interaction only between these two.

$\left(\text{Use } \dfrac{GM}{R} = 6.4\times10^7 \text{ m}^2/\text{s}^2\right)$



Sol. Using conservation of angular momentum:

$$m(v_1 \cos 60°)4R = mv_2 R \Rightarrow \frac{v_2}{v_1} = 2$$

By conservation of energy system

$$-\frac{GMm}{4R} + \frac{1}{2}mv_1^2 = -\frac{GMm}{R} + \frac{1}{2}mv_2^2$$

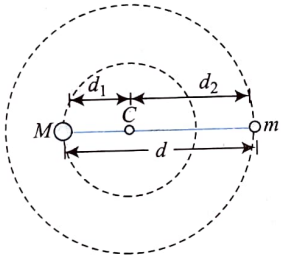
$$\Rightarrow \quad \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = \frac{3}{4}\frac{GM}{R}$$

$$\Rightarrow \quad v_1 = \frac{1}{\sqrt{2}}\sqrt{64\times10^6} = \frac{8000}{\sqrt{2}}\text{ m/s}$$

EXAMPLE 5.4

A pair of stars rotates about a common centre of mass. One of the stars has a mass M and the other m. Their centres are a distance d apart, d being large compared to the size of either star. Derive an expression for the period of revolution of the stars about their common centre of mass. Compare their angular momenta and kinetic energies.

Sol. The centre of mass of the system divides the distance between the stars in the inverse ratio of their masses. If d_1 and d_2 are the distance force for their circular motion,



$$d_1 = \frac{d}{M+m}\times m$$

and $$d_2 = \frac{d}{M+m}\times M \Rightarrow \frac{d_1}{d_2} = \frac{m}{M}$$

the stars will rotate in circles of radii d_1 and d_2 about their centre of mass. The same force of attraction provides the necessary centripetal force for their circular motion.

$$\therefore \quad \frac{GmM}{d^2} = M\omega_1^2 d_1 = m\omega_2^2 d_2$$

or $$\omega_1^2 = \frac{Gm}{d^2 d_1} = \frac{Gm}{d^2}\times\frac{M+m}{d\times m} = \frac{G(M+m)}{d^3}$$

and $$\omega_2^2 = \frac{GM}{d^2 d_2} = \frac{GM}{d^2}\times\frac{M+m}{d\times M} = \frac{G(M+m)}{d^3}$$

$$\omega_1 = \omega_2 = \sqrt{\frac{G(M+m)}{d^3}}$$

From the fact that the moment of momentum is also the angular momentum,

$$\frac{L_M}{L_m} = \frac{(Mv_1)d_1}{(mv_2)d_2} = \frac{M}{m}\times\frac{d_1^2}{d_2^2} \Rightarrow \frac{L_M}{L_m} = \frac{m}{M}$$

$$\Rightarrow \quad \frac{K_M}{K_m} = \frac{\frac{1}{2}Mv_1^2}{\frac{1}{2}mv_2^2} = \frac{M}{m}\frac{\omega_1^2 d_1^2}{\omega_2^2 d_2^2} = \frac{M}{m}\frac{d_1^2}{d_2^2} \quad (\because \omega_1 = \omega_2)$$

Therefore, $$\frac{K_M}{K_m} = \frac{M}{m}\left(\frac{m}{M}\right)^2 = \frac{m}{M}$$

EXAMPLE 5.5

A satellite revolving in a circular equatorial orbit of radius $R = 2.0 \times 10^4$ km from west to east appears over a certain point at the equator every 11.6 h. From these data, calculate the mass of the earth. ($G = 6.67 \times 10^{-11}$ N m^2)

Sol. Angular speed of the satellite relative to the earth

$$\omega_{SE} = \omega_S - \omega_E \quad \Rightarrow \omega_S = \omega_{SE} + \omega_E$$

(because they are moving in same direction west to east)

$$\omega_S = \omega_{SE} + \omega_E = \frac{2\pi}{T_{SE}} + \frac{2\pi}{T_E}$$

Given $T_{SE} = 11.6$ h and $T_E = 24$ h

$$\omega_S = 2\pi\left[\frac{1}{11.6} + \frac{1}{24}\right]\text{h}^{-1} = \frac{2\pi\times(35.6)}{24\times11.6\times3600}\text{ rad s}^{-1}$$

$$= 2.23\times10^{-4}\text{ rad s}^{-1}$$

The condition of the circular orbit around the earth is

$$\frac{GM_e m}{R^2} = m\omega_S^2 R \text{ or } GM_e = \omega_S^2 R^3$$

$$R = 2.0\times10^4\text{ km} = 2.0\times10^7\text{ m}$$

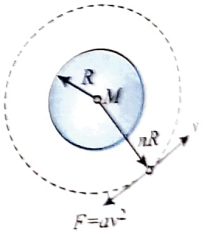
$$M_e = \frac{(2.23\times10^{-4})^2\times(2.0\times10^7)^3}{6.67\times10^{-11}} \approx 6\times10^{24}\text{ kg}$$

EXAMPLE 5.6

An artificial satellite of mass m of a planet of mass M, revolves in a circular orbit whose radius is n times the radius R of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming resistance force on the satellite depends on velocity as $F = av^2$ where a is constant, calculate the time the satellite will stay in orbit before it falls onto the planet's surface.

Sol. $$F = m\frac{dv}{dt} = av^2$$

At $t = 0$, when the satellite is at a distance nR from the centre of the earth. Velocity of the satellite is



$$v_i = \sqrt{\frac{GM}{nR}}$$

At time t when the satellite falls on the earth surface, the velocity of the satellite is $v_f = \sqrt{\frac{GM}{R}}$

From Eq. (i), $m\frac{dv}{v^2} = a\,dt$

Integrating both the sides, we get

$$m\int_{v_i}^{v_f}\frac{dv}{v^2} = a\int_0^t dt \Rightarrow m\left[\frac{v^{-2+1}}{-2+1}\right]_{v_i}^{v_f} = at$$

On solving, we get $t = \frac{m\sqrt{R}(\sqrt{n}-1)}{a\sqrt{(GM)}}$

EXAMPLE 5.7

A satellite of mass m is orbiting the earth in a circular orbit of radius r. it starts losing energy due to small air resistance at the rate of C J s^{-1}. Find the time taken for the satellite to reach the earth.

Sol. Energy of the satellite, $E = -\dfrac{GMm}{2r}$

Rate of change of energy of the satellite,

$$-\frac{dE}{dt} = \frac{GMm}{2}\frac{1}{r^2}\frac{dr}{dt} \qquad ...(i)$$

Negative sign as the satellite is losing its energy,

Rearranging Eq. (i), $dt = \dfrac{GMm}{2C}\cdot\dfrac{dr}{r^2}$

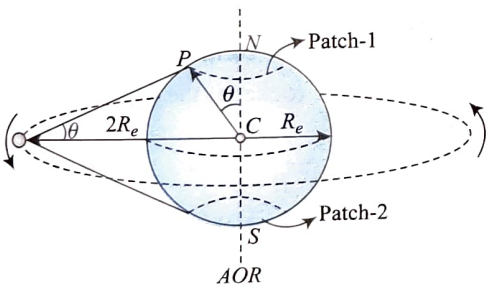
Integrating both sides, $\displaystyle\int_0^t dt = \frac{GMm}{2C}\cdot\int_r^R \frac{dr}{r^2}$

We get, $t = \dfrac{GMm}{2C}\left[\dfrac{1}{R}-\dfrac{1}{r}\right] = \dfrac{GMm}{2C}\left[\dfrac{1}{R}-\dfrac{1}{r}\right]$

EXAMPLE 5.8

A satellite is orbiting around the earth in an orbit in equatorial plane of radius $2R_e$ where R_e is the radius of earth. Find the area on earth, this satellite covers for communication purpose in its complete revolution.

Sol. As shown in figure when satellite S revolves, it covers a complete circular belt on earth's surface for communication.



If the co-latitude of the farthest point on surface up to which signals can be received (point P) is θ then we have

$$\sin\theta = \frac{R_e}{2R_e} = \frac{1}{2} \text{ or } \theta = \frac{\pi}{6}$$

During revolution satellite leaves two spherical patches 1 and 2 on earth surface at north and south poles where no signals can be transmitted due to curvature. The areas of these patches can be obtained by solid angles. The solid angle subtended by a path on earth's centre is

$$\Omega = 2\pi(1-\cos\theta) = \pi(2-\sqrt{3})\,\text{st}$$

Area of path 1 and 2 is $A_p = \Omega R_e^2 = \pi(2-\sqrt{3})R_e^2$

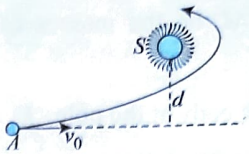
Thus, total area on earth's surface to which communication can be made is

$$A_C = 4\pi R_e^2 - 2A_p = 4\pi R_e^2 - 2\pi(2-\sqrt{3})R_e^2$$

$$= 2\pi R_e^2(2-2+\sqrt{3}) = 2\sqrt{3}\pi R_e^2$$

EXAMPLE 5.9

A cosmic body A moves towards the sun S with velocity v_0 when far from the sun and aiming along a line whose perpendicular distance from the sun is d (figure). Find the minimum distance of this body from the sun. Take M as the mass of the sun.



Sol. Using angular momentum conservation

$$mv_0 d = mv_{max} r_{min} \qquad ...(i)$$

By energy conservation $\dfrac{1}{2}mv_0^2 - 0 = \dfrac{1}{2}mv_{max}^2 - \dfrac{GMm}{r_{min}}$

From Eqs. (i) and (ii),

$$v_0^2 = \left(\frac{v_0 d}{r_{min}}\right)^2 - \frac{2GM}{r_{min}}$$

$$\Rightarrow \quad r_{min}^2 v_0^2 = v_0^2 d^2 - 2GMr_{min}$$

$$\Rightarrow \quad r_{min}^2 + \frac{2GM}{v_0^2} r_{min} - d^2 = 0$$

$$\Rightarrow \quad r_{min} = \frac{-\dfrac{2GM}{v_0^2} \pm \sqrt{\dfrac{4G^2M^2}{v_0^4} + 4d^2}}{2}$$

$$\Rightarrow \quad r_{min} = \frac{Gm}{v_0^2}\left(\sqrt{\left(1+\frac{d^2 v_0^4}{G^2M^2}\right)} - 1\right)$$

EXAMPLE 5.10

A spaceship approaches the moon (mass = M and radius = R) along a parabolic path which is almost tangential to its surface. At the moment of maximum approach, the brake rocket is fired to convert the spaceship into a satellite of the moon. Find the change in speed.

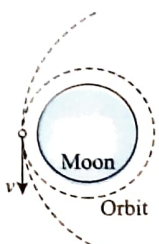
Sol. Figure shows the corresponding situation.

We know a particle follows a parabolic trajectory tangential to a planet, when at the surface of planet it has escape velocity

$$v_{es} = \sqrt{\frac{2GM}{R}}$$

Now to transform it into a circular orbit its speed should be decreased to orbital speed

$$v_0 = \sqrt{\frac{GM}{R}}$$



Thus, change in speed is

$$\Delta v = v_{es} - v_0 = \sqrt{\frac{GM}{R}}\,(\sqrt{2}-1)$$

EXAMPLE 5.11

On the pole of earth a body is imparted velocity v_0 directed vertically up. Knowing the radius of the earth and the free-fall acceleration on its surface, find the height to which the body will ascend. The air drag is to be neglected.

Sol. Let m and M_e be the masses of the body and earth respectively.

Kinetic energy of the body at the pole, $K_i = \frac{1}{2}mv_0^2$

Potential energy of the body at the pole,

$$U_i = -\frac{GMm}{R} \text{ [where } R = \text{radius of the earth]}$$

Potential energy of the body at height h,

$$U_f = -\frac{GMm}{(R+h)}$$

Using the conservation of energy, we have

$$K_i + U_i = U_f$$

or $$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = -\frac{GMm}{(R+h)}$$

or $$v_0^2 = \frac{2GMh}{R(R+h)} = \frac{2GMh}{R^2(1+h/R)}$$

or $$v_0^2 = \frac{2gh}{1+\frac{h}{R}} \qquad \left[\text{as } g = \frac{GM}{R^2}\right]$$

or $$v_0^2 = 2gh - \frac{v_0^2 h}{R} = h\left[\frac{2gR - v_0^2}{R}\right]$$

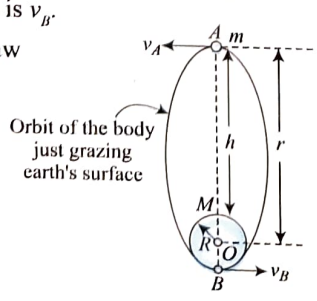
or $$h = \left[\frac{v_0^2 R}{2gR - v_0^2}\right]$$

EXAMPLE 5.12

For a particle projected in a transverse direction from a height h above earth's surface, find the minimum initial velocity so that it just grazes the surface of earth such that path of this particle would be an ellipse with centre of earth as the farther focus, point of projection as the apogee and a diametrically opposite point on earth's surface as perigee.

Sol. Suppose velocity of projection at point A is v_A and at point B, the velocity of the particle is v_B.

Then applying Newton's second law at points A and B, we get



$$\frac{mv_A^2}{\rho_A} = \frac{GM_e m}{(R+h)^2}$$

and $$\frac{mv_B^2}{\rho_B} = \frac{GM_e m}{R^2}$$

where ρ_A and ρ_B are radii of curvature of the orbit at points A and B of the ellipse, but $\rho_A = \rho_B = \rho$ (say).

Now applying conservation of energy at points A and B,

$$\frac{-GM_e m}{R+h} + \frac{1}{2}mv_A^2 = -\frac{GM_e m}{R} + \frac{1}{2}mv_B^2$$

$$\Rightarrow \quad GM_e m\left(\frac{1}{R} - \frac{1}{(R+h)}\right) = \frac{1}{2}(mv_B^2 - mv_A^2)$$

$$= \left(\frac{1}{2}\rho GM_e m\left(\frac{1}{R^2} - \frac{1}{(R+h)^2}\right)\right)$$

or $$\rho = \frac{2R(R+h)}{2R+h} = \frac{2Rr}{R+r}$$

Thus, we get

$$v_A^2 = \frac{\rho GM_e}{(R+h)^2} = 2GM_e\frac{R}{r(r+R)}$$

where r = distance of point of projection from earth's centre $= R + h$.

EXAMPLE 5.13

A rocket is launched vertically from the surface of earth with a velocity u. Find the height up to which the rocket can go from the surface of the earth before falling back to earth.

Sol. Let M and R be the mass and radius of the earth and m, the mass of the rocket. Suppose that the rocket can go up to a height h, when projected with a velocity v. The velocity of the rocket at the highest point will become zero.

According to the principle of conservation of energy, (PE + KE) of the rocket at the surface of earth

= (PE + KE) of the rocket at the highest point

$$-\frac{GMm}{R} + \frac{1}{2}mv^2 = -\frac{GMm}{R+h} + 0$$

($\therefore$ velocity at the highest point is zero)

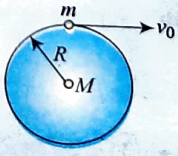
or $$v^2 = 2GM\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

or $$\frac{1}{h} = \left(\frac{2GM}{v^2R^2} - \frac{1}{R}\right)$$

or $$h = \left(\frac{2GM}{v^2R^2} - \frac{1}{R}\right)^{-1}$$

EXAMPLE 5.14

A body of mass m is projected horizontally just above earth's surface with a velocity $v_0 = \sqrt{\frac{4}{3}gR}$. Find the:

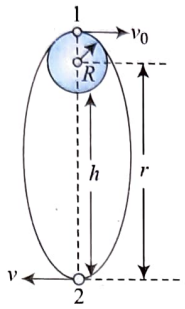


(i) maximum height attained by the body,
(ii) minimum speed of the body,
(iii) work done by gravity in the motion of the body upto its maximum height.

Sol. Applying conservation of angular momentum at 1 and 2

$$mv_0R = mvr$$

$$\Rightarrow \quad v = v_0\frac{R}{r} \qquad ...(i)$$



Conserving energy at 1 and 2

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{r} \qquad ...(ii)$$

Substituting $v_0 = \sqrt{\frac{4}{3}gR}$

and $GM = gR^2$

and value of v from Eq. (i) into Eq. (ii)

$$\frac{1}{2}m\left(\frac{4}{3}gR\right) - \frac{(gR^2)m}{R} = \frac{1}{2}m\left(v_0\frac{R}{r}\right)^2 - \frac{(gR)^2m}{r}$$

on solving we get

$$2\frac{R^2}{r^2} - 3\frac{R}{r} + 1 = 0 \qquad ...(iii)$$

(i) Let $\frac{R}{r} = x$, now Eq. (iii) becomes

$$2x^2 - 3x + 1 = 0 \Rightarrow (2x-1)(x-1) = 0$$

or $\frac{R}{r} = \frac{1}{2}$ and 1

which gives $r = 2R$ or $h + R = 2R$

$$\Rightarrow \quad h = R$$

(ii) From (i),

$$v = \left(\sqrt{\frac{4}{3}gR}\right) \times \frac{1}{2} = \sqrt{\frac{gR}{3}}$$

(iii) Work done by gravity $W_{\text{gravity}} = \Delta U = U_f - U_i$

$$\Rightarrow \quad W_{\text{gravity}} = \left(-\frac{GMm}{r}\right) - \left(-\frac{GMm}{R}\right)$$

But $r = 2R$

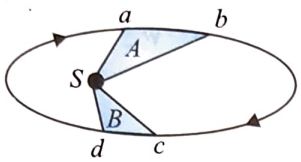
$$\Rightarrow \quad W_{\text{gravity}} = -\frac{GMm}{2R} + \frac{GMm}{R} = \frac{GMm}{2R}$$

But $GM = gR^2$

Hence $W_{\text{gravity}} = \frac{(gR^2)m}{2R} = \frac{1}{2}mgR$

Single Correct Answer Type

1. Figure shows the motion of a planet around the Sun S in an elliptical orbit with the Sun at the focus. The shaded areas A and B are also shown in the figure which can be assumed to be equal. If t_1 and t_2 represent the time taken for the planet to move from a to b and c to d, respectively, then



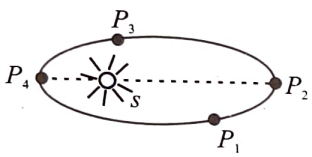
(1) $t_1 < t_2$

(2) $t_1 > t_2$

(3) $t_1 = t_2$

(4) from the given information the relation between t_1 and t_2 cannot be determined

2. Figure shows a planet in an elliptical orbit around the Sun S. Where is the kinetic energy of the planet maximum?



(1) P_1 (2) P_2

(3) P_3 (4) P_4

3. The escape velocity for a body projected vertically upwards from the surface of the earth is 11.2 km s^{-1}. If the body is projected in a direction making an angle 45° with the vertical, the escape velocity will be

(1) $\frac{11.2}{\sqrt{2}}$ km s^{-1} (2) $11.2 \times \sqrt{2}$ km s^{-1}

(3) 11.2×2 km s^{-1} (4) 11.2 km s^{-1}

4. A space station is set up in space at a distance equal to the earth's radius from the surface of the earth. Suppose a satellite can be launched from the space station. Let v_1 and v_2 be the escape velocities of the satellite on the earth's surface and space station, respectively. Then

(1) $v_2 = v_1$ (2) $v_2 < v_1$

(3) $v_2 > v_1$

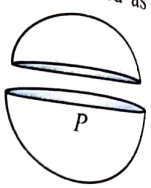
(4) (1), (2) and (3) are valid depending on the mass of satellite

5. A planet is revolving in an elliptical orbit around the Sun. Its closest distance from the Sun is r_{min} and the farthest distance is r_{max}. If the velocity of the planet at the distance of the closest approach is v_1 and that at the farthest distance from the Sun is v_2, then v_1/v_2

(1) $\frac{r_{max}}{r_{min}}$ (2) $\frac{r_{min}}{r_{max}}$

(3) $\frac{r_{min}+r_{max}}{r_{max}-r_{min}}$ (4) none of these

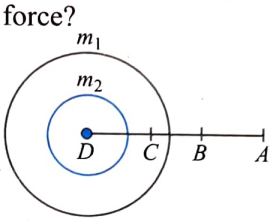
6. A spherical shell is cut into two pieces along a chord as shown in the figure. P is a point on the plane of the chord. The gravitational field at P due to the upper part is I_1 and that due to the lower part is I_2. What is the relation between them?



(1) $I_1 > I_2$ (2) $I_1 < I_2$

(3) $I_1 = I_2$ (4) No definite relation

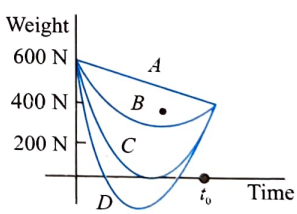
7. Figure shows two shells of masses m_1 and m_2. The shells are concentric. At which point, a particle of mass m shall experience zero force?



(1) A (2) B

(3) C (4) D

8. Suppose, the acceleration due to gravity at the earth's surface is 10 m s^{-2} and at the surface of Mars is 4.0 m s^{-2}. A 60 kg passenger goes from the earth to Mars in a spaceship moving with a constant velocity. Neglect all other objects in the sky. Which part of the figure best represents the weight (net gravitational force) of the passenger as a function of time?



(1) A (2) B

(3) C (4) D

9. If g is the acceleration due to gravity on the earth's surface, the change in the potential energy of an object of mass m raised from the surface of the earth to a height equal to the radius R of the earth is

(1) $\frac{mgR}{2}$ (2) $2mgR$

(3) mgR (4) $-mgR$

10. The orbital velocity of an artificial satellite in a circular orbit just above the earth's surface is v. For a satellite orbiting at an altitude of half of the earth's radius, the orbital velocity is

(1) $\left(\frac{3}{2}\right)v$ (2) $\sqrt{\left(\frac{3}{2}\right)}\,v$

(3) $\sqrt{\left(\frac{2}{3}\right)}\,v$ (4) $\left(\frac{2}{3}\right)v$

11. The radius of the earth is about 6400 km and that of Mars is about 3200 km. The mass of the earth is about 10 times the mass of Mars. An object weighs 200 N on the surface of the earth. Its weight on the surface of mars would be
(1) 6 N (2) 20 N
(3) 40 N (4) 80 N

12. The gravitational force between two objects is proportional to $1/R$ (and not as $1/R^2$) where R is separation between them, then a particle in circular orbit under such a force would have its orbital speed v proportional to
(1) $\frac{1}{R^2}$ (2) R^0
(3) R^1 (4) $\frac{1}{R}$

13. Two particles of equal mass go around a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is
(1) $v = \frac{1}{2R}\sqrt{\left(\frac{1}{Gm}\right)}$ (2) $v = \sqrt{\left(\frac{Gm}{2R}\right)}$
(3) $v = \frac{1}{2}\sqrt{\left(\frac{Gm}{R}\right)}$ (4) $v = \sqrt{\left(\frac{4Gm}{R}\right)}$

14. If the radius of the earth decreases by 10%, the mass remaining unchanged, what will happen to the acceleration due to gravity?
(1) Decreases by 19%
(2) Increases by 19%
(3) Decreases by more than 19%
(4) Increases by more than 19%

15. The maximum vertical distance through which a fully dressed astronaut can jump on the earth is 0.5 m. If mean density of the Moon is two-third that of the earth and radius is one quarter that of the earth, the maximum vertical distance through which he can jump on the Moon and the ratio of the time of duration of the jump on the Moon to hold on the earth are
(1) 3 m, 6:1 (2) 6 m, 3:1
(3) 3 m, 1:6 (4) 6 m, 1:6

16. The distance between the centres of the Moon and the earth is D. The mass of the earth is 81 times the mass of the Moon. At what distance from the centre of the earth, the gravitational force will be zero?
(1) $\frac{D}{2}$ (2) $\frac{2D}{3}$
(3) $\frac{4D}{3}$ (4) $\frac{9D}{10}$

17. The value of g at a particular point is 10 m s^{-2}. Suppose the earth shrinks uniformly to half of its present size without losing any mass. The value of g at the same point (assuming that the distance of the point from the centre of the earth does not change) will now be
(1) 5 m s^{-2} (2) 10 m s^{-2}
(3) 3 m s^{-2} (4) 20 m s^{-2}

18. Two satellites of masses m_1 and m_2 ($m_1 > m_2$) are revolving around the earth in a circular orbit of radii r_1 and r_2 ($r_1 > r_2$), respectively. Which of the following statements is true regarding their speeds v_1 and v_2?
(1) $v_1 = v_2$ (2) $v_1 > v_2$
(3) $v_1 < v_2$ (4) $\frac{v_1}{r_1} = \frac{v_2}{v_2}$

19. Two satellites A and B of the same mass are revolving around the earth in the concentric circular orbits such that the distance of satellite B from the centre of the earth is thrice as compared to the distance of the satellite A from the centre of the earth. The ratio of the centripetal force acting on B as compared to that on A is
(1) $\frac{1}{3}$ (2) 3
(3) $\frac{1}{9}$ (4) $\frac{1}{\sqrt{3}}$

20. The minimum energy required to launch a m kg satellite from the earth's surface in a circular orbit at an altitude $2R$, where R is the radius of earth is
(1) $\frac{5}{3}mgR$ (2) $\frac{4}{3}mgR$
(3) $\frac{5}{6}mgR$ (4) $\frac{5}{4}mgR$

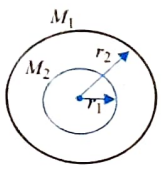
21. Three particles, each of mass M, are placed at the three corners of an equilateral triangle of side l. What is the force due to this system of particles on another particle of mass m placed at the midpoint of any side?
(1) $\frac{3GMm}{4l^2}$ (2) $\frac{4GMm}{3l^2}$
(3) $\frac{GMm}{4l^2}$ (4) $\frac{4GMm}{l^2}$

22. In problem 21, what is the gravitational potential energy of the mass m?
(1) $-\frac{2}{\sqrt{3}}\frac{GMm}{l}(1-2\sqrt{3})$ (2) $-\frac{2}{\sqrt{3}}\frac{GMm}{l}(1+2\sqrt{3})$
(3) $-\frac{\sqrt{3}}{2}\frac{GMm}{l}(1-2\sqrt{3})$ (4) $-\frac{\sqrt{3}}{2}\frac{GMm}{l}(1+2\sqrt{3})$

23. In problem 21, what is the gravitational field strength at the location of m?
(1) $\frac{GM}{l^2}$ (2) $\frac{4GM}{l^2}$
(3) $\frac{4GM}{3l^2}$ (4) $\frac{GM}{3l^2}$

24. If three particles, each of mass M, are placed at the three corners of an equilateral triangle of side a, the forces exerted by this system on another particle of mass M placed (i) at the midpoint of a side and (ii) at the centre of the triangle are, respectively,
(1) $0, \frac{4GM^2}{3a^2}$ (2) $\frac{4GM^2}{3a^2}, 0$
(3) $\frac{3GM^2}{a^2}, \frac{GM^2}{a^2}$ (4) 0, 0

25. Two concentric shells of masses M_1 and M_2 are having radii r_1 and r_2. Which of the following is the correct expression for the gravitational field on a mass m?



(1) $F = \frac{G(M_1 + M_2)}{r^2}$, for $r < r_1$

(2) $F = \dfrac{G(M_1 + M_2)}{r^2}$, for $r < r_2$

(3) $F = \dfrac{GM_2}{r^2}$, for $r_1 < r < r_2$

(4) $F = \dfrac{GM_1}{r^2}$, for $r_1 < r < r_2$

26. The radius of a planet is R. A satellite revolves around it in a circle of radius r with angular velocity ω_0. The acceleration due to the gravity on planet's surface is

(1) $\dfrac{r^3\omega_0}{R}$ (2) $\dfrac{r^3\omega_0^2}{R^2}$

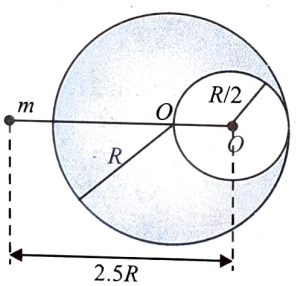
(3) $\dfrac{r^3\omega_0^2}{R}$ (4) $\dfrac{r^3\omega_0^2}{R^2}$

27. A space vehicle approaching a planet has a speed v, when it is very far from the planet. At that moment tangent of its trajectory would miss the centre of the planet by distance R. If the planet has mass M and radius r, what is the smallest value of R in order that the resulting orbit of the space vehicle will just miss the surface of the planet?

(1) $\dfrac{r}{v}\left[v^2 + \dfrac{2GM}{r}\right]^{\frac{1}{2}}$ (2) $vr\left[1 + \dfrac{2GM}{r}\right]$

(3) $\dfrac{r}{v}\left[v^2 + \dfrac{2GM}{r}\right]$ (4) $\dfrac{2GMv}{r}$

28. A solid sphere of radius $R/2$ is cut out of a solid sphere of radius R such that the spherical cavity so formed touches the surface on one side and the centre of the sphere on the other side, as shown. The initial mass of the solid sphere was M. If a particle of mass m is placed at a distance $2.5R$ from the centre of the cavity, then what is the gravitational attraction on the mass m?



(1) $\dfrac{GMm}{R^2}$ (2) $\dfrac{GMm}{2R^2}$

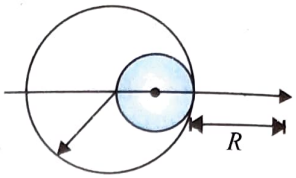
(3) $\dfrac{GMm}{8R^2}$ (4) $\dfrac{23}{100}\dfrac{GMm}{R^2}$

29. A body is released from a point of distance R' from the centre of earth. Its velocity at the time of striking the earth will be $(R' > R_e)$

(1) $\sqrt{2gR_e}$ (2) $\sqrt{R_e g}$

(3) $\sqrt{2g(R' - R_e)}$ (4) $\sqrt{2gR_e\left(1 - \dfrac{R_e}{R'}\right)}$

30. A solid sphere of uniform density and radius R applies a gravitational force of attraction equal to F_1 on a particle placed at a distance $2R$ from the centre of the sphere. A spherical cavity of radius $R/2$ is now made in the sphere as shown in the figure. The sphere with the cavity now applies a gravitational force F_2 on the same particle. The ratio F_1/F_2 is



(1) $\dfrac{1}{2}$ (2) $\dfrac{3}{4}$

(3) $\dfrac{7}{8}$ (4) $\dfrac{9}{7}$

31. Consider two solid uniform spherical objects of the same density ρ. One has radius R and the other has radius $2R$. They are in outer space where the gravitational fields from other objects are negligible. If they are arranged with their surface touching, what is the contact force between the objects due to their traditional attraction?

(1) $G\pi^2R^4$ (2) $\dfrac{128}{81}G\pi^2R^4\rho^2$

(3) $\dfrac{128}{81}G\pi^2$ (4) $\dfrac{128}{87}\pi^2R^2G$

32. A body starts from rest from a point distant r_0 from the centre of the earth. It reaches the surface of the earth whose radius is R. The velocity acquired by the body is

(1) $2GM\sqrt{\dfrac{1}{R} - \dfrac{1}{r_0}}$ (2) $\sqrt{2GM\left(\dfrac{1}{R} - \dfrac{1}{r_0}\right)}$

(3) $GM\sqrt{\dfrac{1}{R} - \dfrac{1}{r_0}}$ (4) $\sqrt{GM\left(\dfrac{1}{R} - \dfrac{1}{r_0}\right)}$

33. Suppose the gravitational force varies inversely as the nth power of the distance. Then, the time period of a planet in a circular orbit of radius R around the Sun will be proportional to

(1) R^n (2) $R^{\frac{(n+1)}{2}}$

(3) $R^{\frac{(n-1)}{2}}$ (4) R^{-n}

34. The escape velocity corresponding to a planet of mass M and radius R is 50 km s^{-1}. If the planet's mass and radius were $4M$ and R, respectively, then the corresponding escape velocity would be

(1) 100 km s^{-1} (2) 50 km s^{-1}

(3) 200 km s^{-1} (4) 25 km s^{-1}

35. The change in the value of g at a height h above the surface of earth is the same as at a depth d below the earth. When both d and h are much smaller than the radius of earth, then which one of the following is correct?

(1) $d = \dfrac{h}{2}$ (2) $d = \dfrac{3h}{2}$

(3) $d = 2h$ (4) $d = h$

36. Four similar particles of mass m are orbiting in a circle of radius r in the same angular direction because of their mutual gravitational attractive force. Velocity of a particle is given by

(1) $\left[\dfrac{Gm}{r}\left(\dfrac{1 + 2\sqrt{2}}{4}\right)\right]^{\frac{1}{2}}$ (2) $\sqrt{\dfrac{Gm}{r}}$

(3) $\sqrt{\dfrac{Gm}{r}(1 + 2\sqrt{2})}$ (4) $\left[\dfrac{1}{2}\dfrac{Gm}{r}\left(\dfrac{1 + 2\sqrt{2}}{2}\right)\right]^{\frac{1}{2}}$

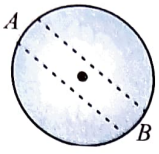
37. Two satellites S_1 and S_2 revolve around a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 hour and 8 hour respectively. The radius of the orbit of S_1 is 10^4 km. The speed of S_2 relative to S_1 when they are closest (in kmh^{-1}) is:

(1) $\frac{1}{2}10^4\pi$ (2) $4 \times 10^4\pi$

(3) $10^4\pi$ (4) $2 \times 10^4\pi$

38. A tunnel has been dug into a solid sphere of non-uniform mass density as shown in the figure. As one moves from A to B, the magnitude of gravitational field intensity

(1) will continuously decrease
(2) will decrease up to the centre of the sphere and then increase
(3) may increase or decrease
(4) will continuously increase



39. A projectile is fired vertically upwards from the surface of the earth with a velocity Kv_e, where v_e is the escape velocity and $k < 1$. If R is the radius of the earth, the maximum height to which it will rise, measured from the centre of the earth, will be (neglect air resistance)

(1) $\frac{1-k^2}{R}$ (2) $\frac{R}{1-k^2}$

(3) $R(1-k)^2$ (4) $\frac{R}{1+k^2}$

40. The two planets with radii R_1, R_2 have densities ρ_1, ρ_2, and atmospheric pressures p_1 and p_2, respectively. Therefore, the ratio of masses of their atmospheres, neglecting variation of g and ρ within the limits of atmosphere, is

(1) $\frac{p_1R_2\rho_1}{p_2R_1\rho_2}$ (2) $\frac{p_1R_2\rho_2}{p_2R_1\rho_1}$

(3) $\frac{p_1R_1\rho_1}{p_2R_2\rho_2}$ (4) $\frac{p_1R_1\rho_2}{p_2R_2\rho_1}$

41. Two bodies with masses M_1 and M_2 are initially at rest and a distance R apart. Then they move directly towards one another under the influence of their mutual gravitational attraction. What is the ratio of the distances travelled by M_1 to the distance travelled by M_2?

(1) $\frac{M_1}{M_2}$ (2) $\frac{M_2}{M_1}$

(3) 1 (4) $\frac{1}{2}$

42. If g is acceleration due to gravity on the earth's surface, the gain in the potential energy of an object of mass m raised from the surface of earth to a height equal to the radius R of the earth is

(1) $\frac{1}{2}mgR$ (2) $2mgR$

(3) mgR (4) $\frac{1}{4}mgR$

43. Imagine a light planet revolving around a very massive star in a circular orbit of radius r with a period of revolution T. If the gravitational force of attraction between the planet and the star is proportional to $r^{5/2}$, then the square of the time period will be proportional to

(1) r^3 (2) r^2

(3) $r^{2.5}$ (4) $r^{3.5}$

44. The masses and radii of the earth and the Moon are M_1, R_1 and M_2, R_2, respectively. Their centres are at distance d apart. The minimum speed with which a particle of mass m should be projected from a point midway the two centres so as to escape to infinity is

(1) $\sqrt{\frac{2G(M_1+M_2)}{d}}$ (2) $\sqrt{\frac{4G(M_1+M_2)}{d}}$

(3) $\sqrt{\frac{4GM_1M_2}{d}}$ (4) $\sqrt{\frac{G(M_1+M_2)}{d}}$

45. A skylab of mass m kg is first launched from the surface of the earth in a circular orbit of radius $2R$ (from the centre of the earth) and then it is shifted from this circular orbit to another circular orbit of radius $3R$. The minimum energy required to place the lab in the first orbit and to shift the lab from first orbit to the second orbit are

(1) $\frac{3}{4}mgR, \frac{mgR}{6}$ (2) $\frac{3}{4}mgR, \frac{mgR}{12}$

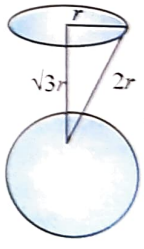
(3) mgR, mgR (4) $2mgR, mgR$

46. Two satellites of the same mass are launched in the same orbit around the earth so as to rotate opposite to each other. If they collide inelastically and stick together as wreckage, the total energy of the system just after collision is

(1) $-\frac{2GMm}{r}$ (2) $-\frac{GMm}{r}$

(3) $\frac{GMm}{2r}$ (4) $\frac{GMm}{4r}$

47. A uniform ring of mass m and radius r is placed directly above a uniform sphere of mass M and of equal radius. The centre of the ring is directly above the centre of the sphere at a distance $r\sqrt{3}$ as shown in the figure. The gravitational force exerted by the sphere on the ring will be



(1) $\frac{GMm}{8r^2}$ (2) $\frac{GMm}{4r^2}$

(3) $\sqrt{3}\frac{GMm}{8r^2}$ (4) $\frac{GMm}{8r^2\sqrt{3}}$

48. A diametrical tunnel is dug across the earth. A ball is dropped into the tunnel from one side. The velocity of the ball when it reaches the centre of the earth is
[Given: gravitational potential at the centre of earth $= -3/(2GM/R)$]

(1) $\sqrt{R}$ (2) $\sqrt{gR}$

(3) $\sqrt{2.5\,gR}$ (4) $\sqrt{7.1\,gR}$

49. Four particles, each of mass M, move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(1) $\frac{GM}{R}$ (2) $\sqrt{2\sqrt{2}\frac{GM}{R}}$

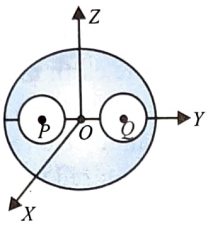
(3) $\sqrt{\frac{GM}{R}(2\sqrt{2}+1)}$ (4) $\sqrt{\frac{GM}{R}\left(\frac{2\sqrt{2}+1}{4}\right)}$

50. Two astronauts have deserted their spaceship in a region of space far from the gravitational attraction of any other body. Each has a mass of 100 kg and they are 100 m apart. They are initially at rest relative to one another. How long will it be before the gravitational attraction brings them 1 cm closer together?

(1) 2.52 days (2) 1.41 days

(3) 0.70 days (4) 1.41 s

51. A solid sphere of uniform density and mass M has radius 4 m. Its centre is at the origin of the coordinate system. Two spheres of radii 1 m are taken out so that their centres are at P (0, –2, 0) and Q (0, 2, 0), respectively. This leaves two spherical cavities. What is the gravitational field at the origin of the coordinate axes?



(1) $\frac{31GM}{1024}$ (2) $\frac{Gm}{1024}$

(3) $31GM$ (4) Zero

52. In problem 51, what is the gravitational field at the centre of the cavities?

(1) $\frac{31GM}{1024}$ (2) $\frac{Gm}{1024}$

(3) $31GM$ (4) GM

53. In problem 51, what is the gravitational potential at any point on the circle $x^2 + z^2 = 6$?

(1) $-\frac{GM}{6}$ (2) $-\frac{GM}{64\sqrt{10}}$

(3) $-\frac{GM}{2}\left[\frac{1}{3}-\frac{1}{32\sqrt{10}}\right]$ (4) $-\frac{GM}{2}\left[\frac{1}{3}+\frac{1}{32\sqrt{10}}\right]$

54. A planet is revolving around the Sun in an elliptical orbit. Its closest distance from the Sun is r and farthest distance is R. If the orbital velocity of the planet closest to the Sun is v, then what is the velocity at the farthest point?

(1) $\frac{vr}{R}$ (2) $\frac{vR}{r}$

(3) $v\sqrt{\frac{r}{R}}$ (4) $v\sqrt{\frac{R}{r}}$

55. A body is fired with a velocity of magnitude $\sqrt{gR} < V < \sqrt{2gR}$ at an angle of 30° with the radius vector of the earth. If at the highest point, the speed of the body is $V/4$, the maximum height attained by the body is equal to

(1) $\frac{V^2}{8g}$ (2) R

(3) $\sqrt{2}\,R$ (4) none of these

56. A system of binary stars of masses m_A and m_B are moving in circular orbits of radii r_A and r_B, respectively. If T_A and T_B are the time periods of masses m_A and m_B, respectively, then

(1) $\frac{T_A}{T_B} = \left(\frac{r_A}{r_B}\right)^{\frac{3}{2}}$ (2) $T_A > T_B$ (if $r_A > r_B$)

(3) $T_A > T_B$ (if $m_A > m_B$) (4) $T_A = T_B$

57. Two spherical bodies of masses M and $5M$ and radii R and $2R$, respectively, are released in free space with initial separation between their centres equal to $12R$. If they attract each other due to gravitational force only, then the distance covered by the smaller body just before collision is

(1) $2.5R$ (2) $4.5R$

(3) $7.5R$ (4) $1.5R$

58. A tunnel is dug along the diameter of the earth (radius R and mass M). There is a particle of mass 'm' at the centre of the tunnel. The minimum velocity given to the particle so that it just reaches to the surface of the earth is

(1) $\sqrt{\frac{GM}{R}}$ (2) $\sqrt{\frac{GM}{2R}}$

(3) $\sqrt{\frac{2GM}{R}}$

(4) it will reach with the help of a negligible velocity

59. A cavity of radius $R/2$ is made inside a solid sphere of radius R. The centre of the cavity is located at a distance $R/2$ from the centre of the sphere. The gravitational force on a particle of mass m at a distance $R/2$ from the centre of the sphere on the line joining both the centres of the sphere and the cavity is (opposite to the centre of the cavity)

[Here $g = (GM)/R^2$, where M is the mass of the sphere]

(1) $\frac{mg}{2}$ (2) $\frac{3mg}{8}$

(3) $\frac{mg}{16}$ (4) none of these

60. A satellite is seen after each 8 h over the equator at a place on the earth when its sense of rotation is opposite to the earth. The time interval after which it can be seen at the same place when the sense of rotation of the earth and the satellite is the same will be

(1) 8 h (2) 12 h

(3) 24 h (4) 6 h

61. The gravitational potential of two homogenous spherical shells A and B of same surface density at their respective centres are in the ratio 3:4. If the two shells coalesce into a single one such that surface charge density remains the same, then the ratio of potential at an internal point of the new shell to shell A is equal to

(1) 3:2 (2) 4:3

(3) 5:3 (4) 5:6

62. A point P is on the axis of a fixed ring of mass M and radius R, at a distance $2R$ from the centre O. A small particle starts from P and reaches O under the gravitational attraction only. Its speed at O will be

(1) zero (2) $\sqrt{\frac{2GM}{R}}$

(3) $\sqrt{\frac{2GM}{R}(\sqrt{5}-1)}$ (4) $\sqrt{\frac{2GM}{R}\left(1-\frac{1}{\sqrt{5}}\right)}$

63. The percentage change in the acceleration of the earth towards the Sun from a total eclipse of the Sun to the point where the Moon is on a side of earth directly opposite to the Sun is

(1) $\frac{M_s}{M_m}\frac{r_2}{r_1} \times 100$ (2) $\frac{M_s}{M_m}\left(\frac{r_2}{r_1}\right)^2 \times 100$

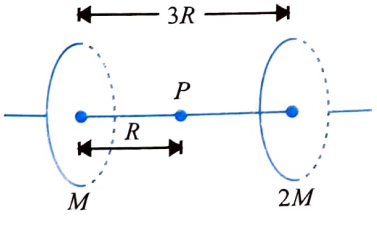
(3) $2\left(\frac{r_1}{r_2}\right)^2 \frac{M_m}{M_s} \times 100$ (4) $\left(\frac{r_1}{r_2}\right)^2 \frac{M_m}{M_s} \times 100$

64. A comet is in highly elliptical orbit around the Sun. The period of the comet's orbit is 90 days. Some statements are given regarding the collision between the comet and the earth. Mark the correct statement. [Mass of the Sun = 2×10^{30} kg, mean distance between the earth and the Sun = 1.5×10^{11} m.]
(1) Collision is there
(2) Collision is not possible
(3) Collision may or may not be there
(4) Enough information is not given

65. Imagine that you are in a spacecraft orbiting around the earth in a circle of radius 7000 km (from the centre of the earth). If you decrease the magnitude of mechanical energy of the spacecraft–earth system by 10% by firing the rockets, then what is the greatest height you can take your spacecraft above the surface of the earth? [R_e = 6400 km]
(1) 6400 km (2) 540 km
(3) 2140 km (4) 3000 km

66. A satellite of mass m is in an elliptical orbit around the earth. The speed of the satellite at its nearest position is $(6GM_e)/(5r)$ where r is the perigee (nearest point) distance from the centre of the earth. It is desired to transfer the satellite to the circular orbit of radius equal to its apogee (farthest point) distance from the centre of the earth. The change in orbital speed required for this purpose is
(1) $0.35\sqrt{\frac{GM_e}{r}}$ (2) $0.085\sqrt{\frac{GM_e}{r}}$
(3) $\sqrt{\frac{2GM_e}{r}}$ (4) zero

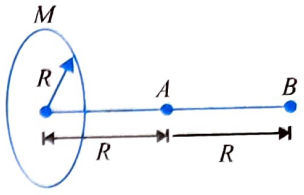
67. Two rings having masses M and $2M$, respectively, having the same radius are placed coaxially as shown in the figure.



If the mass distribution on both the rings is non-uniform, then the gravitational potential at point P is
(1) $-\frac{GM}{R}\left[\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}\right]$ (2) $-\frac{GM}{R}\left[1+\frac{1}{\sqrt{2}}\right]$
(3) zero
(4) cannot be determined from the given information

68. A body is thrown from the surface of the earth with velocity $(gR_e)/2$, where R_e is the radius of the earth at some angle from the vertical. If the maximum height reached by the body is $R_e/4$, then the angle of projection with the vertical is
(1) $\sin^{-1}\left(\frac{\sqrt{5}}{4}\right)$ (2) $\cos^{-1}\left(\frac{\sqrt{5}}{4}\right)$
(3) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (4) none of these

69. A ring having non-uniform distribution of mass M and radius R is being considered. A point mass m_0 is taken slowly towards the ring. In doing so, work done by the external force against the gravitational force exerted by ring is
(1) $\frac{GMm_0}{\sqrt{2}R}$
(2) $\frac{GMm_0}{R}\left[\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{5}}\right]$
(3) $\frac{GMm_0}{R}\left[\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{2}}\right]$



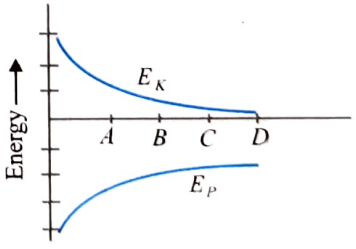
(4) It is not possible to find the required work as the nature of distribution of mass is not known

70. An artificial satellite of the earth is launched in circular orbit in the equatorial plane of the earth and the satellite is moving from west to east. With respect to a person on the equator, the satellite is completing one round trip in 24 h. Mass of the earth is $M = 6 \times 10^{24}$ kg. For this situation, the orbital radius of the satellite is
(1) 2.66×10^4 km (2) 6400 km
(3) 36,000 km (4) 29,600 km

71. A satellite is orbiting around the earth in a circular orbit of radius r. A particle of mass m is projected from the satellite in a forward direction with a velocity v = 2/3 times the orbital velocity (this velocity is given w.r.t. earth). During subsequent motion of the particle, its minimum distance from the centre of earth is
(1) $\frac{r}{2}$ (2) r
(3) $\frac{2r}{3}$ (4) $\frac{4r}{5}$

Multiple Correct Answers Type

1. Figure shows the kinetic energy (E_k) and potential energy (E_p) curves for a two-particle system. Name the point at which the system is a bound system.



(1) A (2) B
(3) C (4) D

2. Suppose universal gravitational constant starts to decrease, then
(1) length of the day on the earth will increase
(2) length of the year will increase
(3) the earth will follow a spiral path of decreasing radius
(4) kinetic energy of the earth will decrease

3. An astronaut, inside an earth satellite, experiences weightlessness because
(1) no external force is acting on him
(2) he is falling freely
(3) no reaction is exerted by the floor of the satellite
(4) he is far away from the earth's surface

4. Choose the incorrect statements from the following:
(1) It is possible to shield a body from the gravitational field of another body by using a thick shielding material between them.
(2) The escape velocity of a body is independent of the mass of the body and the angle of projection.
(3) The acceleration due to gravity increases due to the rotation of the earth.
(4) The gravitational force exerted by the earth on a body is greater than that exerted by the body on the earth.

5. A satellite is orbiting the earth, if its distance from the earth is increased, its
(1) angular velocity would increase
(2) linear velocity would increase
(3) angular velocity would decrease
(4) time period would increase

6. Which of the following are correct?
(1) An astronaut going from the earth to the Moon will experience weightlessness once.
(2) When a thin uniform spherical shell gradually shrinks maintaining its shape, the gravitational potential at its centre decreases.
(3) In the case of a spherical shell, the plot of V versus r is continuous.
(4) In the case of a spherical shell, the plot of gravitational field intensity I versus r is continuous.

7. Which of the following are correct?
(1) Out of electrostatic, electromagnetic, nuclear and gravitational interactions, the gravitational interaction is the weakest.
(2) If the earth were to rotate faster than its present speed, the weight of an object would decrease at the equator but remain unchanged at the poles.
(3) The mass of the earth in terms of g, R and G is (gR^2/G).
(4) If the earth stops rotating in its orbit around the Sun there will be no variation in the weight of a body on the surface of earth.

8. Consider a planet moving in an elliptical orbit around the Sun. The work done on the planet by the gravitational force of the Sun
(1) is zero in any small part of the orbit
(2) is zero in some parts of the orbit
(3) is zero in complete revolution
(4) is zero in no part of the motion

9. Which of the following statements are true about acceleration due to gravity?
(1) g decreases in moving away from the centre if $r > R$.
(2) g decreases in moving away from the centre if $r < R$.
(3) g is zero at the centre of earth.
(4) g decreases if earth stops rotating on its axis.

10. Let V and E denote the gravitational potential and gravitational field at a point, respectively. It is possible to have
(1) $V =$ and $E = 0$ (2) $V = 0$ and $E \neq 0$
(3) $V \neq 0$ and $E = 0$ (4) $V \neq 0$ and $E \neq 0$

11. If a body is projected with a speed lesser than escape velocity, then
(1) the body can reach a certain height and may fall down following a straight line path
(2) the body can reach a certain height and may fall down following a parabolic path
(3) the body may orbit the earth in a circular orbit
(4) the body may orbit the earth in an elliptical orbit

12. Which of the following are not correct?
(1) The escape velocity for the Moon is 6 km s^{-1}.
(2) The escape velocity from the surface of Moon is v. The orbital velocity for a satellite to orbit very close to the surface of Moon is $v/2$.
(3) When an earth satellite is moved from one stable orbit to a further stable orbit, the gravitational potential energy increases.
(4) The orbital velocity of a satellite revolving in a circular path close to the planet is independent of the density of the planet.

13. An orbiting satellite will escape if
(1) its speed is increased by 41%
(2) its speed in the orbit is made $\sqrt{(1.5)}$ times of its initial value
(3) its KE is doubled
(4) it stops moving in the orbit

14. In case of an orbiting satellite, if the radius of orbit is decreased
(1) its KE decreases (2) its PE decreases
(3) its ME decreases (4) its speed decreases

15. If two satellites of different masses are revolving in the same orbit, they have the same
(1) angular momentum (2) energy
(3) time period (4) speed

16. Choose the correct statements from the following:
(1) The gravitational forces between two particles are an action and reaction pair.
(2) Gravitational constant (G) is scalar but acceleration due to gravity (g) is a vector.
(3) The values of G and g are to be determined experimentally.
(4) G and g are constant everywhere.

17. Choose the correct statements from the following:
(1) The magnitude of the gravitational force between two bodies of mass 1 kg each and separated by a distance of 1 m is 9.8 N.
(2) The higher the value of the escape velocity for a planet, the higher is the abundance of the lighter gases in its atmosphere.
(3) The gravitational force of attraction between two bodies of ordinary mass is not noticeable because the value of the gravitational constant is extremely small.
(4) Force of friction arises due to gravitational attraction.

18. Which of the following statements are true for a particle on the surface of the earth?

(1) The linear speed is minimum at the equator.
(2) The angular speed is maximum at the equator.
(3) The linear speed is minimum at the poles.
(4) The angular speed is 7.3×10^{-5} rad s^{-1} at the equator.

19. For two satellites at distances R and $7R$ above the earth's surface, the ratio of their
(1) total energies is 4 and potential and kinetic energies is 2
(2) potential energies is 4
(3) kinetic energies is 4
(4) total energies is 4

20. If both the mass and radius of the earth decrease by 1%, the value of
(1) acceleration due to gravity would decrease by nearly 1%
(2) acceleration due to gravity would increase by 1%
(3) escape velocity from the earth's surface would decrease by 1%
(4) the gravitational potential energy of a body on earth's surface will remain unchanged

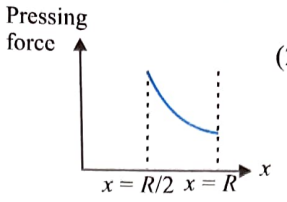
21. An object is taken from a point P to another point Q in a gravitational field:
(1) assuming the earth to be spherical, if both P and Q lie on the earth's surface, the work done is zero
(2) if P is on the earth's surface and Q above it, the work done is minimum when it is taken along the straight line PQ
(3) the work done depends only on the position of P and Q and is independent of the path along which the particle is taken
(4) there is no work done if the object is taken from P to Q and then brought back to P along any path

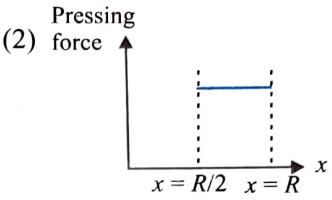
22. A small mass m is moved slowly from the surface of the earth to a height h above the surface. The work done (by an external agent) in doing this is
(1) mgh, for all values of h
(2) mgh, for $h \ll R$
(3) $1/2\ mgR$, for $h = R$
(4) $-1/2\ mgR$, for $h = R$

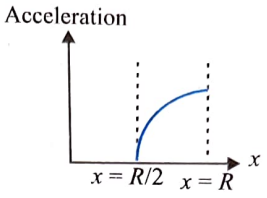
23. Which of the following are correct?
(1) If R is the radius of a planet, g is the acceleration due to gravity, the mean density of the planet is $3g/4\pi GR$.
(2) Acceleration due to gravity is a universal constant.
(3) The escape velocity from a planet which has double the mass of earth and half its radius is 22.4 km s^{-1}.
(4) The ratio of gravitational mass and inertial mass of a body at the surface of earth is 1.

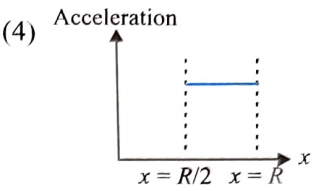
24. A double star is a system of two stars of masses m and $2m$, rotating about their centre of mass only under their mutual gravitational attraction. If r is the separation between these two stars then their time period of rotation about their centre of mass will be proportional to
(1) $r^{\frac{3}{2}}$
(2) r
(3) $m^{\frac{1}{2}}$
(4) $m^{-\frac{1}{2}}$

25. A tunnel is dug along a chord of the earth at a perpendicular distance $R/2$ from the earth's centre. The wall of the tunnel may be assumed to be frictionless. A particle is released from one end of the tunnel. The pressing force by the particle on the wall, and the acceleration of the particle vary with x (distance of the particle from the centre) according to

(1) 

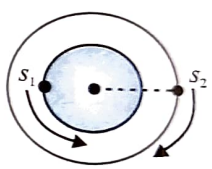
(2) 

(3) 

(4) 

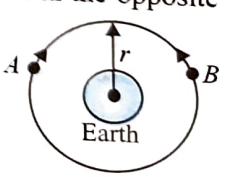
26. A double star consists of two stars having masses M and $2M$. The distance between their centres is equal to r. They revolve under their mutual gravitational interaction. Then, which of the following statements are not correct?
(1) Heavier star revolves in orbit of radius $2r/3$.
(2) Both the stars revolve with the same speed, period of which is equal to $(2\pi/r^3)(2GM^2/3)$.
(3) Kinetic energy of the heavier star is twice that of the other star.
(4) None of the above

27. Two satellites S_1 and S_2 are revolving around the earth in coplanar concentric orbits in the opposite sense. At $t = 0$, the positions of satellites are shown in the diagram. The periods of S_1 and S_2 are 4 h and 24 h, respectively. The radius of orbit of S_1 is 1.28×10^4 km. For this situation, mark the correct statement(s).



(1) The angular velocity of S_2 as observed by S_1 at $t = 12$ h is 0.468π rad s^{-1}.
(2) The two satellites are closest to each other for the first time at $t = 12$ h and then after every 24 h they are closest to each other.
(3) The orbital velocity of S_1 is $0.64\pi \times 10^4$ km.
(4) The velocity of S_1 relative to S_2 is continuously changing in magnitude and direction both.

28. Consider two satellites A and B of equal mass m, moving in the same circular orbit about the earth, but in the opposite sense as shown in figure. The orbital radius is r. The satellites undergo a collision which is perfectly inelastic. For this situation, mark out the correct statement(s). [Take mass of earth as M]



(1) The total energy of the two satellites plus earth system just before collision is $-(GMm)/r$.
(2) The total energy of the two satellites plus earth system just after collision is $-(2GMm)/r$.
(3) The total energy of two satellites plus earth system just after collision is $-(GMm)/2r$.
(4) The combined mass (two satellites) will fall towards the earth just after collision.

29. A particle of mass m is thrown in a parabolic path towards earth having mass M and radius R. It comes to a greatest approach distance $R/2$ from surface of earth. Magnitude of total energy change required to make the particle moving in an elliptical path having eccentricity 1/2 with the greatest approach point either aphelion or perihelion are

(1) $\frac{1}{3}\frac{GMm}{R}$ (2) $\frac{1}{6}\frac{GMm}{R}$

(3) $\frac{1}{2}\frac{GMm}{R}$ (4) $\frac{GMm}{R}$

30. An artificial satellite of mass m is moving in a circular orbit around the earth of mass M and radius R with a speed equal to half of the magnitude of escape velocity on the surface of the earth (g = acceleration due to gravity on the surface of earth). Then

(1) If the satellite is stopped suddenly in its orbit and is allowed to fall freely towards the earth, the speed with which it hits the surface of earth is $\sqrt{gR}$.

(2) The escape velocity with respect to the earth in the same orbit is $\sqrt{\frac{GM}{R}}$.

(3) The energy spent in shifting this satellite to an orbit at a further height $R/2$ from the present orbit is $\frac{GM}{12}\frac{m}{R}$.

(4) The energy spent in shifting this satellite to an orbit at a further height $R/2$ from the present orbit is $\frac{GM}{20}\frac{m}{R}$.

31. Two particles of mass m and $4m$ are at rest at an infinite separation. They move towards each other under mutual gravitational attraction. If G is the universal gravitational constant. Then at separation r, (Assume zero reference potential energy at infinite separation)

(1) the total energy of the two object is zero

(2) their relative velocity of approach is $\left(\frac{10Gm}{r}\right)^{1/2}$ in magnitude

(3) the total kinetic energy of the object is $\frac{4Gm^2}{r}$

(4) Net angular momentum of both the particles is zero about any point

32. A satellite is orbiting round the earth's surface in an orbit as close as possible to the surface of the earth. Then,

(1) the time period of revolution of satellite is independent of its mass and is maximum

(2) the orbital speed of satellite is maximum

(3) the kinetic energy of the satellite is minimum

(4) the total energy of the "earth plus satellite" system is maximum.

33. Two spherical planets have the same mass but densities in the ratio 1 : 8. For these planets, the

(1) acceleration due to gravity will be in the ratio 4 : 1

(2) acceleration due to gravity will be in the ratio 1 : 4

(3) escape velocities from their surfaces will be in the ratio $\sqrt{2}:1$

(4) escape velocities from their surfaces will be in the ratio $1:\sqrt{2}$

Linked Comprehension Type

For Problems 1–3

The orbit of Pluto is much more eccentric than the orbits of the other planets. That is, instead of being nearly circular, the orbit is noticeably elliptical. The point in the orbit nearest to the Sun is called the perihelion and the point farthest from the Sun is called the aphelion.

1. At perihelion, the gravitational potential energy of Pluto in its orbit has

(1) its maximum value

(2) its minimum value

(3) the same value as at every other point in the orbit

(4) the value which depends on the sense of rotation

2. At perihelion, the mechanical energy of Pluto's orbit has

(1) its maximum value

(2) its minimum value

(3) the same value as at every other point in the orbit

(4) the value which depends on the sense of rotation

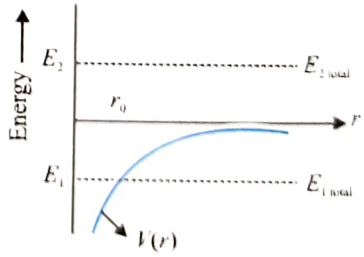
3. As Pluto moves from the perihelion to the aphelion, the work done by gravitational pull of Sun on Pluto is

(1) is zero (2) is positive

(3) is negative (4) depends on sense of rotation

For Problems 4–6

In the graph shown, the PE of earth–satellite system is shown by a solid line as a function of distance r (the separation between earth's centre and satellite). The total energy of the two objects which may or may not be bounded to the earth are shown in the figure by dotted lines.



Based on the above information, answer the following questions.

4. Mark the correct statement(s).

(1) The object having the total energy E_1 is bounded one.

(2) The object having the total energy E_2 is bounded one.

(3) Both the objects are bounded.

(4) Both the objects are unbounded.

5. If object having total energy E_1 is having the same PE curve as shown in the figure, then
   (1) r_0 is the maximum distance of the object from the earth's centre
   (2) this object and the earth system is bounded one
   (3) the KE of the object is zero when $r = r_0$
   (4) all the above

6. If both the objects have the same PE curve as shown in the figure, then
   (1) for objects having total energy E_2, all values of r are possible
   (2) for the object having total energy E_2, values of $r < r_0$ are only possible
   (3) for the object having total energy E_1, all values of r are possible
   (4) none of the above

For Problems 7–8

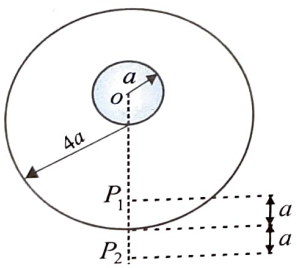
The minimum and maximum distances of a satellite from the centre of the earth are $2R$ and $4R$, respectively, where R is the radius of the earth and M is the mass of the earth.

7. The minimum and maximum speeds are
   (1) $\sqrt{\frac{GM}{9R}}, \sqrt{\frac{2GM}{R}}$ (2) $\sqrt{\frac{GM}{5R}}, \sqrt{\frac{3GM}{2R}}$
   (3) $\sqrt{\frac{GM}{6R}}, \sqrt{\frac{2GM}{3R}}$ (4) $\sqrt{\frac{GM}{3R}}, \sqrt{\frac{5GM}{2R}}$

8. The radius of curvature at the point of minimum distance is
   (1) $\frac{8R}{3}$ (2) $\frac{5R}{3}$
   (3) $\frac{4R}{3}$ (4) $\frac{7R}{3}$

For Problems 9–10

A uniform metal sphere of radius a and mass M is surrounded by a thin uniform spherical shell of equal mass and radius $4a$. The centre of the shell falls on the surface of the inner sphere.



9. Find the gravitational field intensity at point P_1.
   (1) $\frac{GM}{16a^2}$ (2) $\frac{GM}{8a^2}$
   (3) $\frac{GM}{2a^2}$ (4) $\frac{GM}{4a^2}$

10. Find the gravitational field intensity at point P_2.
   (1) $\frac{21GM}{900a^2}$ (2) $\frac{61GM}{450a^2}$
   (3) $\frac{61GM}{900a^2}$ (4) $\frac{61GM}{1800a^2}$

For Problems 11–15

The satellites when launched from the earth are not given the orbital velocity initially, a multistage rocket propeller carries the spacecraft up to its orbit and during each stage rocket has been fired to increase the velocity to acquire the desired velocity for a particular orbit. The last stage of the rocket brings the satellite in circular/elliptical (desired) orbit.

Consider a satellite of mass 150 kg in a low circular orbit. In this orbit, we cannot neglect the effect of air drag. This air opposes the motion of satellite and hence the total mechanical energy of earth–satellite system decreases. That means the total energy becomes more negative and hence the orbital radius decreases which causes the increase in KE. When the satellite comes in the low energy orbit, excessive thermal energy generation due to air friction may cause the satellite to burn up. Based on the above information, answer the following questions.

11. It has been mentioned in the passage that as r decreases, E decreases but K increases. The increase in K is [E = total mechanical energy, r = orbital radius, K= kinetic energy]
   (1) due to increase in gravitational PE
   (2) due to decrease in gravitational PE
   (3) due to work done by air friction force
   (4) both (2) and (3)

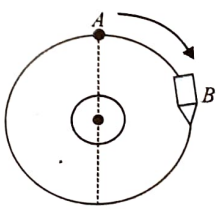
12. If due to air drag, the orbital radius of the earth decreases from R to $R - \Delta R$, $\Delta R \ll R$, then the expression for increase in the orbital velocity Δv is
   (1) $\frac{\Delta R}{2}\sqrt{\frac{GM}{R^3}}$ (2) $-\frac{\Delta R}{2}\sqrt{\frac{GM}{R^3}}$
   (3) $\Delta R\sqrt{\frac{GM}{R^3}}$ (4) $-\Delta R\sqrt{\frac{GM}{R^3}}$

13. For information given in problem 12, the change in KE, ΔK is
   (1) $\frac{GMm}{R^2} \times \Delta R$ (2) $-\frac{GMm}{R^2} \times \Delta R$
   (3) $\frac{GMm}{2R^2} \times \Delta R$ (4) $-\frac{GMm}{2R^2} \times \Delta R$

14. For information given in problem 12, the change in PE, ΔU is
   (1) $\frac{GMm}{R^2} \times \Delta R$ (2) $-\frac{GMm}{R^2} \times \Delta R$
   (3) $\frac{GMm}{2R^2} \times \Delta R$ (4) $-\frac{GMm}{2R^2} \times \Delta R$

15. For information given in problem 12, the work done by air friction force W is
   (1) $\frac{GMm}{R^2} \times \Delta R$ (2) $-\frac{GMm}{R^2} \times \Delta R$
   (3) $\frac{GMm}{2R^2} \times \Delta R$ (4) $-\frac{GMm}{2R^2} \times \Delta R$

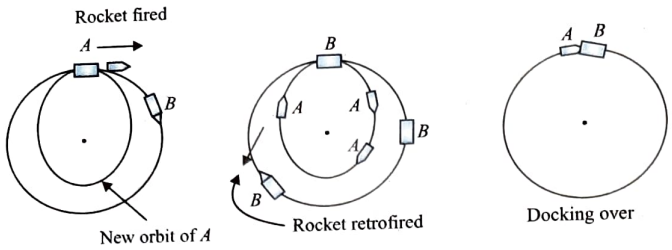
For Problems 16–20



An unmanned satellite A and a spacecraft B are orbiting around the earth in the same circular orbit as shown.

The spacecraft is ahead of the satellite by some time. Let us consider that some technical problem has arisen in the satellite and the astronaut from B has made it correct. For this to be done docking of two (A and B) is required (in layman

terms connecting A and B). To achieve this, the rockets of A have been fired in forward direction and docking takes place as shown in the figure below:



Take mass of the earth = 5.98×10^{24} kg
Radius of the earth = 6400 km
Orbital radius = 9600 km
Mass of satellite A = 320 kg
Mass of spacecraft = 3200 kg
Assume that initially spacecraft B leads satellite A by 100 s, i.e., A arrives at any particular position after 100 s of B's arrival. Based on the above information answer the following questions.

16. To dock A and B in the above-described situation, one can use the rocket system of either one, i.e., either of A or of B. To accomplish docking in the minimum possible time which is the best way?
(1) To use rocket system of A.
(2) To use rocket system of B.
(3) Either (1) or (2).
(4) Information insufficient.

17. The initial total energy and time period of satellite are, respectively,
(1) -6.65×10^{10} J, 9358 s (2) -6.65×10^{9} J, 9358 s
(3) -6.65×10^{10} J, 9140 s (4) -6.65×10^{9} J, 9140 s

18. If from the base station, the rocket system of A has been operated, so that its rocket has been fired in forward direction thus reducing the speed of A by 0.5%, then which of the following will happen?
(1) Its orbit becomes elliptical with semi-major axis 9505.3 km.
(2) Its total energy becomes -6.714×10^{9} J.
(3) Its new time period becomes 9219.67 s.
(4) All of the above.

19. By doing the above operation, now by how much time is the satellite leading the spacecraft?
(1) 38 s (2) 138 s
(3) Lags by 38 s (4) Lags by 138 s

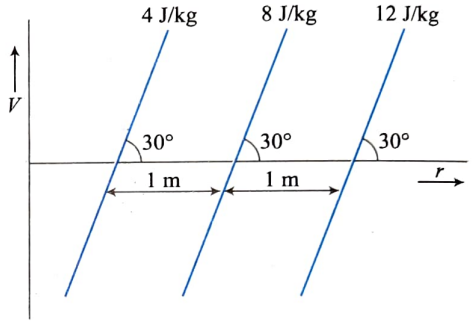
20. After once returning to the original point, i.e., the place from where the rockets have been fired, in which direction and with what extent the rockets have to be fired from the satellite to again come back in the original orbit?
(1) Forward direction with the same extent.
(2) Backward direction with the same extent.
(3) Forward direction with the higher extent.
(4) Backward direction with the higher extent.

For Problems 21–22

Gravitational force is a conservative and medium independent force. Its nature is attractive. Gravitational field intensity and gravitational potential gives information about gravitational field in vector and scalar form respectively. Actually gravitational field intensity is equal to the negative of the potential gradient. Potential energy is defined for only conservative force. It is also equal to the total energy in escaping condition. Gravitational potential is either negative or zero but can never be positive due to attractive nature of gravitational force, if potential at infinity is taken zero.

21. A person brings a mass of 1 kg from infinity to a point A. Initially the mass was at rest but it moves with a speed of 2 m/s as it reaches A. The work done by the person on the mass is –3J. The potential of A is
(1) –3 J/kg (2) –2 J/kg
(3) –5 J/kg (4) –7 J/kg

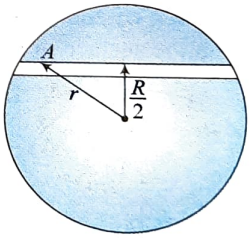
22. Gravitational potential V versus distance r graph is represented in figure. The magnitude of gravitational field intensity is equal to



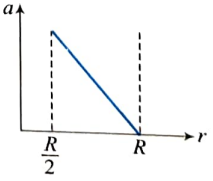
(1) 8 N/kg (2) 4 N/kg
(3) 6 N/kg (4) 2 N/kg

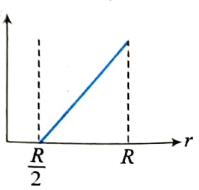
For Problem 23–24

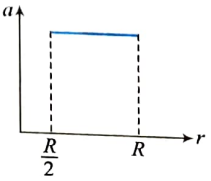
A frictionless tunnel is dug along a chord of the earth of mass M_e at a perpendicular distance $R/2$ from the centre of earth (where R is radius of earth). An object is released from one end of the tunnel.

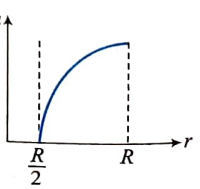


23. The correct graph, showing the variation of magnitude of acceleration of object with its distance from centre of earth is

(1) 

(2) 

(3) 

(4) 

24. Time period of oscillation of the object is

(1) $T = 2\pi\sqrt{\frac{R^3}{GM_e}}$ (2) $T = 2\pi\sqrt{\frac{2R^3}{GM_e}}$

(3) $T = 2\pi\sqrt{\frac{R^3}{2GM_e}}$ (4) $T = 2\pi\sqrt{\frac{R^3}{4GM_e}}$

Matrix Match Type

1. Considering the earth to be a homogeneous sphere but keeping in mind its spin, match the following:

Column I	Column II
i. Acceleration due to gravity	a. may change from point to point
ii. Orbital angular momentum of the earth as seen from a distant star	b. does not depend on direction of projection
iii. Escape velocity from the earth	c. remains constant
iv. Gravitational potential due to earth at a particular point	d. depend on direction of projection

2.

Column I	Column II
i. Geostationary satellite	a. Circular orbit
ii. Total energy of earth–satellite system is constant in	b. Elliptical orbit
iii. Angular momentum of satellite about centre of earth is constant in	c. Equatorial plane orbit
iv. Orbital speed of satellite is/ may be constant	d. Non-equatorial plane orbit

3. An artificial satellite is in circular orbit around the earth. One of the rockets of the satellite is momentarily fired, the direction of firing of rocket is mentioned in Column I and corresponding change(s) are given in Column II. Match the entries of Column I with the entries of Column II.

Column I	Column II
i. Towards the earth's centre	a. Orbit changes and becomes elliptical
ii. Away from the earth's centre	b. Orbit plane changes
iii. At right angle to the plane or orbit	c. Semi-major axis of orbit increases
iv. In forward direction	d. Energy of earth–satellite system increases

4. A satellite is revolving around the earth in a circular orbit of radius a with velocity v_0. A particle is projected from the satellite in forward direction with relative velocity $v = [(\sqrt{5}/4) - 1]\, v_0$. During the subsequent motion of the particle, match the following:

Column I	Column II
i. Total energy of particle	a. $(3GM_em)/a$
ii. Minimum distance of particle from the earth	b. $(5GM_em)/a$
iii. Maximum distance of particle from the earth	c. $5a/3$
	d. $2a$
	e. $2a/3$
	f. a

5. Let V and E denote the gravitational potential and gravitational field, respectively, at a point due to certain uniform mass distribution described in four different situations of Column I. Assume the gravitational potential at infinity to be zero. The values of E and V are given in Column II. Match the statement in column I with the results in Column II.

Column I	Column II
i. At the centre of thin spherical shell	a. $E = 0$
ii. At the centre of solid sphere	b. $E \neq 0$
iii. A solid sphere has a non-concentric spherical cavity. At the centre of the spherical cavity	c. $V \neq 0$
iv. At the centre of the joining two point masses of equal magnitude	d. $V = 0$

6. For a planet orbiting about the Sun in a elliptical orbit, some incomplete statements regarding physical quantities are given in Column I, which can be completed by using the entries of Column II. Match the entries of Column I with the entries of Column II.

Column I	Column II
i. Maximum PE of the Sun–planet system	a. is at perihelioin
ii. Maximum speed of the planet	b. is at aphelion
iii. Minimum PE of Sun–planet system	c. is independent of mass of planet
iv. Minimum kinetic energy of planet	d. is independent of semi-major axis of orbit

7.

Column I	Column II
i. Gravitational potential	a. On the surface of planets with density ratio 1 : 2
ii. Escape velocity	b. Conservation of angular momentum
iii. Ratio of the acceleration due to gravity 1 : 3	c. Varies with the reference point
iv. Orbiting satellite	d. Does not depend on the angle
	e. Similar to an atom

8. If rotational speed of earth is reduced, match the columns:

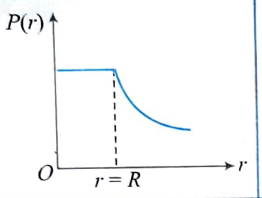
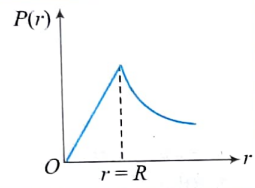
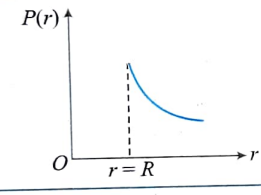
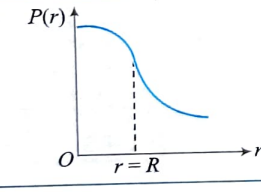
Column I	Column II
i. Distance of geostationary satellite	a. will increase
ii. Energy of geostationary satellite	b. will decrease
iii. Value of g at equator	c. will remain same
iv. Value of g at pole	d. unpredictable

Now match the given columns and select the correct option from the codes given below.

Codes

	i	ii	iii	iv
(1)	c	c	d	a
(2)	a	d	b	c
(3)	a	a	a	c
(4)	a	c	d	d

9. $P(r)$ is magnitude of physical quantity as a function of r (distance from center of spherical distribution of radius R). match the following columns and select the correct option from the codes given below.

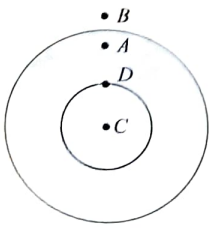
Column I	Column II
i. 	**a.** $P(r)$ is variation of gravitational potential for a uniform volumetric spherical distribution of mass.
ii. 	**b.** $P(r)$ is variation of gravitational potential for a uniform spherical shell.
iii. 	**c.** $P(r)$ is variation of gravitational field intensity for a uniform volumetric spherical distribution of mass.
iv. 	**d.** $P(r)$ is variation of gravitational field intensity for a uniform spherical shell.

Now match the given columns and select the correct option from the codes given below.

Codes

	i	ii	iii	iv
(1)	d	c	b	a
(2)	b	d	c	a
(3)	b	c	d	a
(4)	c	d	a	b

10. Two concentric spherical shells are as shown in the figure. Match the following columns and select the correct option from the codes given below.



Column I	Column II
i. Potential at A	**a.** greater than B
ii. Gravitational field at A	**b.** less than B
iii. As one moves from C to D	**c.** potential remains constant
iv. As one moves from D to A	**d.** gravitational field decreases
	e. none of these

Now match the given columns and select the correct option from the codes given below.

Codes

	i	ii	iii	iv
(1)	b	e	c	d
(2)	a	b	c	d
(3)	b	c	d	a
(4)	c	d	e	b

Numerical Value Type

1. The earth (mass = 10^{24} kg) revolves round the Sun with an angular velocity 2×10^{-7} rad s^{-1} in a circular orbit of radius 1.5×10^{8} km. Find the force exerted by the Sun on the earth (in $\times 10^{21}$ N).

2. The density of a newly discovered planet is twice that of the earth. The acceleration due to gravity at the surface of the planet is equal to that at the surface of the earth. If the radius of the earth is R and the radius of the planet is R'. Then what is the value of R/R'?

3. Imagine a new planet having the same density as that of the earth but it is 3 times bigger than the earth in size. If the acceleration due to gravity on the surface of the earth is g and that on the new planet is g', then what is the value of g'/g?

4. A particle of mass m is subjected to an attractive central force of magnitude k/r^2, k being a constant. If at the instant when the particle is at an extreme position in its closed orbit at a distance a from the centre of force, its speed is $(k/2ma)$, if the distance of other extreme position is b. Find a/b.

5. A small 2 kg mass moved slowly from the surface of earth to a height of 6.4×10^{6} m above the earth. Find the work done [in mega joule].

6. The sun's mass is about 3.2×10^{5} times the earth's mass. The sun is about 400 times as far from the earth as the earth is from the moon. Assume that the sun-moon distance is constant and equal to sun-earth distance. The ratio of the magnitudes of the gravitational pull of the sun on the moon (F_{ms}) and of the earth on the moon (F_{em}) will be $\frac{F_{ms}}{F_{me}} = \text{'*'}$. Find '*'.

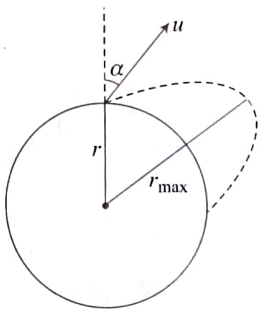
7. If the radius of the earth were to shrink by 2% its mass remaining the same, by how much percentage would the

acceleration due to gravity on the earth's surface would increase?

8. A particle is fixed vertically upwards with a speed of 15 km s^{-1}. The speed of the particle (in km s^{-1}), when it goes out of the earth's gravitational pull is 5α. Find the value of α. [Given: $v_e^2 = 125$ (km s^{-1})^2]

9. Gravitational acceleration on the surface of the planet is $\frac{\sqrt{6}}{11}g$, where g is the acceleration due to gravity on the surface of earth. The average mass density of the planet is 2/3 time that of the earth. If the escape speed on the surface of the earth is taken to be 11 km s^{-1}, then find the escape speed on the surface of the planet (in km s^{-1}).

10. A projectile of mass m is fired from the surface of earth at an angle $\alpha = 60°$ from the vertical. The initial speed of the projectile is $u = \sqrt{\frac{GM}{R}}$, where M and R are mass and radius of the earth. Find the height attained by the projectile in km. Neglect air resistance and the rotation of earth. (Given: $R = 6400$ km)



11. Suppose earth's orbital motion around the sun is suddenly stopped. Find the time (in days) taken by the earth to fall into the sun.

12. A ring of radius $R = 4$ m made of a highly dense material has a mass $m_1 = 5.4 \times 10^9$ kg distributed uniformly over its circumference. A highly dense particle of mass $m_2 = 6 \times 10^8$ kg is placed on the axis of the ring at a distance 3 m from the center of the ring. Find the speed of the particle (in cm/s), when the particle is at the center of the ring. Except mutual gravitational interaction of the two, neglect all other forces.

13. A body weighs 64 N on the surface of the earth. What is the gravitational force (in N) on it due to the earth at a height equal to one-third of the radius of the earth?

14. Find the percentage decrease in weight of a body, when taken 16 km below the surface of the earth. Take radius of the earth as 6,400 km.

15. The radius of the earth is about 6,400 km and that of mars is about 3,200 km. The mass of the earth is about 10 times the mass of mars. An object weighs 80 N on the surface of earth. What will be its weight (in N) on the surface of mars?

16. The value of acceleration due to gravity at the surface of the earth is 9.8 m s^{-2} and its mean radius is about 6.4×10^6 m. Assuming that we could get more soil from somewhere, estimate how thick (in km)would an added uniform outer layer on the earth have to have the value of acceleration due to gravity 10 m s^{-2} exactly ? (Given the density of the earth's soil, $\rho = 5{,}520$ kg m^{-3}.)

17. A particle of mass 50 g is kept on the surface of a uniform sphere of mass 120 kg and radius 50 cm. Find the work to be done against the gravitational force between them (in nJ) to take the particle far away from the sphere. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

18. The small dense stars rotate about their common centre of mass as a binary system, each with a period of 1 year. One star has mass double than that of the other, while mass of the lighter star is one-third the mass of the Sun. The distance between the two stars is r and the distance of the earth from the Sun is R; find the ratio r/R.

19. Taking the earth to be a uniform sphere of radius 6400 km and the value of g at the surface to be 10 m s^{-2}, calculate the energy needed (in $\times 10^{10}$ J) to raise a satellite of mass 2000 kg to a height of 800 km above the earth's surface and to set it into circular orbit at that altitude.

20. A sky laboratory of mass 3×10^3 kg has to be lifted from one circular orbit of radius $2R$ into another circular orbit of radius $3R$. Calculate the minimum energy(in$\times 10^{10}$ J) required if the radius of earth is $R = 6.4 \times 10^6$ m and $g = 10$ m s^{-2}.

21. The gravitational field in a region is given by $\vec{E} = (3\hat{i} - 4\hat{j})$ N kg^{-1}. Find out the work done (in joule) in displacing a particle by 1 m along the line $4y = 3x + 9$.

22. A man can throw a ball at a speed on the earth which can cross a river of width 10 m. The man reaches on an imaginary planet whose mean density is twice that of the earth. Find out the maximum possible radius of the planet (in km) so that if the man throws the ball at the same speed it may escape from the planet. Given radius of the earth $= 6.4 \times 10^6$ m.

23. The earth is assumed to be a sphere of radius R. A platform is arranged at a height 3 R from the surface of the earth. The escape velocity of a body from this platform is $f v_e$, where v_e is its escape velocity from the surface of the earth. Find the value of f.

24. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth. If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed (in m/s) with which it hits the surface of the earth.

Archives

JEE Main

Single Correct Answer Type

1. Two bodies of masses m and $4m$ are placed at a distance r. The gravitational potential at a point on the line joining them where the gravitational field is zero is

(1) zero (2) $-\frac{4Gm}{r}$

(3) $-\frac{6Gm}{r}$ (4) $-\frac{9Gm}{r}$ **(AIEEE 2011)**

2. The mass of a spaceship is 1000 kg. It is to be launched from the earth's surface out into free space. The value of 'g' and 'R' (radius of earth) are 10 m/s^2 and 6400 km, respectively. The required energy for this work will be

(1) 6.4×10^{11} J (2) 6.4×10^{8} J

(3) 6.4×10^{9} J (4) 6.4×10^{10} J

(AIEEE 2012)

3. What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of $2R$?

(1) $\frac{2GmM}{3R}$ (2) $\frac{GmM}{2R}$

(3) $\frac{GmM}{3R}$ (4) $\frac{5GmM}{6R}$

(JEE Main 2013)

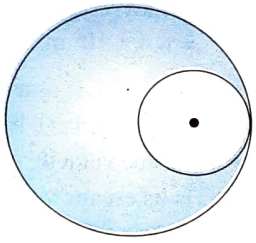
4. Four particles, each of mass M and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(1) $\sqrt{\frac{GM}{R}(1+2\sqrt{2})}$ (2) $\frac{1}{2}\sqrt{\frac{GM}{R}(1+2\sqrt{2})}$

(3) $\sqrt{\frac{GM}{R}}$ (4) $\sqrt{2\sqrt{2}\frac{GM}{R}}$

(JEE Main 2014)

5. From a solid sphere of mass M and radius R, a spherical portion of radius $R/2$ is removed, as shown in the figure. Taking gravitational potential $V = 0$ at $r = \infty$, the potential at the centre of the cavity thus formed is (G = gravitational constant)



(1) $\frac{-GM}{2R}$ (2) $\frac{-GM}{R}$

(3) $\frac{-2GM}{3R}$ (4) $\frac{-2GM}{R}$

(JEE Main 2015)

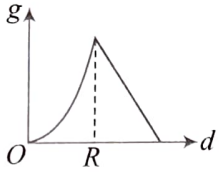
6. A satellite is revolving in a circular orbit at a height 'h' from the earth's surface (radius of earth R; $h << R$). The minimum increases in its orbital velocity required, so that the satellite could escape from the earth's gravitational field, is close to: (Neglect the effect of atmosphere.)

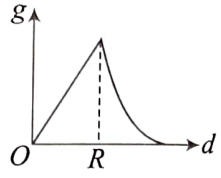
(1) $\sqrt{2gR}$ (2) $\sqrt{gR}$

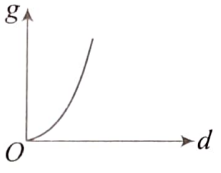
(3) $\sqrt{gR/2}$ (4) $\sqrt{gR}\left(\sqrt{2}-1\right)$

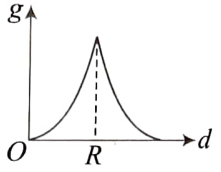
(JEE Main 2016)

7. The variation of acceleration due to gravity g with distance d from centre of the earth is best represented by (R = Earth's radius)

(1) 

(2) 

(3) 

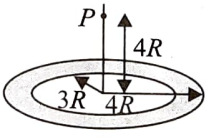
(4) 

(JEE Main 2017)

JEE Advanced

Single Correct Answer Type

1. A thin uniform annular disc (see figure) of mass M has an outer radius $4R$ and an inner radius $3R$. The work required to take a unit mass from point P on its axis to infinity is



(1) $\frac{2GM}{7R}(4\sqrt{2}-5)$ (2) $-\frac{2GM}{7R}(4\sqrt{2}-5)$

(3) $\frac{GM}{4R}$ (4) $\frac{2GM}{5R}(\sqrt{2}-1)$

(IIT-JEE 2010)

2. A satellite is moving with a constant speed V in a circular orbit about the earth. An object of mass m is ejected from the satellite such that it just escapes from the gravitational pull of the earth. At the time of its ejection, the kinetic energy of the object is

(1) $\frac{1}{2}mV^2$ (2) mV^2

(3) $\frac{3}{2}mV^2$ (4) $2\,mV^2$ **(IIT-JEE 2010)**

3. A planet of radius $R = 1/10$ (radius of the Earth) has the same mass density as the Earth. Scientists dig a well of depth $R/5$ on it and lower a wire of the same length and of linear mass density 10^{-3} kgm^{-1} into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of Earth $= 6 \times 10^6$ m and the acceleration due to gravity of Earth is 10 ms^{-2})

(1) 96 N (2) 108 N

(3) 120 N (4) 150 N

(JEE Advanced 2014)

4. A rocket is launched normal to the surface of the Earth, away from the Sun, along the line joining the Sun and the Earth. The Sun is 3×10^5 times heavier than the Earth and is at a distance 2.5×10^4 times larger than the radius of the Earth. The escape velocity from Earth's gravitational field is $v_e = 1.2$ km s^{-1}. The minimum initial velocity (v_s) required for the rocket to be able to leave the Sun–Earth system is closest to (Ignore the rotation and revolution of the Earth and the presence of any other planet)

(1) $v_s = 22$ km s^{-1} (2) $v_s = 72$ km s^{-1}

(3) $v_s = 42$ km s^{-1} (4) $v_s = 62$ km s^{-1}

(JEE Advanced 2017)

Multiple Correct Answers Type

1. Two spherical planets P and Q have the same uniform density ρ, masses M_P and M_Q, and surface areas A and $4A$, respectively. A spherical planet R also has uniform density ρ and its mass is $(M_P + M_Q)$. The escape velocities from the planets P, Q and R, are V_P, V_Q and V_R. respectively. Then

(1) $V_Q > V_R > V_P$ (2) $V_R > V_Q > V_P$

(3) $V_R/V_P = 3$ (4) $V_P/V_Q = 1/2$

(IIT-JEE 2012)

2. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the midpoint of the line joining their centres, perpendicular to the line. The gravitational constant is G. The correct statement(s) is (are)

(1) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $4\sqrt{\frac{GM}{L}}$

(2) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $2\sqrt{\frac{GM}{L}}$

(3) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $\sqrt{\frac{2GM}{L}}$

(4) The energy of the mass m remains constant

(JEE Advanced 2013)

3. A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If $P(r)$ is the pressure at $r(r < R)$, then the correct option(s) is (are)

(1) $P(r = 0) = 0$ (2) $\frac{P(r = 3R/4)}{P(r = 2R/3)} = \frac{63}{80}$

(3) $\frac{P(r = 3R/5)}{P(r = 2R/3)} = \frac{16}{21}$ (4) $\frac{P(r = R/2)}{P(r = R/3)} = \frac{20}{27}$

(JEE Advanced 2015)

Matrix Match Type

1. A planet of mass M, has two natural satellites with masses m_1 and m_2. The radii of their circular orbits are R_1 and R_2 respectively. Ignore the gravitational force between the satellites. Define v_1, L_1, K_1 and T_1 to be, respectively, the orbital speed, angular momentum, kinetic energy and time period of revolution of satellite 1 and v_2, L_2, K_2 and T_2 to be the corresponding quantities of satellite 2. Given $m_1/m_2 = 2$ and $R_1/R_2 = 1/4$, match the ratios in List-I to the numbers in List-II.

List-I		List-II	
P.	$\frac{v_1}{v_2}$	1.	1/8
Q.	$\frac{L_1}{L_2}$	2.	1
R.	$\frac{K_1}{K_2}$	3.	2
S.	$\frac{T_1}{T_2}$	4.	8

(1) $P \to 4; Q \to 2; R \to 1; S \to 3$

(2) $P \to 3; Q \to 2; R \to 4; S \to 1$

(3) $P \to 2; Q \to 3; R \to 1; S \to 4$

(4) $P \to 2; Q \to 3; R \to 4; S \to 1$

(JEE Advanced 2018)

Numerical Value Type

1. Gravitational acceleration on the surface of a planet is $\frac{\sqrt{6}}{11}g$, where g is the gravitational acceleration on the surface of the earth. The average mass density of the planet is 2/3 times that of the earth. If the escape speed on the surface of the earth is taken to be 11 kms^{-1}, then find the escape speed on the surface of the planet in kms^{-1}.

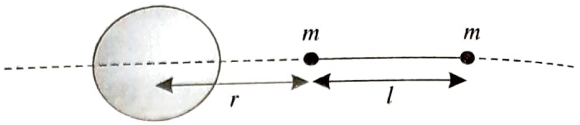
(IIT-JEE 2010)

2. A bullet is fired vertically upwards with velocity v from the surface of a spherical planet. When it reaches its maximum height, its acceleration due to the planet's gravity is $1/4^{th}$ of its value at the surface of the planet. If the escape velocity from the planet is $v_{esc} = v\sqrt{N}$, then the value of N is _____. (Ignore energy loss due to atmosphere)

(JEE Advanced 2015)

3. A large spherical mass M is fixed at one position and two identical point masses m are kept on a line passing through

the centre of M (see figure). The point masses are connected by a rigid massless rod of length l and this assembly is free to move along the line connecting them. All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance $r = 3l$ from M, the tension in the rod is zero for $m = k\left(\frac{M}{288}\right)$. The value of k is



(JEE Advanced 2015)

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	**2.** (4)	**3.** (4)	**4.** (2)	**5.** (1)
6. (3)	**7.** (4)	**8.** (3)	**9.** (1)	**10.** (3)
11. (4)	**12.** (2)	**13.** (3)	**14.** (4)	**15.** (1)
16. (4)	**17.** (2)	**18.** (3)	**19.** (3)	**20.** (3)
21. (2)	**22.** (2)	**23.** (3)	**24.** (2)	**25.** (3)
26. (4)	**27.** (1)	**28.** (4)	**29.** (4)	**30.** (4)
31. (2)	**32.** (2)	**33.** (2)	**34.** (1)	**35.** (3)
36. (1)	**37.** (3)	**38.** (3)	**39.** (2)	**40.** (4)
41. (2)	**42.** (1)	**43.** (4)	**44.** (2)	**45.** (2)
46. (1)	**47.** (3)	**48.** (2)	**49.** (4)	**50.** (2)
51. (4)	**52.** (1)	**53.** (3)	**54.** (1)	**55.** (2)
56. (4)	**57.** (3)	**58.** (1)	**59.** (2)	**60.** (3)
61. (3)	**62.** (4)	**63.** (3)	**64.** (2)	**65.** (3)
66. (2)	**67.** (1)	**68.** (1)	**69.** (2)	**70.** (1)
71. (1)				

Multiple Correct Answers Type

1. (1),(2),(3),(4)	**2.** (2),(4)	**3.** (2),(3)
4. (1),(3),(4)	**5.** (3),(4)	**6.** (1),(2),(3)
7. (1),(2),(3),(4)	**8.** (2),(3)	**9.** (1),(3)
10. (1),(3),(4)	**11.** (1),(2),(3),(4)	**12.** (1),(4)
13. (1),(3)	**14.** (2),(3)	**15.** (3),(4)
16. (1),(2),(3)	**17.** (2),(3)	**18.** (3),(4)
19. (2),(3),(4)	**20.** (2),(4)	**21.** (1),(3),(4)
22. (2),(3)	**23.** (1),(3),(4)	**24.** (1),(4)
25. (2),(3)	**26.** (1),(3)	**27.** (1),(2),(3),(4)
28. (1),(2),(4)	**29.** (2),(3)	**30.** (1),(2),(3)
31. (1),(2),(3),(4)	**32.** (1),(2),(3)	**33.** (2),(4)

Linked Comprehension Type

1. (2)	**2.** (3)	**3.** (3)	**4.** (1)	**5.** (4)
6. (1)	**7.** (3)	**8.** (1)	**9.** (1)	**10.** (3)
11. (2)	**12.** (1)	**13.** (3)	**14.** (2)	**15.** (4)
16. (3)	**17.** (2)	**18.** (4)	**19.** (1)	**20.** (2)
21. (3)	**22.** (1)	**23.** (4)	**24.** (1)	

Matrix Match Type

1. i. → a., b.; ii. → c.; iii. → a., b.; iv. → c.
2. i. → a., c.; ii. → a., b., c., d.; iii. → a., b., c., d.; iv. → a., c., d.
3. i.→ a.; ii. → a., c., d.; iii. → a., b., c., d.; iv. → a.
4. i. → a.; ii. → f.; iii. → c.
5. i. → a., c.; ii. → a., c.; iii. → a., c.; iv. → a., c.
6. i. → b., d.; ii. → a., c., d.; iii. → a., d.; iv. → b., c.
7. i. → c., d.; ii. → d.; iii. → a.; iv. → b., c.
8. (3) **9.** (3) **10.** (1)

Numerical Value Type

1. (6)	**2.** (2)	**3.** (3)	**4.** (3)	**5.** (64)
6. (2)	**7.** (4)	**8.** (2)	**9.** (3)	**10.** (3200)
11. (65.00)	**12.** (18.00)	**13.** (36.00)	**14.** (0.25)	**15.** (32.00)
16. (130.0)	**17.** (0.80)	**18.** (1)	**19.** (7.10)	**20.** (1.60)
21. (0)	**22.** (04.00)	**23.** (0.50)	**24.** (8000)	

ARCHIVES

JEE Main

Single Correct Answer Type

1. (1)	**2.** (4)	**3.** (4)	**4.** (2)	**5.** (2)
6. (4)	**7.** (2)			

JEE Advanced

Single Correct Answer Type

1. (1)	**2.** (2)	**3.** (2)	**4.** (3)

Multiple Correct Answers Type

1. (2),(4)	**2.** (2),(4)	**3.** (2),(3)

Matrix Match Type

1. (2)

Numerical Value Type

1. (3)	**2.** (2)	**3.** (7)

6 Fluid Mechanics

INTRODUCTION

A fluid is a substance, such as a liquid or a gas, that has no rigidity like solids. Liquids are distinguished from the gases by the presence of a surface. As the fluids have no rigidity, they fail to support a shearing stress. When a fluid is subjected to a shearing stress, the fluid layers slide relative to each other. This characteristic gives fluid the ability to flow or change shapes. However, in this chapter, we confine ourselves to the study of liquids only.

HYDROSTATICS

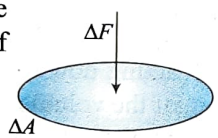
PRESSURE

While considering the equilibrium of solids we usually think in terms of forces exerted by one body on another body. But in case of liquids, we usually speak in terms of pressure instead of forces. If a uniform force F is exerted normal to an area A, then the pressure P is defined as

$$P = \frac{F}{A}$$

Note: The area A in the above expression is the actual area of contact. In case of solids, the actual area of contact between two surfaces is less than the apparent area of contact. Therefore, the concept of pressure is of no use in case of solids. On the contrary, the contact between a liquid and a solid surface or between two liquid surfaces is rather a smoother one, that is, the actual area of contact is equal to the apparent area of contact. Therefore, one can easily use the concept of pressure.

If the applied force F is not uniform, then the pressure at a point is defined as the intensity of force at that point. That is,



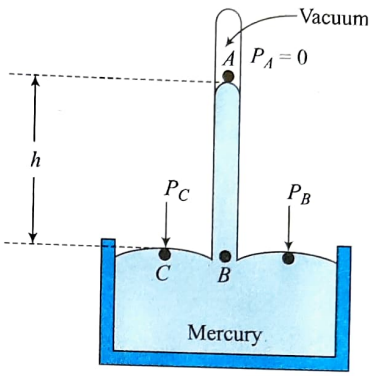
$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

Pressure is a scalar quantity. Pressure acts normal to a surface and it is always compressive in nature. Therefore, only its magnitude is required for its complete description. The SI unit of pressure is N/m^2 and is also called pascal (Pa). The other common units of pressure are atmosphere and bar.

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} \Rightarrow 1 \text{ bar} = 10^5 \text{ Pa}$$

MEASURING PRESSURE

Figure shows a mercury barometer which was invented by Torricelli. It is a device used for measuring pressure. A long glass tube (may be 1 m long) is filled with mercury and inverted with its open end in a dish of mercury. The mercury level in the tube falls and comes to rest at a vertical height of 76 cm above the mercury level in the dish.



The space above mercury in the tube is almost a vacuum. So $P_A = 0$. At point C, there is only atmospheric pressure. So $P_C = P_B$ which is equal to the pressure at B, because points B and C are at the same level.

Thus, $P_C = P_B = P_A + \rho gh$

$\therefore \quad P_A = 0, \quad \because \quad P_C = \rho gh$

For mercury barometer, $h = 76 \text{ cm} = 0.76 \text{ m}$,

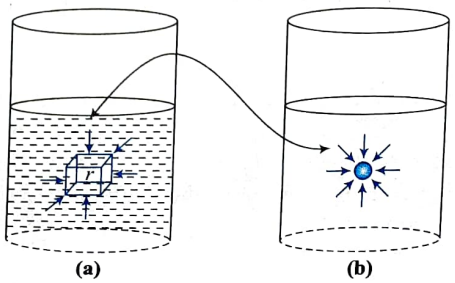
$\rho = 13.6 \times 10^3 \text{ kg/m}^3$

$P_C = P_B = 0.76 \times 13.6 \times 10^3 \times 9.81 = 1.013 \times 10^5 \text{ N/m}^2$

Note: The space above the free surface of mercury in the tube is vacuum. In fact, there is a little vapour of mercury present in that free space due to the evaporation of mercury, but it does not give a considerable pressure. In this way, the vapour pressure of mercury in the free space (or physical vacuum) can be neglected. Hence, $P = 0$ in the free space.

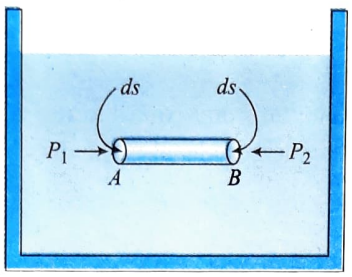
SOME PROPERTIES OF PRESSURE

1. **Pressure is isotropic:** The pressure exerted by a liquid at a point is the same in all directions. Figure (a) shows a small cubic element of liquid with a force acting on each face of this element. Since the element is in equilibrium, the force acting on each lateral face of the cube must be equal.



As the element reduces to a point, the top and bottom forces also become equal. Thus, every point within the liquid is equally compressed from all directions, as shown in Fig. (b).

2. **Pressure is uniform on a horizontal plane:** For a liquid at rest (or moving with uniform velocity or vertical acceleration), the pressure at different points on the same horizontal level within the liquid remains the same. Consider a cylindrical volume of liquid within the liquid contained in a vessel. Let 'ds' be the elemental cross-sectional area of the horizontal cylinder. The force F_1 on face A is $P_1\,ds$ and force F_2 on face B is $P_2 ds$. Here, P_1 and P_2 are the respective pressures on the faces A and B.

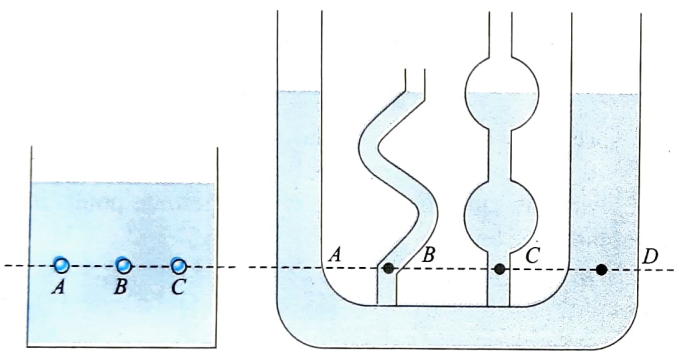


For equilibrium, $F_1 = F_2$

$\Rightarrow \quad P_1 ds = P_2 ds$

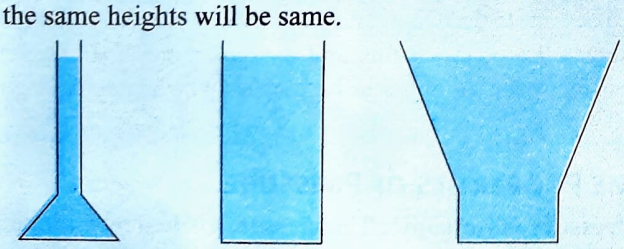
$\Rightarrow \quad P_1 = P_2$

On any horizontal plane in a stationary liquid, pressure is the same at all the points. In figure, pressures at A, B, C and D are equal, i.e., $P_A = P_B = P_C = P_D$.



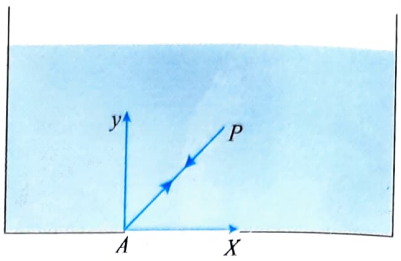
Important Point:

The pressure exerted by a liquid at a point depends on the vertical depth of the point and the density of the liquid only. It does not depend upon the shape of the container. The pressure at the bottom of the three vessels containing the same liquid up to the same heights will be same.



This apparently surprising fact that the different amounts of liquid exert the same pressure is called the hydrostatic paradox. But there is nothing paradoxical as the pressure at a point is determined by the depth and not by the total quantity of the liquid.

3. **Fluids in equilibrium exert pressure at right angles to the surface of contact:** Let us consider some liquid contained in a beaker as shown in figure. Consider the force exerted by the liquid at any point A of the container. Let it act along some direction PA.

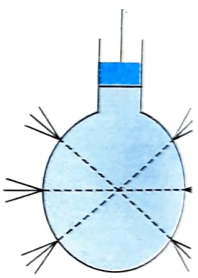


Fluid in equilibrium

Since action and reaction are equal in magnitude and opposite in direction, the surface of the container will give an equal reaction to water along AP. Let us resolve the reaction force into two rectangular components—along AX and AY. As liquid cannot resist any tangential force, the liquid near A should begin to flow along AX parallel to the surface. But the liquid is at rest. So, there cannot be any component along AX. Thus, the following points can be noted.

1. The force acting on a liquid in equilibrium has to be perpendicular to the surface of the container.
2. The force exerted by a liquid on any surface in contact with it is called thrust of the liquid.
3. The thrust exerted by a liquid (at rest) per unit area of the surface in contact with the liquid is called pressure.

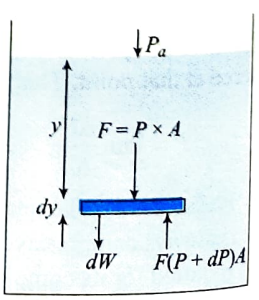
The fact that a liquid exerts pressure at right angles to the surface of the container at every point can be demonstrated in a simple mechanical manner. Take a globe which has a number of holes in its surface and is fitted with a piston (figure).



Fill the globe and a part of the barrel with water. Now push the piston inwards. Water will be seen to flow out radially from all the holes. This demonstrates that water exerts pressure at right angles to the walls of the container.

4. **Pressure varies with depth and height:** Let us look at figure and consider a small volume element in between the liquid at a depth y.

Let A be the area of cross section of this volume element, dy be its thickness, P_0 be the atmospheric (i.e., outside) pressure acting on the free surface of liquid, F be the total force acting downward on the top face of the volume element $= P \times A$, F' be the total upward force acting on its bottom face $= (P + dP) \times A$, dW be the weight of this volume element, acting downwards.



As this volume element is in equilibrium,

$$(P + dP)A - PA = dW$$

$$dP \times A = (A \times dy)\rho g$$

or $\quad dP = \rho g dy$

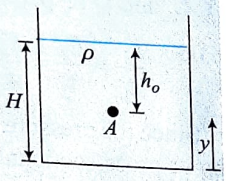
$$\therefore \quad \int_{P_a}^{P} dP = \int_0^h \rho g dy$$

$$[P]_{P_a}^{P} = \rho g [y]_0^h \Rightarrow P = P_a + \rho g h$$

Note:

- The liquid pressure is the same on all points at the same depth.
- The total (or absolute) pressure at a depth h below the free liquid surface is more than the outside atmospheric pressure by an amount ρgh.
- The pressure at the bottom of the container will not depend upon the shape or size of the container.
- **Pressure increases with depth linearly:** The rate of increase of pressure with depth is equal to the specific weight of the liquid at that point.
  The specific weight is defined as the weight per unit volume, i.e., ρg. Thus,
  $$\frac{dp}{dy} = \rho g$$
- **Pressure decreases with height linearly:** The rate of decrease of pressure with height is constant and is given by
  $$\frac{dp}{dy} = -\rho g$$

The pressure at the point A is obtained as follows.



$$dp = -\rho g\, dy$$

$$\Rightarrow \int dp = -\rho g \int dy$$

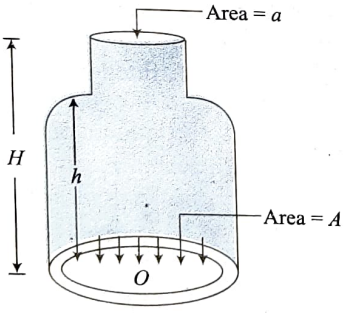
$$\Rightarrow p = -\rho g y + c$$

Now, at $y = H$, $p = p_0$. Therefore, $c = p_0 + \rho gH$

Thus, $p = \rho g (H - y) + p_0$

From figure, since $H - y = h_0$ Hence $p = \rho g h_0 + p_0$

5. **A liquid may exert thrust greater than its weight:** Consider a cylindrical vessel of base area A and height h, with a thin cylindrical stem of cross-sectional area 'a'. If a liquid of density ρ is poured in it to a height $H (H > h)$, the thrust exerted by the liquid on the base of the vessel = pressure at the base due to liquid column × base area.



$$F_{base} = (\rho gH)A \quad \text{...(i)}$$

The weight of the liquid contained in the vessel is given by

Volume × density × $g = [AH - (A - a)(H - h)]\rho g$ ...(ii)

$$W_{liquid} = [Ah + a(H - h)]\rho g \quad \text{...(iii)}$$

Comparing Eqs. (i) and (ii) it is evident that the force exerted by the liquid on the base of the vessel exceeds the weight of the liquid.

To understand the situation, let us draw the pressure diagram on the horizontal faces of the vessel.

The thrust exerted by the liquid on the bottom face is

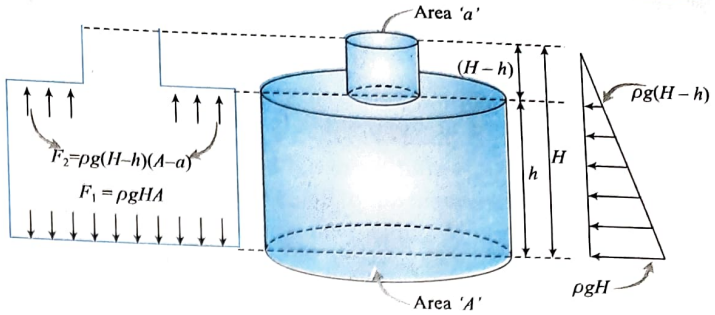
$F_1 = \rho gHA$ (downwards)

and that on the top face is

$F_2 = \rho g(H - h)(A - a)$ (upwards)

Therefore, the net thrust exerted by the liquid on the horizontal faces of the cylinder is

$F_1 - F_2 = [Ah - a(H - h)]\rho g$ (downwards)

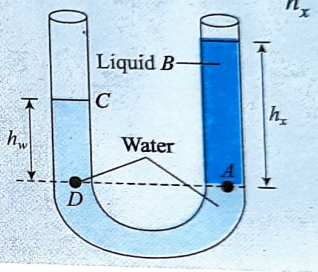


which is exactly the same as the weight of the liquid [Eq. (iii)].

Thus, it follows that the total thrust on the lower face (ρgHA) is a consequence of the weight of the liquid $[Ah + (H - h)a]\rho g$ and the downward normal reaction of the upper horizontal face $[(A - a)(H - h)\rho g]$.

ILLUSTRATION 6.1

Figure shows an open tube which contains some water and a less dense liquid poured in the right-hand side. If the density of the unknown liquid is ρ_x, show that $\rho_x = \dfrac{\rho_w h_w}{h_x}$.



Sol. Consider levels D and A in the tube. If the pressures at D and A are not equal, then water would flow. Since water does not flow, the pressures at D and A are equal.

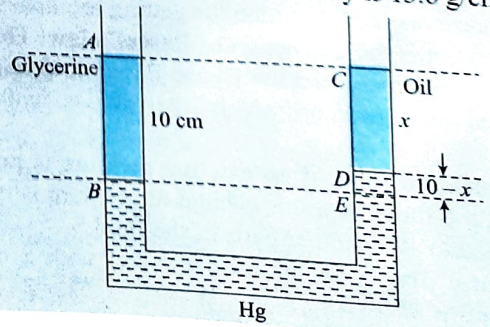
$$P_A = P_{atm} + \rho_w g h_w \text{ and } P_D = P_{atm} + \rho_x g h_x$$

Equating P_A and P_D, we obtain

$$\rho_x = \frac{\rho_w h_w}{h_x}$$

ILLUSTRATION 6.2

A vertical U tube of uniform cross section contains mercury in both of its arms. A glycerine (d = 1.3 g/cm^3) column of length 10 cm is introduced into one of the arms. Oil of density 0.8 g/cm^3 is poured in the other arm until the upper surfaces of the oil and glycerine are in the same horizontal level. Find the length of oil column. Density of mercury is 13.6 g/cm^3.



Sol. Let the length of oil column = $CD = x$ cm.

Length of glycerine = $AB = 10$ cm

$\therefore \quad DE = (10 - x)$ cm

Equating pressure at horizontal level BE in both arms, we have

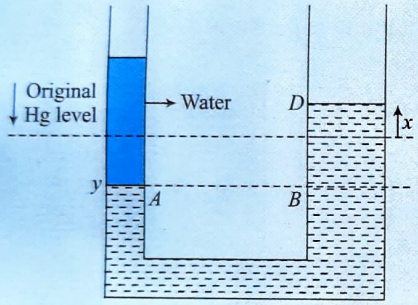
$$P_{oil} + P_{Hg} = P_{glycerine}$$

$$x d_{oil} g + (10 - x)\rho_{Hg} g = 10 d_{glycerine} g$$

or $\quad x(0.8) + (10 - x)13.6 = 10(1.3) \Rightarrow x = \dfrac{136 - 13}{12.8} = 9.6$ cm

ILLUSTRATION 6.3

Two communicating cylindrical tubes contain mercury. The diameter of one vessel is four times larger than the diameter of the other. A column of water of height 70 cm is poured into the narrow vessel. How much will the mercury level rise in the other vessel and how much will it sink in the narrow one? How much will the mercury level rise in the narrow vessel, if a column of water of the same height is poured into the broad vessel?



Sol. Let x = rise of Hg level in the broad vessel, y = drop of Hg level in the narrow vessel. Then,

$$xA_1 = yA_2 \Rightarrow 16x = y$$

(because volume displaced will be the same)

Equating pressures at A and B on the same horizontal plane, we get

$$70 \times 1 \times g = (x + y) \times 13.6 \times g \Rightarrow x + y = \frac{70}{13.6}$$

Solving the equations, we get $x = 0.3$ cm, $y = 4.8$ cm.

If 70 cm water column is poured into the broad vessel, then following the same pattern, we get the equations

$$x' = 16y' \text{ and } (x' + y') \times 13.6 = 70 \times 1$$

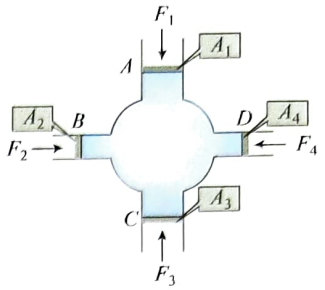
$$\Rightarrow \quad x' = 4.8 \text{ cm and } y' = 0.3 \text{ cm}$$

PASCAL'S LAW

If you squeeze one end of a tube for getting toothpaste out from the other end, it is the live action of **Pascal's law**. This law was formulated by French scientist Blaise Pascal and is one of the fundamental principles on which hydrostatics is founded. It states that:

"In an enclosed fluid, if an external pressure is produced in any part of the fluid, then this change of pressure is transmitted undiminished to all the other parts of the fluid."

Experimental proof: Let us take a vessel with a number of openings of different cross-sectional areas A_1, A_2, A_3 and A_4, all of which are fitted with water-tight movable pistons A, B, C and D respectively as shown in figure. Let the vessel be completely filled with water and pistons be kept in position. If any of the pistons, say A, be now pressed inwards, all other pistons will be found to move outwards which shows that the external pressure is transmitted in all directions.



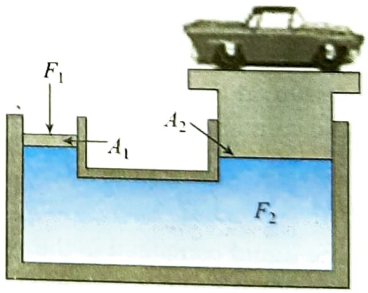
Again, if a force F_1 be applied on the piston A, a pressure F_1/A_1 is exerted by it on the liquid. In order to prevent the motion of other pistons, additional forces F_2, F_3, F_4 have to be applied on B, C, and D respectively. It is found that

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} = \frac{F_3}{A_3} = \frac{F_4}{A_4} \qquad \text{...(i)}$$

Since the pressure on any piston is the force acting on it divided by its cross-sectional area, it follows from Eq. (i) that each piston is acted upon by the same pressure as has been applied to A. Hence, we conclude that the change in pressure is transmitted undiminished in all directions.

Application of Pascal's Principle in the Hydraulic Lift

An important application of Pascal's law is the hydraulic lift. A simple diagram of the hydraulic lift is shown in figure. It consists of two cylinders A and B joined together by a connecting pipe. The cylinders and the connecting pipe are filled with a liquid such as oil. Cylinder B has a larger cross-sectional area than that of the cylinder A. P_1 and P_2 are two pistons of cross-sectional areas A_1 and A_2 fitted into these cylinders.



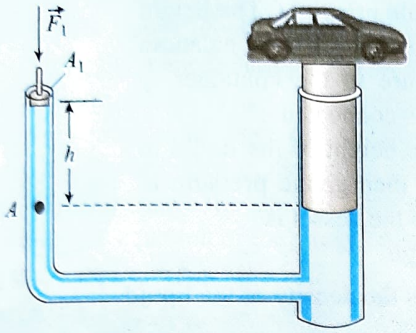
In operation, let an external force of magnitude F_1 be directed downward on the left-hand (or input) piston, whose surface area is A_1. The pressure is transmitted through an incompressible liquid to a larger piston of surface area A_2, the device then produces an upward force of magnitude F_2 on the right-hand (or output) piston. Because the increase in pressure (Δp) must be the same on both sides,

$$\Delta p = \frac{F_1}{A_1} = \frac{F_2}{A_2}, \text{ so } F_2 = F_1 \frac{A_2}{A_1}.$$

If area A_2 is larger than area A_1, depending on the ratio of the areas A_2/A_1, a large output force can be applied by means of a small input force. Hydraulic brakes, car lifts, hydraulic jacks, and forklifts all make use of this principle.

ILLUSTRATION 6.4

In the hydraulic car lift as shown in figure, the input piston on the left has a radius of $r_1 = 0.010$ m and a negligible weight. The output plunger on the right has a radius of $r_2 = 0.20$ m. The combined weight of the car and the plunger is 20000 N. Suppose that the bottom surfaces of the piston and plunger are at the same level, so that $h = 0$ m in figure. What is the magnitude F_1 of the input force needed?



Sol. The output force should have a magnitude of $F_2 = 20000$ N to supports the car. Initially the piston and the plunger are at the same height, so the fluid pressure at the piston is equal to the fluid pressure at the plunger $\frac{F_1}{A_1} = \frac{F_2}{A_2}$ where A_1 is the area of the piston and A_2 is the area of the plunger.

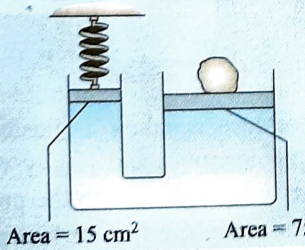
Therefore, the magnitude of the force $F_1 = F_2\left(\frac{A_1}{A_2}\right)$

Using $A = \pi r^2$ for the circular areas of the piston and plunger.

We find that, $F_1 = F_2\left(\frac{A_1}{A_2}\right) = F_2\left(\frac{\pi r_1^2}{\pi r_2^2}\right) = 20000\left(\frac{0.01}{0.2}\right)^2 = 50$ N

ILLUSTRATION 6.5

The drawing shows a hydraulic chamber with a spring (spring constant = 1600 N/m) attached to the input piston and a rock of mass 40.0 kg resting on the output plunger. The piston and plunger are nearly at the same height, and each has a negligible mass. By how much is the spring compressed from its unstrained position?



Sol. Let F_1 and F_2 are the magnitudes of the force the spring exerts on the piston and the rock exerts on the plunger respectively. Initially the piston and the plunger are at the same height, so the fluid pressure at the piston is equal to the fluid pressure at the plunger $\frac{F_1}{A_1} = \frac{F_2}{A_2}$ where A_1 is the area of the piston and A_2 is the area of the plunger. Therefore, the magnitude of the force F_2 that the rock exerts on the plunger is given by

$$F_2 = F_1\left(\frac{A_2}{A_1}\right)$$

Here $F_1 = kx$, where k is the spring constant of the spring, and x is the amount by which the spring is compressed from its unstrained position. and the magnitude F_2 of the force the rock exerts on the plunger is equal to the magnitude $W = mg$ where m is the rock's mass and g is the magnitude of the acceleration due to gravity.

Solving $F_2 = F_1\left(\frac{A_2}{A_1}\right)$ for F_1 and substituting $F_2 = mg$ yields

$$F_1 = F_2\left(\frac{A_1}{A_2}\right) = mg\left(\frac{A_1}{A_2}\right) \quad \text{...(i)}$$

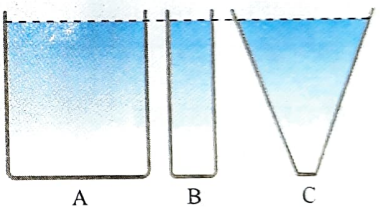
Solving for x, we obtain, $x = \frac{F_1}{k}$ ...(ii)

Substituting Eq. (i) into Eq. (ii), we find that

$$x = \frac{F_1}{k} = \frac{mg}{k}\left(\frac{A_1}{A_2}\right) = \frac{(40\text{ kg})(10\text{ m/s}^2)}{1600\text{ N/m}}\left(\frac{15\text{ cm}^2}{75\text{ cm}^2}\right) = 5.0\times10^{-2}\text{ m}$$

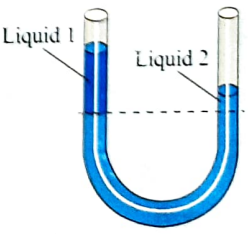
CONCEPT APPLICATION EXERCISE 6.1

1. Three containers filled to the same height with the fluid of same density as shown in figure. In which container, if any, is the pressure at the bottom greatest?



(a) Container A, because its bottom has the greatest base area.
(b) Container C, because its bottom has the least base area.
(c) Container A, because it the volume of fluid is greatest.
(d) Container B, because it the volume of fluid is least.
(e) All three containers have the same pressure at the bottom.

2. In a U-tube which is open at both ends, two liquids, 1 and 2, are in equilibrium, as shown in the figure. If the liquids do not mix and liquid 1 rests on top of liquid 2, then how is the density ρ_1 of liquid 1 related to the density ρ_2 of liquid 2?



(a) ρ_1 is equal to ρ_2 because the liquids are in equilibrium,
(b) ρ_1 is less than ρ_2.
(c) ρ_1 is greater than ρ_2.
(d) There is not enough information to tell which liquid has the greater density.

3. A hole of area 5 cm^2 is formed in the side of a ship 3 m below the water level. What minimum force is required

to hold on a patch covering the hole from the inside of the ship?

4. What will be the length of mercury column in a barometer tube, when the atmospheric pressure is 75 cm of mercury and the tube is inclined at an angle of 60° to the vertical?
5. How high would water rise in the pipes of a building, if the water pressure gauge shows the pressure at the ground floor to be 270 kPa?
6. Torricelli's barometer used mercury. Pascal duplicated it using alcohol of density 984 kg m^{-3}. Determine the height of the alcohol column for normal atmospheric pressure. (Given normal atmospheric pressure, $P_{atm} = 1.013 \times 10^5$ Nm^{-2})
7. The gauge pressure in both the tyres of a bicycle is 6.0 $\times 10^5$ Pa. If the bicycle and the person riding it have a combined mass of 90 kg, what is the area of contact of each tyre with the ground?
8. A hydraulic automobile lift has input and output pistons with diameters of 10 cm and 30 cm. The lift is used to hold up a car with a weight of 144×10^4 N.
   (a) What is the force on the input piston?
   (b) What pressure is applied on the input piston?

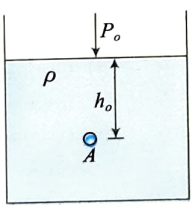
ANSWERS

1. (e) **2.** (b) **3.** 15 N **4.** 150 cm **5.** 27 m
6. 10.5 m **7.** 7.5×10^{-4} m^2
8. (a) 1.6×10^3 N (b) 2.04×10^5 Nm^{-2}

MEASUREMENT OF PRESSURE

Absolute Pressure and Gauge Pressure

Absolute pressure is the total pressure at a point including the contribution of the liquid as well as that of the atmosphere. Gauge pressure is the pressure measured at a point relative to the local atmospheric pressure. In other words, it does not include the contribution of atmospheric pressure.



Absolute pressure and gauge pressure are related as

$$p_{absolute} = p_{gauge} + p_{atm}$$

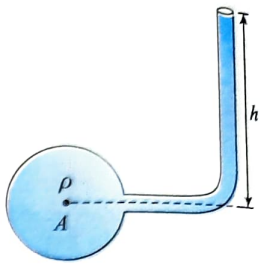
Consider a point A at a depth h from the free surface of a liquid of density ρ, as shown in figure. The gauge pressure at the point A is $p_{gauge} = \rho gh$ and the absolute pressure is $p_{absolute} = \rho gh + p_0$.

Note:
- Absolute pressure is always positive and is never equal to zero.
- Gauge pressure may be positive, zero or negative.
- Positive gauge pressure means that pressure at a point is more than the atmospheric pressure.
- Negative gauge pressure means that pressure at a point is less than the atmospheric pressure.
- Practically, the exact value of absolute pressure cannot be measured due to the uncertainty in the value of atmospheric pressure. The atmospheric pressure varies from place to place and at a place from time to time.

Manometers: Measurement of Gauge Pressure

The manometers are simple devices used for measuring gauge pressure of fluids. There are two types of manometers as discussed in the following.

1. **Simple manometer:** It consists of a vertical glass tube, the lower end of which is attached to the point where the pressure is to be measured and the upper end is left open to the atmosphere. It works on a very simple principle. The height of the liquid in the tube balances the pressure in the container to which it is connected.



If h is the height of the liquid in the tube, then gauge pressure at point A in the vessel is

$$P_A = \rho gh$$

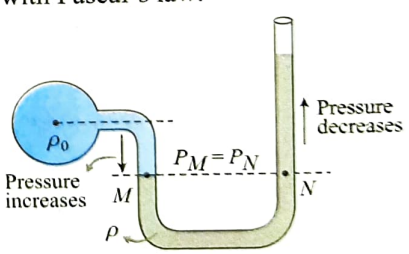
where ρ is the density of the liquid.

Note: Limitations of a simple manometer are as follows:
- It cannot measure pressure in a gas.
- It can measure positive gauge pressures in liquids only.
- It cannot measure very large pressures as it may require very long tube.

2. **U-tube manometer:** An U-tube manometer consists of two tubes connected at the bottom to form a U-shaped tube. It can measure positive and negative gauge pressures in liquids as well as in gases. It works on Pascal's law. The pressure at a point in a vessel is obtained by the manometric equation.

Writing Manometric Equation

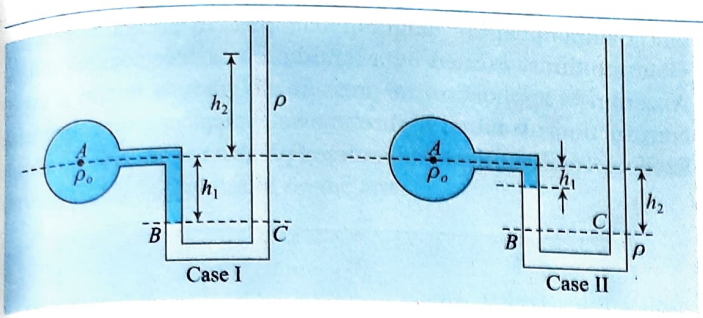
We start from a point where pressure is to be obtained and move towards the free end through the tube. As we move down in the tube pressure increases, as we move up in the tube pressure decreases and as we move horizontally pressure does not change in accordance with Pascal's law.



The increase in pressure is taken as positive, the decrease in pressure is taken as negative, and finally we reach the free end of the tube where we equate it with the pressure at the free surface. If we put pressure at the free surface equal to zero, we obtain gauge pressure at that point in the vessel; and if we put pressure at free surface equal to atmospheric pressure, we obtain the absolute pressure at that point in the vessel.

ILLUSTRATION 6.6

A container 'A' is having some fluid of density θ_0. It is connected with a U-tube manometer shown in figure. Find the pressure of the container in both the cases given in figure.



Sol. **Case I:** Let the gauge pressure at A be p_A. Starting from container move along the manometer by a height h_1 downwards. The change in pressure will be $+\rho_0 g h_1$. The pressure at B and C will be the same. To move to the open end, we have to move by a distance $(h_1 + h_2)$ upward. Hence, change in pressure will be $-\rho g(h_1 + h_2)$. Finally, we reach the top, i.e., open to atmosphere. Hence, the manometric equation of the gauge pressure will be

$$p_A + \rho_0 g h_1 - \rho g (h_1 + h_2) = 0$$

$$\Rightarrow \quad p_A = \rho g (h_1 + h_2) - \rho_0 g h_1$$

In terms of absolute pressure at A, the manometric equation will be

$$p'_A + \rho_0 g h_1 - \rho g (h_1 + h_2) = p_0$$

$$p'_A = (h_1 + h_2)\rho g - \rho_0 g h_1 + p_0$$

Case II: Gauge pressure at A, $p_A + \rho_0 g h_1 + \rho g(h_2 - h_1) = 0$

Absolute pressure at A, $p'_A + \rho_0 g h_1 + \rho g(h_2 - h_1) = p_0$

Note:

- Pressures at B and C are equal.
- Absolute pressure is always greater than or equal to zero, while gauge pressure can be negative also.
- In the same liquid, pressure will be same at all points at the same level. For example, in figure, $P_1 \neq P_2$, $P_3 = P_4$, $P_5 = P_6$.

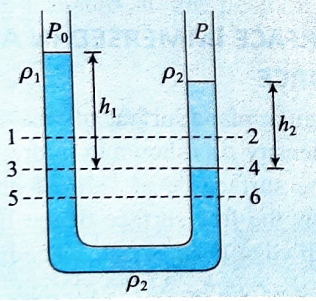
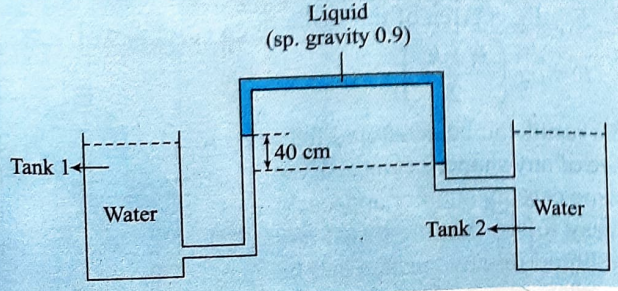


ILLUSTRATION 6.7

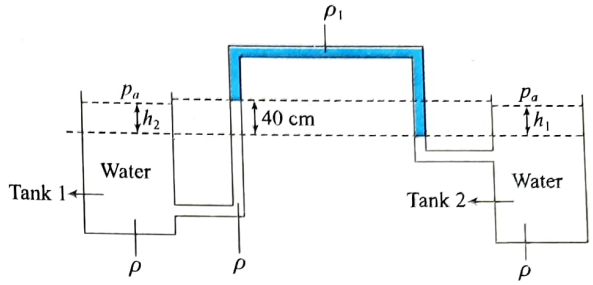
An inverted U-tube manometer shown in figure is used to measure the difference in water level between the two tanks. Calculate this difference for the conditions indicated.



Sol. The inverted U-tube manometer is used for finding the difference of water level in two tanks. Draw a horizontal straight line passing through the lower level and upper level liquid in the manometer as shown in figure. Starting from top of tank 2 and moving through the tube and reaching the top of tank 1, the manometric equation can be written as

$$p_a + h_1 \rho g - 40\rho_1 g + 40\rho g = p_a + h_2 \rho g$$

$$h_2 \rho g - h_1 \rho g = 40 \rho g - 40 \rho_1 g$$



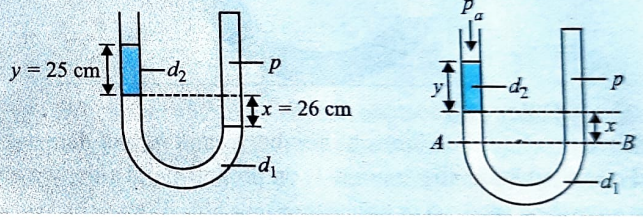
But we are given, $\rho_1 = 0.9\rho$

$$\therefore \quad (h_2 - h_1)\rho g = 40\rho g - 36\rho g$$

or $\quad h_2 - h_1 = 4$ cm

ILLUSTRATION 6.8

In a given U-tube open at left end and closed at right end, if the pressure above the liquid in the right arm is p, find the value of p. Given $d_2 = 2 \times 13.6$ g/cm^3, $d_1 = 13.6$ g/cm^3. Atmospheric pressure $p_a = 76$ cm of Hg.



Sol. The line AB is passing through the same liquid; hence, the pressure in the liquid at the same level is the same, i.e., at A and B, the pressures will be equal. The manometric equation is

$$p_a + d_2 y g + d_1 x g = p$$

$$\Rightarrow \quad p_a + 13.6 \times 2 \times 0.25 \times g + 13.6 \times 0.26 \times g = p$$

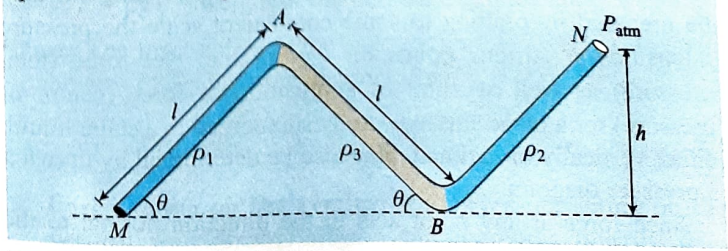
$$p_a + 13.6 \times g/100[50 + 26] = p$$

$$\Rightarrow \quad p = 2 \times 13.6 \times g \times \frac{76}{100} \qquad [\because p_a = 13.6 \times g \times 76/100]$$

$$= 2p_a$$

ILLUSTRATION 6.9

Find out pressures at points A and B. Also find angle 'θ'. A zig-zag tube open at N, having liquids of densities ρ_1, ρ_2 and ρ_3, is placed in a vertical plane as shown in figure. (The pressure at M is equal to atmospheric pressure.)



Sol. Pressure at A is $P_A = P_{atm} - \rho_1 g l \sin\theta$

Pressure at B is $P_B = P_{atm} + \rho_2 g h$

But P_B is also given by $P_B = P_A + \rho_3 gl \sin\theta$

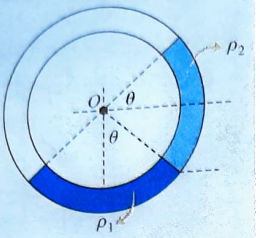
Hence, $P_{atm} + \rho_2 gh = P_A + \rho_3 gl \sin\theta$

$\Rightarrow \quad P_{atm} + \rho_2 gh = P_{atm} - \rho_1 gl \sin\theta + \rho_3 gl \sin\theta$

$\Rightarrow \quad \sin\theta = \dfrac{\rho_2 h}{(\rho_3 - \rho_1)l}$

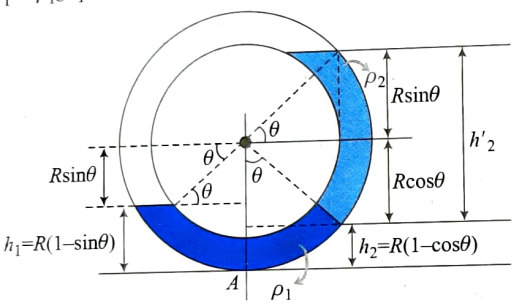
ILLUSTRATION 6.10

A circular tube of uniform cross section is filled with two liquids of densities ρ_1 and ρ_2 such that half of each liquid occupies a quarter of volume of the tube. If the line joining the free surfaces of the liquid makes an angle θ with horizontal, find the value of θ.



Sol. Let us find the pressure at the lowest point 1. Since the liquid has density ρ_2 and height of liquid column is h'_2, on the right-hand side of point 1, we have

$$P_1 = \rho_1 g h_1 \qquad ...(i)$$



Since two liquid columns of heights h_1 and h_2 and densities ρ_1 and ρ_2 are situated above point 1, on the left-hand side, we have

$$P_2 = \rho_1 g h_2 + \rho_2 g h'_2 \qquad ...(ii)$$

Equating P_1 and P_2 from Eqs. (i) and (ii), we have

$$\rho_1 h_2 + \rho_2 h'_2 = \rho_1 h_1 \qquad ...(iii)$$

Substituting $h'_2 = R\sin\theta + R\cos\theta$, $h_2 = R(1-\cos\theta)$ and $h_1 = R(1-\sin\theta)$ in Eq. (iii), we have

$$\rho_1 R(1-\cos\theta) + \rho_2 R(\sin\theta + \cos\theta) = \rho_1 R(1-\sin\theta)$$

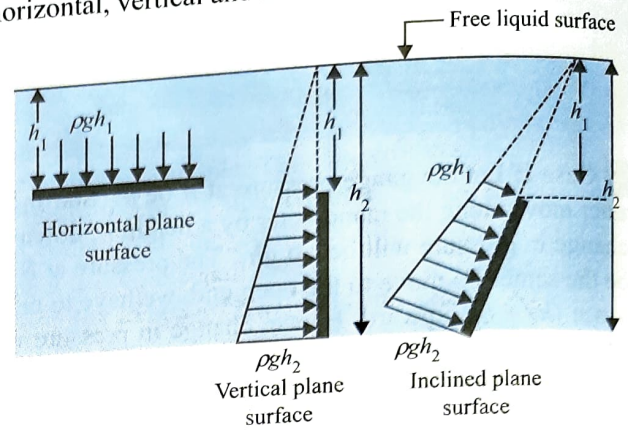
This gives, $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \dfrac{\rho_1}{\rho_2} \Rightarrow \tan\theta = \dfrac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$

Then $\theta = \tan^{-1}\left(\dfrac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\right)$

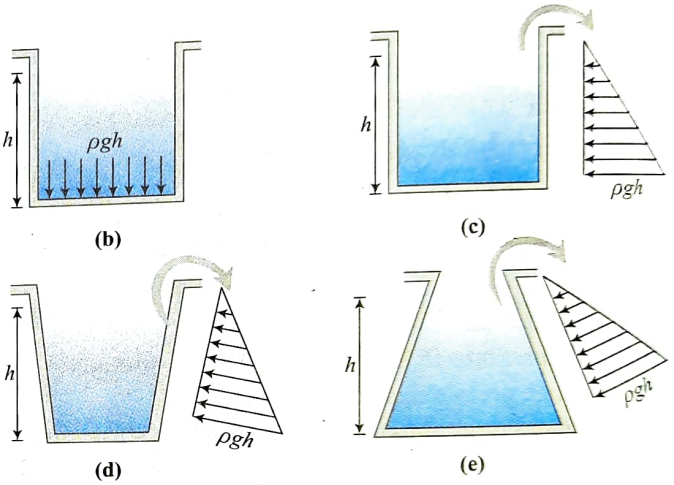
PRESSURE DIAGRAMS AND FORCE ON BOUNDARIES

A pressure diagram is a graphical representation of the variation of the pressure intensity over a surface. Such a diagram may be prepared by plotting to some convenient scale the pressure intensities at various points on the surface. Net force (total pressure) as well as point of application of force (centre of pressure) for a plane surface wholly submerged in a static liquid, either vertically or inclined, may also be determined by drawing a pressure diagram.

Since force at any point acts in the direction normal to the surface, the pressure intensities at various points on the surface are plotted normal to the surface. A diagram representing the magnitude of pressure at various points, to some suitable scale, is known as pressure diagram. The areas of pressure diagrams yield the thrust exerted by a liquid on a surface per unit width. A common application of pressure diagram is in the field of construction of dams. Figure shows typical pressure diagrams for horizontal, vertical and inclined plane surfaces.



(a)



FORCE ON SURFACE IMMERSED IN A LIQUID AND CENTRE OF FORCE

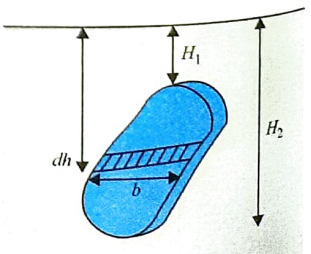
Consider a rectangular plane surface of area 'A' held vertically in a static liquid of density ρ as shown in figure. Let top and bottom edges of the plane surface be at vertical depths of h_1 and h_2, respectively, below the free surface of the liquid. The pressure intensity at the top edge is $P_1 = \rho g h_1$. The pressure intensity at the bottom edge is $P_2 = \rho g h_2$. As the pressure intensity increases linearly from $\rho g h_1$ to $\rho g h_2$, so the average pressure intensity over the entire surface is

$$P_{av} = \frac{P_1 + P_2}{2} = \rho g\left(\frac{h_1 + h_2}{2}\right)$$

Thus, the net force on the vertical surface is

$F = P_{av} \times$ Area of the surface

or $\quad F = \rho g\left(\dfrac{h_1 + h_2}{2}\right) \times A$



This result can be generated for a surface of any shape. Let us analyse the same case by another approach. It is easy to see that the thrust due to the liquid on any surface can be obtained as

Thrust = (Area) × (Hydrostatic pressure at the centroid of the area)

Consider a plane surface of any arbitrary shape, immersed in a liquid of density ρ. Let us subdivide the area into infinitesimally thin horizontal strips. Consider one such strip of width 'b' and thickness dh, at a depth h below the free liquid surface. Normal thrust on this strip is dF = Pressure × Area = $\rho gh\,(bdh)$. Total thrust exerted by the liquid on the area is

$$F = \Sigma dF = \Sigma \rho ghbdh = \rho g \sum_{h=h_1}^{h_2} bhdh$$

Now, if $\bar{h}$ be the depth of the centroid of the area, below the free surface and A is the total area, then

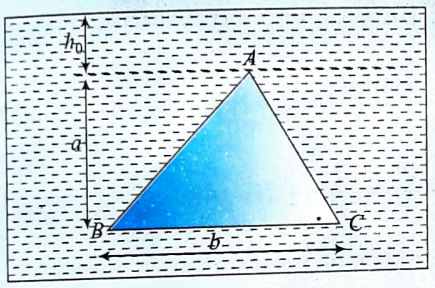
$$\bar{h} = \sum_{h=h_1}^{h_2} \frac{(bdh)h}{A} \Rightarrow \sum_{h=h_1}^{h_2} (bdh)h = A\bar{h}$$

$$\therefore \quad F = (\rho g \bar{h})A \qquad \ldots(i)$$

Hence force on a vertical immersed surface is the product of the pressure on the centroid of the surface and the area of the surface.

ILLUSTRATION 6.11

Figure shows a triangular plate submerged in a liquid of density ρ.



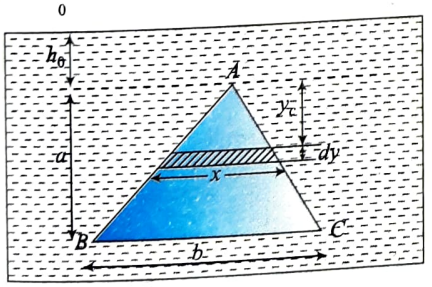
What is the hydrostatic force on one face of the plate?

Sol. **Method 1:** Pressure at the position differential area element

$$P = \rho g(y + h_0).$$

Force on this element, $dF = \rho g\,(y + h_0).x\,dy$

$$F = \int dF = \int_0^a \rho g\,(y + h_0).x\,dy \qquad \ldots(i)$$



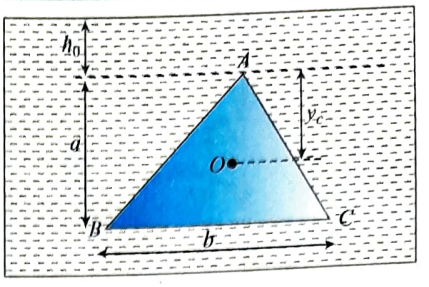
$$F = \int dF = \int_0^a \rho g\,(y + h_0).x\,dy \qquad \left[\frac{x}{b} = \frac{y}{a} \Rightarrow x = \frac{b}{a}y\right]$$

$$F = \int dF = \int_0^a \rho g\,(y + h_0)\left(\frac{b}{a}y\right)dy$$

$$= \rho g \frac{b}{a}\int_0^a (y^2 dy + h_0 y dy) = \rho gb\left[\frac{a^2}{3} + \frac{h_0 a}{2}\right]$$

Method 2: The horizontal force on the plate due to water

F_H = Pressure at the centroid of plate × Area of the plate



Pressure at the centroid of the plate, $P_O = \rho g\left(h_0 + \frac{2a}{3}\right)$

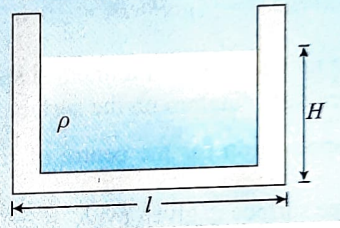
Area of the plate, $A = \frac{1}{2}ab$

The horizontal force on the plate,

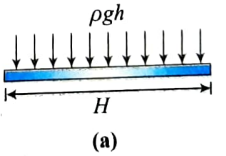
$$F = P_O A = \rho g\left(\frac{2a}{3} + h_0\right) \times \frac{1}{2}ab = \rho gb\left[\frac{a^2}{3} + \frac{h_0 a}{2}\right]$$

ILLUSTRATION 6.12

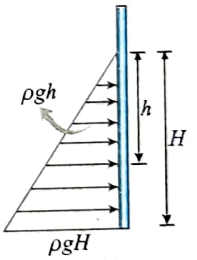
Consider a rectangular tank of size ($l \times b$) filled with a liquid of density ρ to a height H as shown in figure. Find the force at the base and on the walls of the tank.



Sol. Whenever a liquid comes in contact with solid boundaries it exerts a force on it. The force on the boundary may be obtained by integrating the pressure over the entire area of the boundary. The variations of liquid pressure acting at the base and at the wall are shown in Figs. (a) and (b), respectively.



(a)



(b)

1. **Force at the base**: Since the pressure is uniform at the base, force acting at the base is given by

   $F = \rho \times$ (area of the base)

   Since $p = \rho gH$, therefore

   $F = \rho gH(lb) = \rho g(lbH)$

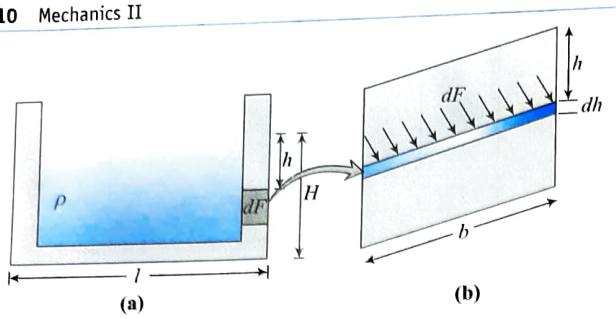
   Since $lbH = V$ (volume of the liquid), so

   $F = \rho gV$ = weight of the liquid inside the tank

Note: The force acting at the base per unit width (F/b) is equal to the area of the pressure diagram. That is,

$$\frac{F}{b} = pl = \rho gHl$$

2. **Force acting on the vertical wall**: Pressure acting on the vertical wall is not uniform but increases linearly with depth. Pressure at a depth h from the free surface is $p = \rho gh$.



Force dF acting on a differential element of height dh is

$$dF = p(bdh) = (\rho gh)(bdh) = \rho gbhdh$$

The total force is

$$F = \rho gb\int_0^H hdh = \frac{1}{2}\rho gbH^2$$

The total force acting per unit width of the vertical wall is

$$\frac{F}{b} = \frac{1}{2}\rho gH^2$$

Note: The force acting per unit width is equal to the area of the pressure diagram shown in Fig. (b).

$$\frac{F}{b} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(\rho gH)(H) = \frac{1}{2}\rho gH^2$$

Hence, force per unit width of an immersed surface is equal to the area of the pressure diagram on the surface.

The point of application (or the centre of force) of the total force from the free surface is given by $h_C = \frac{1}{F}\int_0^H hdF$.

Here, $\int_0^H h\,dF = \int_0^H h(\rho gbh\,dh) = \rho gb\int_0^H h^2\,dh = \frac{1}{3}\rho gbH^3$

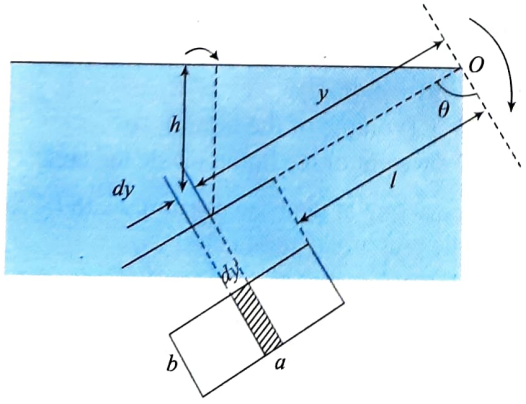
$$F = \frac{1}{2}\rho gbH^2 \Rightarrow h_C = \frac{2}{3}H$$

The total force acts at a depth 2/3 H from the force diagram.

Note: The total force acts through the centroid of the pressure diagram, at a depth 2/3H from the top of the liquid level.

HYDROSTATIC FORCES ON INCLINED SURFACES IMMERSED IN LIQUID

Method 1: Consider a plane surface of length a and width b which is inside a liquid of density ρ as shown in figure.



Choose an element of length (dy) at a distance y from O. The depth of the element $h = y\sin\theta$. The intensity of pressure at the position of the element is $P = \rho gh = \rho g\,(y\sin\theta)$.

The force on the element is $dF = P(bdy)$

$= \rho g(y\sin\theta)\,bdy = \rho gb\sin\theta\; y\,(dy)$

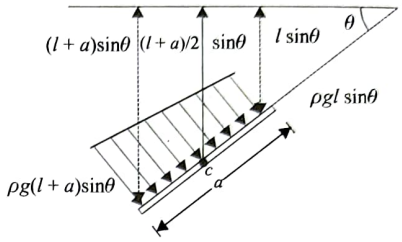
The force on the entire surface is

$$F = \int_l^{(l+a)} dF = \rho gb\sin\theta\int_l^{(l+a)} ydy$$

$$= \rho g\frac{b\sin\theta}{2}\left|y^2\right|_l^{(l+a)} = \rho gb\frac{\sin\theta}{2}\left[(l+a)^2 - l^2\right]$$

$$= \frac{\rho gb\sin\theta}{2}\left[a^2 + 2al\right] = \frac{\rho gab}{2}[a+2l]\sin\theta$$

Method 2: Force per unit width on the surface can be found out by taking area of pressure diagram on the surface. Figure shows the pressure diagram on the surface.



Area of pressure diagram = Force per unit width of surface

$$\frac{F}{b} = \frac{1}{2}[\rho g(l+a)\sin\theta + \rho gl\sin\theta]a$$

$$= \frac{\rho ga\sin\theta}{2}[l+a+l]$$

$$F = \frac{\rho gab\sin\theta}{2}[a+2l]$$

Method 3: Force on inclined surface is given by

$F = P_{av} \times$ area of surface

$$= \frac{1}{2}[\rho gl\sin\theta + \rho g(l+a)\sin\theta]\times ab$$

$$= \frac{1}{2}\rho gab\sin\theta\,[a+2l]$$

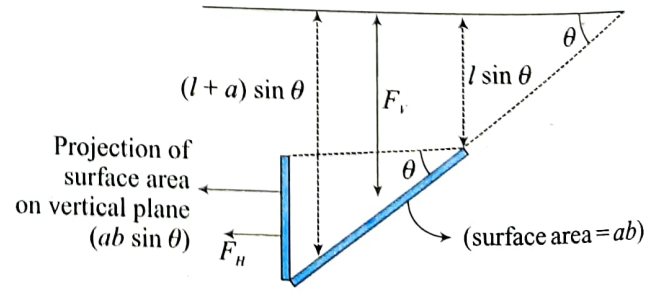
Method 4: Force on inclined surface is given by

$F =$ Pressure at the centroid of the surface $\times$ Area of surface

$$= \left[\rho g\left(l + \frac{a}{2}\right)\sin\theta\right]ab$$

$$= \frac{\rho gab\sin\theta}{2}[a+2l]$$

Method 5:



Horizontal component of total force,

$F_H =$ Force on the area projected on the vertical plane

$$= \frac{1}{2}[\rho gl\sin\theta + \rho g(l+a)\sin\theta]\,(ab\sin\theta)$$

$$= \frac{1}{2}\rho g\, ab \sin^2\theta\ (2l + a) \qquad ...(i)$$

Vertical component of the force,

F_V = Weight of water prism above the inclined surface

$$= \frac{1}{2}[l\sin\theta + (l+a)\sin\theta]\, a\cos\theta\, b\rho g$$

$$= \frac{1}{2}\rho g\, ab \sin\theta\cos\theta(2l + a) \qquad ...(ii)$$

Hence, net force on the surface,

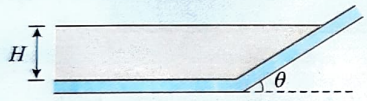
$$F = \sqrt{F_H^2 + F_V^2}$$

$$= \sqrt{\left(\frac{1}{2}\rho g\, ab \sin\theta\ (2l+a)\right)^2 [\sin^2\theta + \cos^2\theta]}$$

$$= \frac{1}{2}\rho g\, ab \sin\theta(2l + a)$$

ILLUSTRATION 6.13

Consider a container having an inclined wall, inclined an angle θ with horizontal as shown in figure. The length of the wall is b (into the plane of paper, not shown in figure). If the container is filled with a liquid of density ρ, calculate the horizontal and vertical components of force applied by liquid per unit width on the wall.

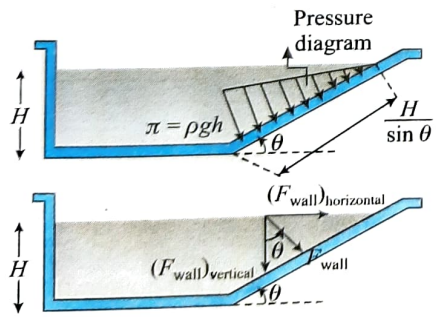


Sol. **Method 1:** The total force acting per unit width of the inclined wall = Area of pressure diagram

$$\frac{F_{wall}}{b} = \frac{1}{2}\rho gH \times \frac{H}{\sin\theta} = \frac{1}{2}\rho gH^2 \operatorname{cosec}\theta$$

The total force acting on the inclined wall

$$F_{wall} = \frac{1}{2}\rho gH^2 b \operatorname{cosec}\theta$$



The vertical component of force acting per unit width of the inclined wall acting on the inclined wall

$$\left(\frac{F_{wall}}{b}\right)_{vertical} = \left(\frac{F_{wall}}{b}\right)\cos\theta = \left(\frac{1}{2}\rho gH^2 \operatorname{cosec}\theta\right).\cos\theta$$

$$\Rightarrow \quad \left(\frac{F_{wall}}{b}\right)_{vertical} = \frac{1}{2}\rho gH^2 \cot\theta$$

The horizontal component of force acting per unit width of the inclined wall acting on the inclined wall

$$\left(\frac{F_{wall}}{b}\right)_{horizontal} = \left(\frac{F_{wall}}{b}\right)\sin\theta = \left(\frac{1}{2}\rho gH^2 \operatorname{cosec}\theta\right).\sin\theta$$

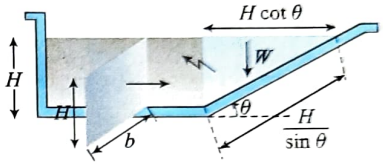
$$\Rightarrow \quad \left(\frac{F_{wall}}{b}\right)_{horizontal} = \frac{1}{2}\rho gH^2$$

Method 2: The vertical component of the force applied by water on the wall should be equal to the weight of water wedge above the inclined wall as shown in figure.

$$W = \frac{1}{2}H.(H\cot\theta).b\rho g = \frac{1}{2}\rho gH^2 b\cot\theta$$

$$\Rightarrow \quad \left(\frac{F_{wall}}{b}\right)_{vertical} = \frac{1}{2}\rho gH^2\cot\theta$$

The horizontal force F_x acts on the vertical projection of the inclined wall as shown in the figure.



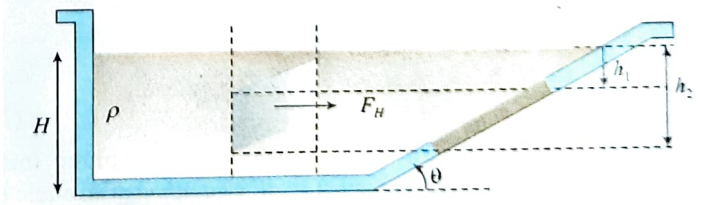
F_H = Force applied by water on the projection of gate on vertical plane.

= Average pressure on the projection of gate on vertical plane × Area.

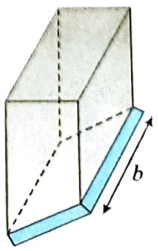
$$F_H = \left(\frac{0 + \rho gH}{2}\right)Hb = \frac{1}{2}\rho gH^2 b \Rightarrow \left(\frac{F_{wall}}{b}\right)_{horizontal} = \frac{1}{2}\rho gH^2$$

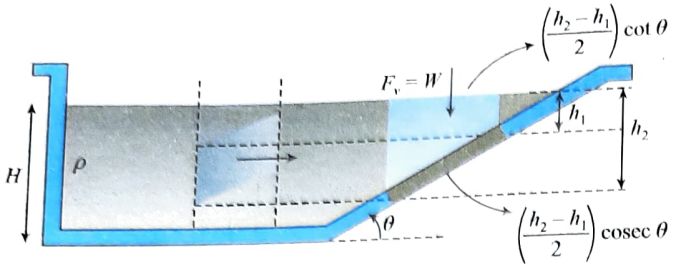
ILLUSTRATION 6.14

Consider the container of previous illustration, a window is made on the inclined wall as shown in figure, calculate the horizontal and vertical components of force applied by liquid per unit width on the window.



Sol. The vertical component of the force applied by water on the window should be equal to the weight of water prism above the inclined wall as shown in figure.





The weight of water prism above the window

$$F_v = W = \frac{1}{2}\left(\frac{h_1+h_2}{2}\right)\cdot\left(\frac{h_2-h_1}{2}\right)\cot\theta.b\rho g$$

$$= \left(\frac{h_2^2-h_1^2}{8}\right)b\cot\theta\rho g$$

The horizontal component of force acting on the gate

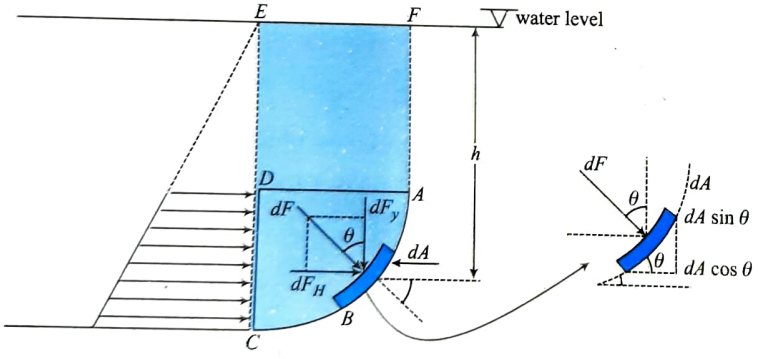
= Force applied by water on the projection of gate on vertical plane.

= Average pressure on the projection of gate on vertical plane × Area

$$F_H = \left(\frac{\rho g h_1 + \rho g h_2}{2}\right)\cdot(h_2-h_1)b = \frac{\rho bg}{2}(h_2^2-h_1^2)$$

FORCE ON CURVED SURFACE

Figure shows a curved surface inside a static liquid of density ρ, which extends in the direction normal to the plane of the paper. As liquid imparts the thrust normal to the point of contact, at any point on the curved surface the force acts normal to the surface. Let us take a small element of area dA of the curved surface lying at a vertical depth of h below the free surface of the liquid. The force on the element is



$$dF = P(dA) = \rho gh(dA) \quad ...(i)$$

The force dF acting on the element can be resolved into horizontal and vertical components. Let dF is inclined at an angle θ with the vertical.

$$dF_H = dF\sin\theta = \rho gh(dA)\sin\theta \quad ...(ii)$$

$$dF_V = dF\cos\theta = \rho gh(dA)\cos\theta \quad ...(iii)$$

The total horizontal and vertical components of the force can be obtained by integrating Eqs. (ii) and (iii). So

$$F_H = \int dF_H = \rho g\int h(dA)\sin\theta \quad ...(iv)$$

$$F_V = \int dF_v = \rho g\int h(dA)\cos\theta \quad ...(v)$$

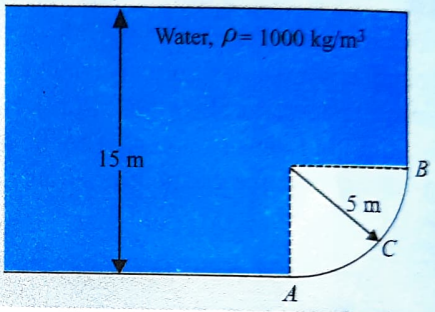
In Eq. (iv), $dA\sin\theta$ is the vertical projection of the area dA and in Eq. (v), $dA\cos\theta$ is the horizontal projection of the elementary area dA. Thus, $\rho g\int h(dA)\sin\theta$ represents the total force on the vertical projection of the curved surface, i.e., the horizontal component of the net force is equal to the total force on the projected area of the curved surface on the vertical plane, the trace of which is represented by CD.

Further, $\rho g\int h(dA)\sin\theta$ represents the total force on the horizontal projection of the curved surface, and it is equal to the weight of the liquid contained in the portion extending above the curved surface in the portion $ABCDEFA$, i.e., vertical component of net force is equal to the weight of the liquid contained in the portion extending vertically above the curved surface up to the free surface of the liquid.

The resultant force on the curved surface is $F = \sqrt{F_H^2 + F_V^2}$.

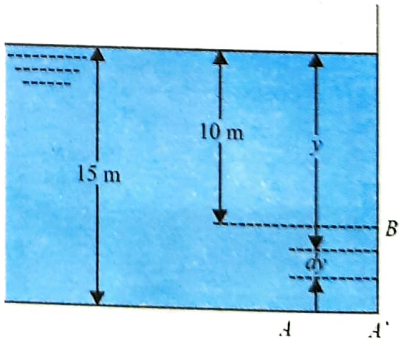
ILLUSTRATION 6.15

Water is retained by a curved gate AB. Compute the horizontal and vertical components of the resultant of the hydrostatic pressure distribution on the gate AB, which is a quarter of a cylinder. Assume that the gate is 4 m wide.



Sol. Horizontal component: The horizontal component of the resultant force on the gate is equal to the force on the vertical projection BA' of the gate. The pressure at a depth y from the free surface of the water is $P = \rho gy$. The force on the strip of area $4dy$ of the gate is

$$dF_H = P(4dy) = \rho gy(4dy) = 4\rho gydy$$

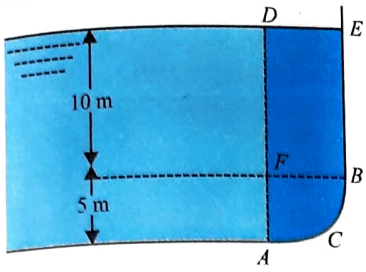


Total horizontal force is

$$F_H = 4\rho g\int_{10}^{15} ydy = 4\rho g\left|\frac{y^2}{2}\right|_{10}^{15}$$

$$= 4\times1000\times10\left(\frac{15^2-10^2}{2}\right) = 250\times10^4\,\text{N}$$

Vertical component: It is equal to the weight of the water above the curved surface. Dividing the volume above the curved surface into a rectangular parallelepiped and a quarter cylinder, we thus have



F_V = Weight of water in $DEFB$ + Weight of water in $ACBF$

$$= \rho[V_1 + V_2]g = 1000\left[10\times4\times4 + \frac{\pi(5)^2\times4}{4}\right]\times10$$

$$= [8\times\pi]\times10^4\ \text{N}$$

After calculating the values of F_V and F_H we can calculate the resultant force on the gate $F = \sqrt{F_H^{\,2} + F_V^{\,2}}$.

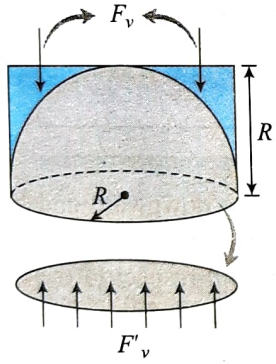
ILLUSTRATION 6.16

A solid hemisphere of radius R is made to just sink in a liquid of density ρ. Find **(a)** the vertical thrust on the curved surface, **(b)** the vertical thrust on the flat surface, **(c)** the side thrust on the hemisphere, **(d)** the total hydrostatic force acting on the hemisphere.

Sol.

(a) Vertical thrust of the liquid is equal to weight of the liquid column above the curved (spherical) surface, i.e.,

$F_V = V\rho g$



where V = Volume of the shaded portion

= Volume of the cylinder − Volume of the hemisphere

$$= (\pi R^2)R - \frac{2}{3}\pi R^3 = \frac{\pi R^3}{3}$$

Substituting V in the equation $F_V = V\rho g$, we have

$$F_V = \frac{\pi R^3}{3}\rho g \ \text{(down)}$$

(b) The upward thrust on the base of the hemisphere is

$F'_V = V'\rho g$

where V' = Volume of the liquid column above the base

$= (\pi R^2)R = \pi R^3$. Then, we have $F'_V = \pi R^3\rho g$ (upward)

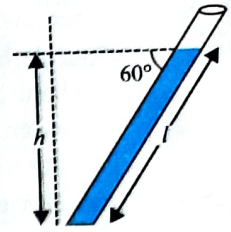
(c) Side thrust $F_h = 0$ (as discussed earlier)

(d) The total hydrostatic force is

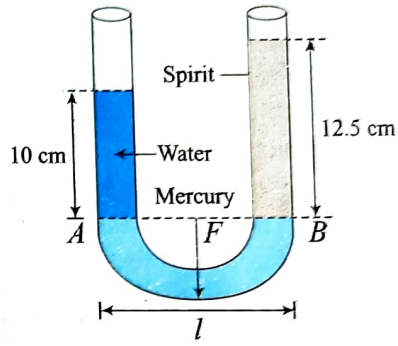
$$F = F'_V - F_V = \pi R^3\rho g - \frac{\pi R^3}{3}\rho g = \frac{2}{3}\pi R^3\rho g \ \text{(upward)}$$

CONCEPT APPLICATION EXERCISE 6.2

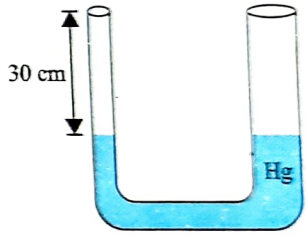
1. What is the absolute pressure on a swimmer 10 m below the surface of a lake? Take atmospheric pressure 1×10^5 N/m^2.
2. What will be the length of mercury column in a barometer tube when the atmospheric pressure is 75 cm of mercury and the tube is inclined at an angle of 60° with the horizontal direction?



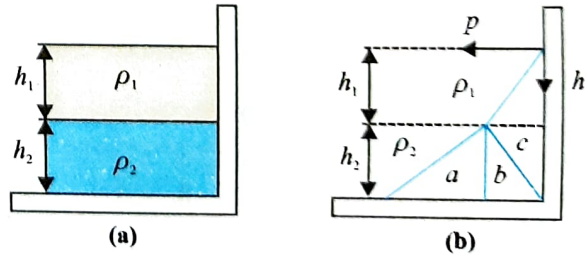
3. A U-tube contains water and methylated spirit separated by mercury. The mercury columns in the two arms are in the same level with 10 cm of water in one arm and 12.5 cm of spirit in the other. What is the specific gravity of spirit?



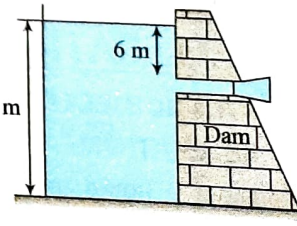
4. A U-tube in which the cross-sectional area of the limb on the left is one-third of the limb on the right contains mercury (density 13.6 g/cm^3). The level of mercury in the narrow limb extends to a distance of 30 cm from the upper end of the tube. What will be the rise in the level of mercury in the right limb, if the left limb is filled to the top with water (neglect surface tension effects)?



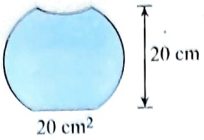
5. Figure (b) shows the variation of pressure (p) with depth (h) along a vertical wall supporting two liquids of different densities as shown in Fig. (a). Identify the correct variation of pressure in the liquid of density ρ_2.



6. Water stands at a depth of 15 m behind a reservoir dam. A horizontal pipe 4 cm in diameter passes through the dam 6 m below the surface of water as shown. There is a plug which secures the pipe opening. Then find the friction between the plug and pipe wall.



7. A glass full of water has a bottom of area 20 cm^2, top of area 20 cm^2, height 20 cm and volume half a litre.

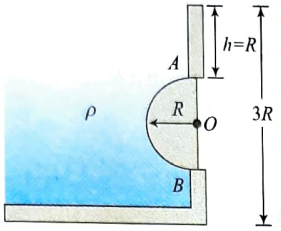


   (a) Find the force exerted by the water on the bottom.

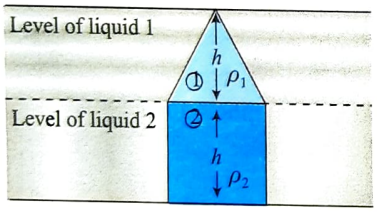
   (b) Considering the equilibrium of the water, find the resultant force exerted by the sides of the glass on the water. Atmospheric pressure = 1.0×10^5 N/m^2. Density of water = 1000 kg/m^3 and g = 10 m/s^2.

8. An isosceles triangular plate of base 3 m and altitude 3 m is immersed in oil vertically with its base coinciding with the free surface of the oil of relative density 0.8. Determine the total thrust.

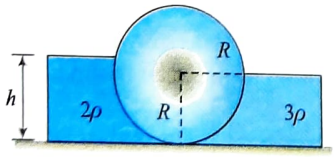
9. Figure shows a semi-cylindrical massless gate pivoted at the point O holding a stationary liquid of density ρ. The length of the cylinder is l. Calculate, the horizontal force exerted by the liquid on the gate.



10. A composite plate comprises a triangular area (1) and rectangular area (2) as shown in the figure. The triangular plate experiences forces due to the liquids of densities ρ_1 and the rectangular plate experiences the force due to liquid of density ρ_2. Find the hydrostatic forces on **(a)** area 1, **(b)** area 2, **(c)** total composite area of the plate. Assume $\rho_1 = \rho$ and $\rho_2 = 2\rho$.



11. In the figure shown, the heavy cylinder (radius R) resting on a smooth surface separates two liquids of densities 2ρ and 3ρ. Find the height 'h' for the equilibrium of cylinder.



ANSWERS

1. 2.00 N/m^2 **2.** 86.6 cm **3.** 0.8 **4.** 0.58 cm

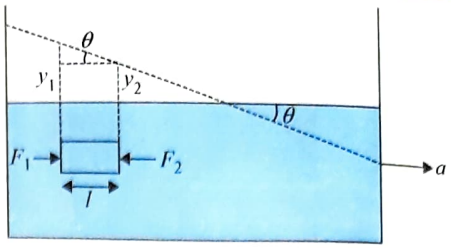
6. 74 N **7.** (a) 4 N (b) 1 N **8.** 36 kN **9.** $4\rho g R^2 l$

10. (a) $\frac{\rho g h^2 b}{3}$ (b) $\rho g h^2 b$ (c) $\frac{7}{3}\rho g h^2 b$ **11.** $h = \sqrt{\frac{3}{2}}R$

FLUID IN UNIFORMLY ACCELERATING MOTION

LIQUID SUBJECTED TO CONSTANT HORIZONTAL ACCELERATION

Consider a liquid in a tank which is moving on a horizontal surface with constant acceleration a. The free surface of the liquid takes the shape as shown in figure.



Consider a cylinder of liquid of length l and area of cross section A. The force on the left face of the cylinder is $F_1 = P_1 A$ and force on the right face of the cylinder is $F_2 = P_2 A$. Here $P_1 = \rho g y_1$ and $P_2 = \rho g y_2$. Mass of the liquid cylinder is $m = A l \rho$. Using Newton's second law for the liquid cylinder,

$$F_1 - F_2 = ma$$

or $$P_1 A - P_2 A = ma \text{ or } (\rho g y_1 - \rho g y_2)A = Al\rho a$$

or $$\frac{y_1 - y_2}{l} = \frac{a}{g}$$

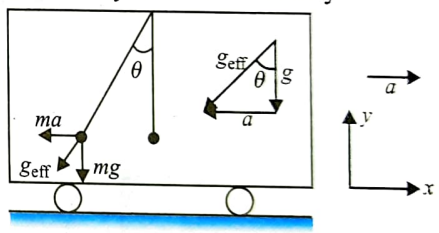
From figure,

$$\frac{y_1 - y_2}{l} = \tan\theta$$

$$\therefore \quad \tan\theta = \frac{a}{g}$$

Comparison of an Accelerated Fluid with an Accelerated Pendulum

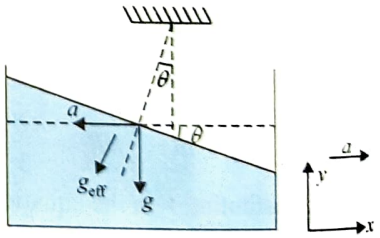
The behaviour of a liquid contained in a horizontally accelerated container can be compared to a pendulum suspended from the ceiling of a horizontally accelerated trolley.



The weight (mg) acts downward and the pseudo-force (ma) on the pendulum acts horizontally. Hence, it attains a new equilibrium position with respect to the trolley. The pendulum orients itself in the direction of net effective gravity (g_{eff}).

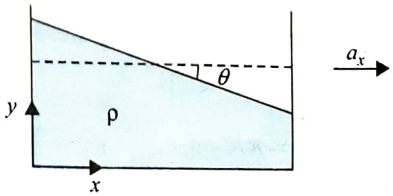
$$\tan\theta = \frac{a}{g}$$

Every fluid element attains an equilibrium position under the action of gravity and pseudo-force. The free surface of the liquid orients itself perpendicular to the direction of net effective gravity.



$$\tan\theta = \frac{a}{g}$$

In a horizontally accelerated fluid, pressure varies in the horizontal direction. In the arrangement shown in figure, rate of change of pressure in x direction is



$$\frac{dp}{dx} = -\rho a_x$$

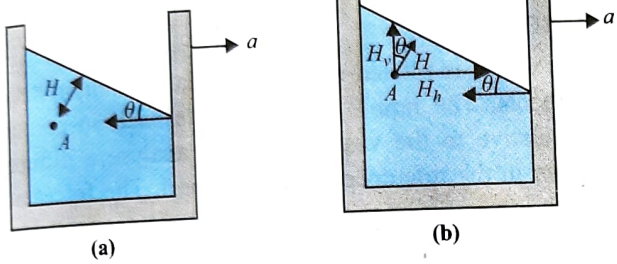
The negative sign shows that **pressure increases in a direction opposite to the direction of acceleration.**

Analysis of Pressure in Horizontal Accelerated Liquid

Now from equilibrium of liquid we can state that pressure at every point in a liquid layer parallel to the free surface (which is not horizontal) remains the same. For example, pressure at a point A in the accelerated container as shown in Fig. (a) is given as

$$P_A = P_0 + H\rho\sqrt{a^2 + g^2} \quad ...(i)$$

where H is the depth of the point A below the free surface of liquid along effective gravity and P_0 is the atmospheric pressure acting on the free surface of the liquid.



(a) (b)

The pressure at point A can also be obtained in another way as shown in Fig. (b). If H_V and H_h are the vertical and horizontal distances of point A from the free surface of liquid, then pressure at point A can also be given.

According to Pascal's law, the pressure at point A due to the vertical height of liquid above A is given as

$$P_a = P_0 + H_V \rho g \quad ...(ii)$$

Here we can write H_V as $H_V = H\sec\theta = \dfrac{H\sqrt{a^2+g^2}}{g}$

From Eq. (ii), $P_A = P_0 + H\rho\sqrt{a^2+g^2}$ ...(iii)

This equation is same as Eq. (i).

Similarly, if we consider the horizontal distance of point A from the free surface of liquid, which is H_h, then due to pseudo-acceleration of container the pressure at point A is given as

$$P_A = P_0 + H_h\rho a \quad ...(iv)$$

Here, H_h is given as $H_h = H\,\text{cosec}\,\theta = \dfrac{H\sqrt{g^2+a^2}}{a}$

From Eq. (iv), we have $P_A = P_0 + H\rho\sqrt{g^2+a^2}$ ...(v)

This equation is also same as Eq. (i).

Equipressure Lines

We have seen that pressure at different points at same horizontal line will be different. Since the pressure increases vertically downward and decreases in the direction of acceleration of the vessel, you can expect equal pressure between any two points A and B on the straight line AB.

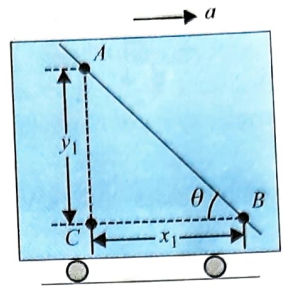
Since $P_C = P_A + \rho g y_1$ and $P_B = P_C - \rho a x_1$, we have

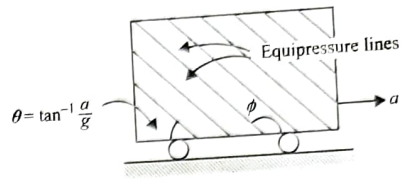
$$P_B - P_A = \rho(g y_1 - a x_1)$$

Substituting $P_B = P_A$, we have $y_1 = \dfrac{a}{g}x_1$

$$\Rightarrow \quad \frac{y_1}{x_1} = \tan\theta$$

Hence, $\theta = \tan^{-1}\dfrac{a}{g}$





This equation tells us that the equipressure lines are the parallel straight lines making an obtuse angle $\phi = 180 - \tan^{-1}(a/g)$ with the direction of acceleration of the liquid. The angle ϕ does not depend on the density of liquid.

Fluid Subjected to Constant Vertical Acceleration

Pressure Difference When Liquid is Accelerating in Vertical Direction

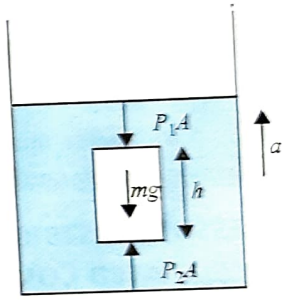
Consider a cylindrical element of height h and area A. The force on the top face of the element is P_1A and force on the bottom face is P_2A. If a is the acceleration of the liquid, then

$$P_2A - (mg + P_1A) = ma$$

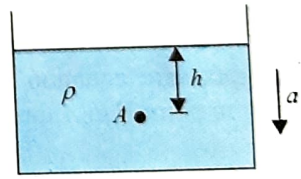
Here m is the mass of the element of the liquid which is equal to $hA\rho$. Thus, we have

$$P_2A - (hA\rho g + P_1A) = (hA\rho)\,a$$

After simplification, we get $P_2 - P_1 = \rho(g + a)h = \rho g_{eff}h$.



Constant Downward Acceleration ($a < g$)



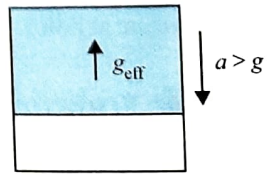
The free surface remains horizontal (figure). Pressure decreases at every point below the surface. At the point A,

$$P = \rho(g - a)h = \rho g_{eff}h$$

Pressure becomes zero everywhere when $a = g$.

Constant Downward Acceleration ($a > g$)

In this case, the fluid occupies the upper part of the container as shown in figure.

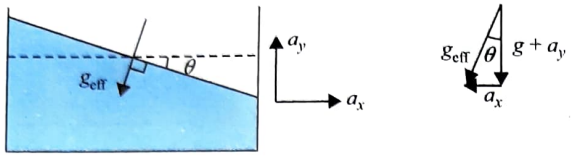


$g_{eff} = a - g$

Fluid Subjected to Combined Horizontal and Vertical Acceleration

In general, when the fluid has both horizontal and vertical acceleration, the inclination of the free surface with the horizontal may be obtained as

$$\tan\theta = \frac{dy}{dx} = \frac{dp/dx}{dp/dy} = \frac{a_x}{g + a_y}$$



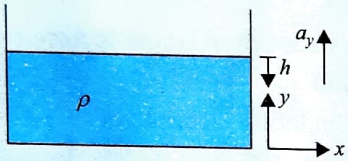
Note: In a non-accelerating fluid, pressure increases with depth and decreases with height:

$$\frac{dp}{dh} = \rho g \Rightarrow \frac{dp}{dy} = -\rho g$$

But when a fluid is subjected to a vertical acceleration, the above two equations may be modified as

Variation with depth: $\frac{dp}{dh} = \rho(g + a_y)$

Variation with height: $\frac{dp}{dy} = -\rho(g + a_y)$



A container accelerated in the vertical direction

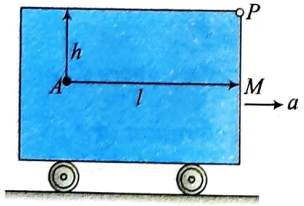
a_y is positive for upward acceleration and a_y is negative for downward acceleration.

Pressure Distribution in a Closed Accelerated Container

Consider the situation shown in figure. A closed tank car filled with water is accelerating on a horizontal track with an acceleration a. In the body, the least pressure point is P. If container is completely closed, then P can be taken as a zero-pressure point. If we wish to find pressure at a general point A at a depth h below the top surface of container and at a distance l from the right wall of container, then this can again be obtained directly by writing the pressure equation for the liquid with respect to the tank car from point P to A through point M. It can be written as

$$0 + h\rho g + l\rho a = P_A$$

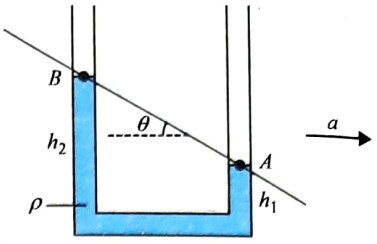
or $$P_A = h\rho g + l\rho a \quad \text{...(i)}$$



Thus, we can simply get pressure at point A by using the pressure equation in a static fluid.

Modified Manometric Equation

For an accelerated manometric tube the manometric equation is modified at a particular point, that is, when we move horizontally pressure does not remain constant; it increases in the opposite direction of acceleration and decreases in the direction of acceleration. For example, for an arrangement shown in figure the manometric equation may be written as follows:



A U-tube accelerated in the horizontal direction

Starting from point A, $P_A + \rho g h_1 + \rho a l - \rho g h_2 = P_B$

Starting from point B, $P_B + \rho g h_2 - \rho a l - \rho g h_1 = P_A$

Note down the positive and negative signs of the term $\rho a l$.

Liquid in a Vessel Rotating with Constant Angular Velocity

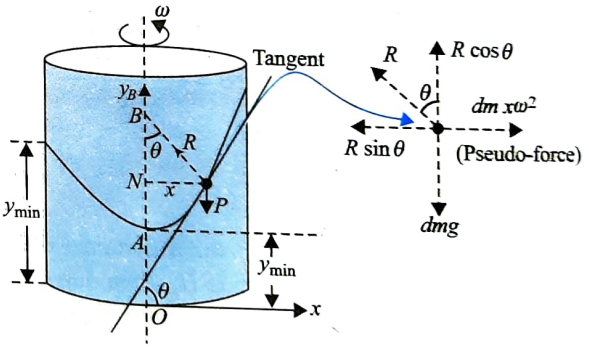
When the vessel is rotated there is no relative motion between fluid elements. The liquid surface orients itself in permanent position relative to the vessel. We will analyse a differential element dm in the reference frame of vessel. The forces acting on it are shown in Figs. (a) and (b). R is the force exerted on the element due to neighbouring elements of fluid.

$$R\cos\theta = dmg \quad \text{...(i)}$$

$$R\sin\theta = dmx\omega^2 \quad \text{...(ii)}$$

Dividing Eq. (ii) by Eq. (i), we have

$$\tan\theta = \frac{\omega^2 x}{g} \quad \text{...(iii)}$$



(a) (b)

At the position of the element, the slope of the tangent to curve is given by dy/dx.

$$\tan\theta = \frac{dy}{dx} = \frac{\omega^2 x}{g} \Rightarrow dy = \frac{\omega^2 x}{g}dx$$

which on integration yields $y = \frac{\omega^2 x^2}{2g} + c$

which represents a parabola. Thus, liquid surface is a paraboloid of revolution.

At $x = R$, $y = y_{max} = \frac{R^2\omega^2}{2g} + c$

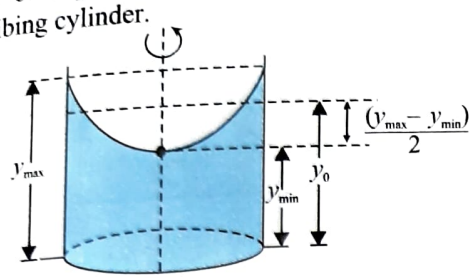
At $x = 0$, $y = y_{min} = c$.

Therefore, $y_{max} = \frac{R^2\omega^2}{2g} + y_{min}$

and the general equation can be written as

$$y = \frac{x^2\omega^2}{2g} + y_{min} \quad ...(iv)$$

Volume of a paraboloid of revolution is half that of the circumscribing cylinder.



Hence, volume of a paraboloid $= \frac{(y_{max} - y_{min})}{2} A$

If originally liquid filled up to height y_0, then

$$y_0 = y_{min} + \frac{y_{max} - y_{min}}{2} = \frac{y_{max} + y_{min}}{2}$$

$$y_{max} = y_0 + \frac{\omega^2 R^2}{4g} \quad \text{(using Eq. (iv))}$$

ILLUSTRATION 6.17

A cylindrical tank of radius 20 cm and height 50 cm has water up to 30 cm of height. What will be the rise in level of liquid at the periphery if the cylinder be given an angular velocity of 10 rad s^{-1}? Also determine the frequency of rotation when water just starts spilling over the sides of the vessel.

Sol. At an angular speed of 10 rad s^{-1}, let us assume that the water does not spill. We have

$$h_2 = H + \frac{\omega^2 r^2}{4g} \Rightarrow h_2 - H = \frac{\omega^2 r^2}{4g} = \frac{(10)^2 \times (20)^2}{4 \times 1000} = 10 \text{ cm}$$

Therefore, rise in the liquid level at the periphery is 10 cm.

For water to just spill over the sides, the maximum height (h_{max}) is 50 cm.

Again, $h_{max} = y_0 + \frac{\omega^2 R^2}{4g}$

Here, $h_{max} = 50$ cm, $y_0 = 30$ cm, $R = 20$ cm, $g = 10$ m s^{-2}

$$\therefore \quad 50 \times 10^{-2} = 30 \times 10^{-2} + \frac{\omega^2 (20 \times 10^{-2})^2}{4 \times 10}$$

$$\Rightarrow \quad \omega^2 \times 10^{-3} = 20 \times 10^{-2}$$

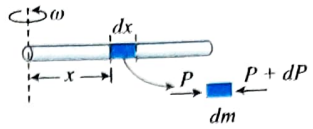
$$\omega^2 = 200 \Rightarrow \omega = 10\sqrt{2} \text{ rad s}^{-1}$$

Hence, frequency of rotation is $f = \frac{\omega}{2\pi} = \frac{5\sqrt{2}}{\pi} s^{-1}$.

FINDING PRESSURE DIFFERENCE IN ROTATING FLUIDS

In case of accelerating fluids, the pressure increases in the direction opposite to the direction of the acceleration. The rotating fluid is also the case of accelerating fluid. So in this case the pressure increases in moving away from the rotational axis.

Suppose that liquid of density ρ kept inside a tube of area of cross-section A is rotating with an angular velocity 'ω' as shown.



Let us take a small element of length 'dx' at a distance x from the axis of rotation. Mass of this element is,

dm = (density) (volume) or $dm = (\rho A dx)$

This element is rotating in a circle of radius 'x'. So, this is accelerated towards centre with a centripetal acceleration

$a = x\omega^2$ (as $a = R\omega^2$)

To provide this acceleration, pressure on right hand side of the element should be more.

$$\therefore \quad (p + dp)\, A - (p)\, A = (dm)a = (\rho A dx)\,(x\omega^2)$$

or $\quad dp = (\rho x \omega^2) dx$

$$\therefore \quad \Delta p = \int_0^x (\rho\omega^2) x dx \text{ or } \Delta p = \frac{\rho\omega^2 x^2}{2}$$

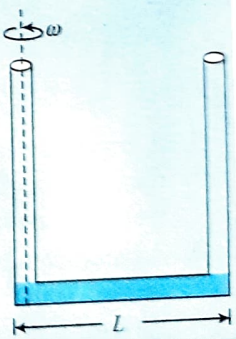
It means at a distance 'x' from the rotational axis, increases in pressure is $\frac{\rho\omega^2 x^2}{2}$

Important Point:

Take $\Delta p = +\frac{\rho\omega^2 x^2}{2}$ in moving away from the rotational axis, as pressure increases in this direction and take $\Delta p = -\frac{\rho\omega^2 x^2}{2}$ in moving towards the rotational axis.

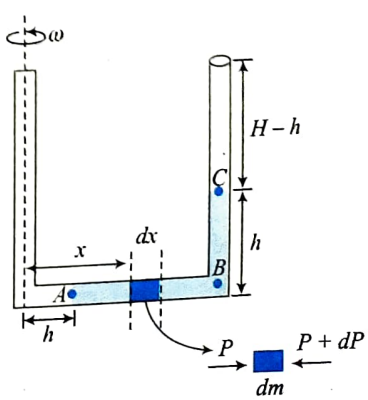
ILLUSTRATION 6.18

Length of a horizontal arm of a U-tube is L and ends of both the vertical arms are open to atmospheric pressure P_0. A liquid of density ρ is poured in the tube such that liquid just fills the horizontal part of the tube as shown in the figure. Now one end of the opened ends is sealed and the tube is then rotated about a vertical axis passing through the other vertical arm with angular speed ω_0. If length of each vertical arm is H and in the sealed end liquid rises to a height h, find pressure in the sealed tube during rotation.



Sol. When tube is rotated, liquid starts to flow radially outward and air in sealed arm is compressed. Let the shift of liquid be y as shown in figure.

Let the cross-sectional area of tube be S. Here the pressure difference between point A and B can be given by integrating the pressure difference across an element of width dx, which is given as



$$(P+dP)A - PA = dm\omega^2 x = (\rho A dx)\omega^2 x$$

$$\Rightarrow \quad dP = \rho\omega^2 x\,dx$$

Now integrating from A to B, we get

$$P_B - P_A = \int_x^L \rho\omega^2 x\,dx = \frac{\rho\omega^2}{2}(L^2 - h^2)$$

$$P_B = P_0 + \frac{\rho\omega^2}{2}(L^2 - h^2)$$

Thus pressure at point C can be given as

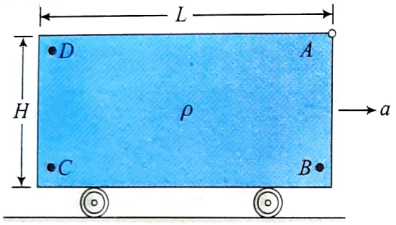
$$P_C = P_B - h\rho g$$

and at point A, pressure is atmospheric, thus we have

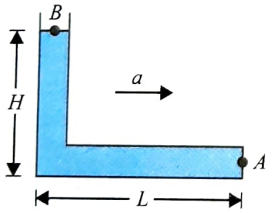
$$P_C = \frac{\rho\omega^2}{2}(L^2 - h^2) + P_0 - h\rho g$$

CONCEPT APPLICATION EXERCISE 6.3

1. A rectangular box is completely filled with a liquid of density ρ, as shown in figure. The box is accelerated horizontally with a constant acceleration a. Determine the gauge pressures at the four points A, B, C and D.



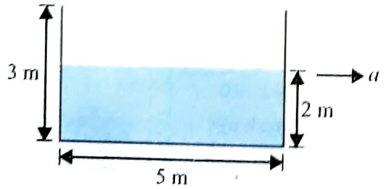
2. Figure shows an L-shaped tube filled with a liquid to a height H. What should be the horizontal acceleration a of the tube so that the pressure at the point A becomes atmospheric.



3. A barometer kept in an elevator accelerating upwards reads 76 cm of Hg. What will be the possible air pressure inside the elevator?

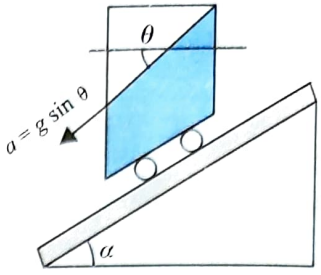
4. A barometer kept in an elevator reads 76 cm when it is at rest. What will be the barometric reading when the elevator accelerates upwards?

5. A container shown in figure is accelerated horizontally with acceleration a. Find the minimum value of a at which liquid just starts spilling out.

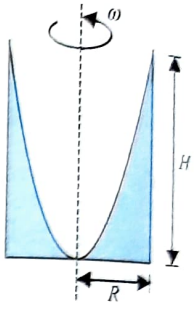


6. A closed rectangular tank 10 m long, 5 m wide and 3 m deep is completely filled with an oil of specific gravity 0.92. Find the pressure difference between the rear and front corners of the tank, if it is moving with an acceleration of 3 m/s^2 in the horizontal direction.

7. A trolley containing a liquid slides down a smooth inclined plane of angle α with the horizontal. Find the angle of inclination θ of the free surface with the horizontal.



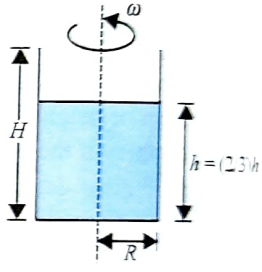
8. A cylinder of radius R and height H is filled with a liquid to an unknown height h. When it is rotated at an unknown constant angular velocity ω, the base of the cylinder gets exposed when the liquid just starts spilling out, as shown in figure.



(a) Find the height h of the liquid.

(b) Find the angular speed ω of the cylinder.

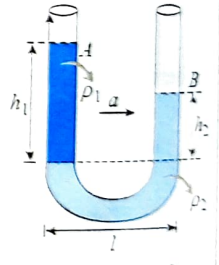
9. A cylinder of radius $R = 1$ m and height $H = 3$ m, two-thirds filled with water, is rotated about its vertical axis, as shown in figure. Determine the speed of rotation when



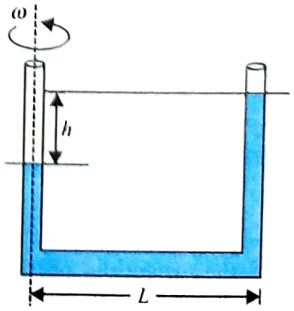
(a) the water just starts spilling over the rim.

(b) the point at the centre of the base is just exposed.

10. A U-tube shown in figure containing two liquids of specific gravities ρ_1 and ρ_2 is given an acceleration a in the horizontal direction. What is the ratio $\rho_1:\rho_2$?



11. A U-tube of length L contains liquid. It is mounted on a horizontal turn table rotating with an angular speed ω about one of its arms as shown in figure. Find the difference in heights (h) between the liquid columns in two vertical arms.



ANSWERS

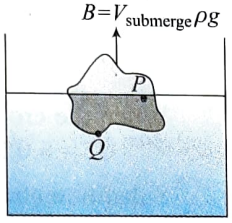
2. $g\left(\frac{H}{L}\right)$ 5. 4m/s² 6. 27.6 kPa 7. $\theta = \alpha$

8. (a) $\frac{H}{2}$ (b) $\sqrt{\frac{2gH}{R}}$ 9. (a) $\sqrt{40}$ rad/s (b) $\sqrt{60}$ rad/s

10. $\frac{la + h_2 g}{h_1 g}$ 11. $\frac{\omega^2 L^2}{2g}$

ARCHIMEDES' PRINCIPLE

Consider an irregularly shaped object at rest within a fluid. Liquid exerts a thrust on sides of the solid surface in contact with it due to its pressure. Because the liquid pressure increases with depth, the force exerted by the liquid on the surface of the object is greater for those portions which are more deeply immersed. In figure, liquid thrust at point Q is greater than that at P. The net effect on the entire body is an upward or lifting force which is called the buoyant force.



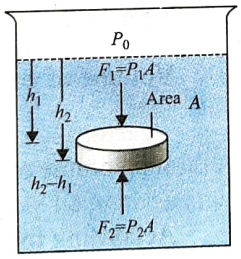
The magnitude of this upward buoyant force (B) is given by Archimedes' principle. We have $B = v_{\text{submerge}}\rho g$, where v_{submerge} is the volume occupied by the solid inside the liquid and ρ is the density of the liquid.

A body that is partially or entirely submerged in a fluid feels an upward force equal in magnitude to the weight of the displaced fluid.

CALCULATION OF BUOYANT FORCE

The force on the bottom of a submerged object is greater than that on its top because pressure increases with depth. Therefore, the fluid exerts a net upward buoyant force on the object.

Consider a submerged cylinder as shown in figure. The top of the cylinder is at depth h_1 and its bottom is at depth h_2. The horizontal forces on the cylinder acting at any depth between h_1 and h_2 cancel out, because for each force on one side there is a force of equal magnitude but opposite direction on the other.



$F_1 = P_1 A$, where $P_1 = P_0 + \rho g h_1$

From the bottom, $F_2 = P_2 A$, where $P_2 = P_0 + \rho g h_2$

Net buoyant force which acts upward on the cylinder is

$$F_b = F_2 - F_1 = P_2 A - P_1 A$$

$$= (P_0 + \rho g h_2)A - (P_0 + \rho g h_1)A \quad = \rho g A (h_2 - h_1)$$

$A(h_2 - h_1)$ is the volume of the cylinder or in other words it is the volume of the fluid displaced by the cylinder V.

Therefore, $F_b = (\rho V) g$

where ρV is the mass of the fluid displaced and we have

$F_b = mg$ (the weight of the fluid displaced)

which is Archimedes' principle. It states that 'the buoyant force acting on a submerged volume is equal to the weight of an equivalent volume of the fluid displaced'.

Important Points:

- Buoyancy force = $V_{\text{bouyant}} = V_{\text{displaced}}\rho_{\text{liquid}}\, g$
- It is independent of all factor of the body such as its mass, size, density etc.
- It only depends upon the volume of body inside the fluid.
- It depends on the nature of fluid.
- It depends on acceleration due to gravity. In gravitation free space no buoyant force will act.
- The buoyant force acts through the centroid of the displaced fluid.
- The buoyant force acts perpendicular to free water surface.

ILLUSTRATION 6.19

A cylinder is floating in a beaker which contains the liquids. The height h_1 and h_2 of the cylinder is submerged in the liquids of densities ρ_1 and ρ_2 respectively as shown in the figure. What will be change in submerged length of cylinder in fluid-2, if we increase the height of fluid-1 above fluid-2?

Sol. Net up-thrust force (buoyant force) acting on the cylinder

$$= \rho_1 l_1 A g + \rho_2 l_2 A g$$

It means buoyant force acting on the cylinder is free from the depth of the liquid above cylinder (h).

Hence no change in the submergence if we increase the height of fluid-1 above fluid-2.

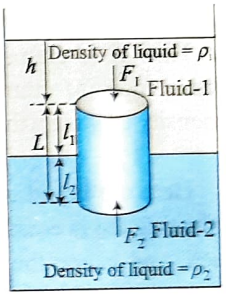


ILLUSTRATION 6.20

A beaker containing water is placed on the pan of a balance which shows a reading of M. A lump of sugar of mass m and volume v is now suspended by a thread (from an independent support) in such a way that it is completely immersed in water without touching the beaker and without any overflow of water. How will the reading change as time passes on?

Sol. Initially, the balance will show the weight of M plus thrust. But we know that

$$\text{Thrust} = v\sigma g = \frac{m}{\rho} \times 1 \times g \quad \left[\text{as } v = \frac{m}{\rho} \text{ and } \sigma = 1\right]$$

where, ρ is the density of sugar and σ is the density of water.

$$\therefore \quad W_1 = \left(M + \frac{m}{\rho}\right)g = \left(M + \frac{m}{2\rho} + \frac{m}{2\rho}\right)g \qquad \ldots(i)$$

Now as the sugar dissolves, the thrust and hence the reading of the balance should decrease. However, as dissolved sugar comes in solution, the reading of the balance should increase. To find which of these effects predominates, consider the situation when half the sugar has dissolved. In this situation the thrust reduces to $(mg/2\rho)$ from (mg/ρ) while weight of the solution increases by $(m/2)g$ (due to sugar dissolved). So reading of the balance will become

$$W_2 = \left(M + \frac{m}{2\rho} + \frac{m}{2}\right)g \qquad \ldots(ii)$$

Finally, when all the sugar is dissolved, thrust will become zero and weight of the solution will increase by mg. So the reading will become

$$W_3 = (M+m)g = \left(M + \frac{m}{2} + \frac{m}{2}\right)g \quad \ldots(iii)$$

Now as $\rho > 1$, $m/2 > m/2\rho$. So comparing Eqs. (i), (ii) and (iii), we find that the reading of the balance will gradually increase till all the sugar dissolves in water and finally will become constant equal to $(M+m)g$.

ILLUSTRATION 6.21

A piece of ice is floating in water. What will happen to the level of water when all ice melts? What will happen if the beaker is filled not with water but with liquid **(a)** denser than water, **(b)** lighter than water?

Sol. M grams of ice is floating in a liquid of density σ_L, then for its equilibrium, weight of the ice = thrust

or $Mg = V_D \sigma_L g$

So the volume of the liquid displaced by the floating ice is

$$V_D = (M/\sigma_L) \quad \ldots(i)$$

Now if M grams of ice melts completely, water formed will have mass M grams (as mass is conserved). Now if σ_w is the density of water, the volume of water formed will be

$$V_F = (M/\sigma_W) \quad \ldots(ii)$$

Here the liquid is water, i.e., $\sigma_L = \sigma_W$; so water displaced by floating ice is equal to the water formed by melting of the whole ice, and hence the level of water will remain unchanged. The following two points may be noted:

(a) If $\sigma_L > \sigma_W$, then $M/\sigma_L < M/\sigma_W$, i.e., $V_D < V_F$
i.e., liquid displaced by the floating ice will be lesser than water formed and so the level of the liquid in the beaker will rise.

(b) If $\sigma_L < \sigma_W$, $M/\sigma_L > M/\sigma_W$, i.e., $V_D > V_F$
i.e., liquid displaced by the floating ice will be more than water formed and so the level of the liquid in the beaker will fall.

ILLUSTRATION 6.22

A jeweler claims that he sells ornaments made of pure gold. He sells a crown of 9.0 N weight to a customer. In order to check the purity, the clever customer measures the weight of the crown inside water and finds it to be 8.0 N weight. **(a)** Is the ornament made of pure gold? Given, relative density of pure gold = 9.3. **(b)** In case the jeweler has mixed copper (relative density of copper = 8.9), find the mass of pure gold in the crown.

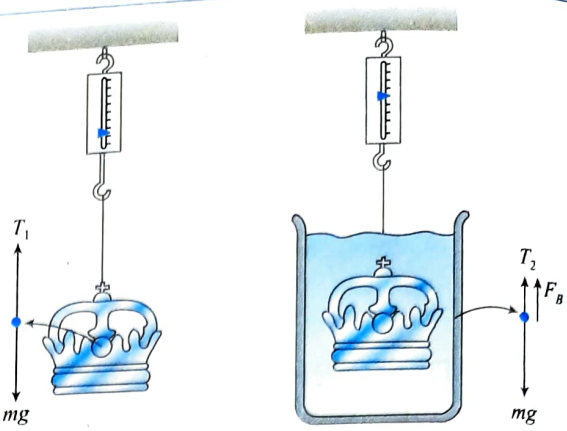
Sol.

(a) When the crown is suspended in air, the scale reads the true weight $T_1 = mg = 9.0$ N (neglecting the small buoyant force due to the surrounding air).

When the crown is immersed in water, the buoyant force $\vec{B}$ reduces the scale reading to an apparent weight,

$$T_2 = mg - B.$$

Hence, $B = mg - T_2$



FBD of crown in air FBD of crown in water

The density of the crown,

$$\rho_c = \frac{m}{V_c} = \frac{mg}{V_c g} = \frac{mg}{(B/\rho_w)} = \frac{mg\rho_w}{B} = \frac{mg\rho_w}{mg - T_2}$$

When the crown is immersed in water, apparent weight is $T_2 = 8.0$ N

Substitute numerical values:

$$\rho_c = \frac{(9.0\text{ N})(1000\text{ kg/m}^3)}{9.0\text{ N} - 8.0\text{ N}} = 9.0 \times 10^3\text{ kg/m}^3$$

Given the density of gold, $\rho = 9.3$ g cm^{-3}

Since the calculated density of the necklace is less than that of the pure gold, the necklace is not made of the pure gold.

(b) The mass of the crown in air, $m = \frac{9\text{ N}}{10} = 0.90\text{ kg} = 900\text{ g}$

The mass of the crown in water, $m' = \frac{8\text{ N}}{10} = 0.80\text{ kg} = 800\text{ g}$

Let m be mass of the pure gold in the necklace.

Then, mass of copper in the necklace = $900 - m$

Volume of gold in the necklace $\frac{m}{\rho_1} = \frac{m}{9.3}$

and volume of copper in the necklace $= \frac{900-m}{\rho_2} = \frac{900-m}{8.9}$

According to Archimedes' principle,

loss of weight of the necklace inside water = weight of the water displaced by it = buoyant force

or $(900-800)g = \left(\frac{m}{9.3} + \frac{900-m}{8.9}\right) \times 1 \times g$

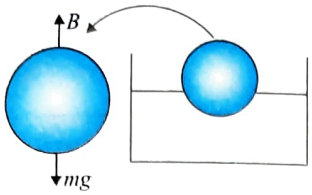
or $100 = \frac{m}{9.3} + \frac{900}{8.9} - \frac{m}{8.9}$

or $\frac{m}{8.9} - \frac{m}{9.3} = 1.1236 \Rightarrow m = 232.5\text{g}$

FLOATING BODY

Whenever the buoyant force on a body equals its weight, the net force acting on the body becomes zero. Hence, the body floats in equilibrium. Let a body of total volume V and density d be floating on a liquid of density ρ. Let v be the volume of the body inside the liquid.

Then $B = mg$ (for a floating body)
or $v\rho g = Vdg$



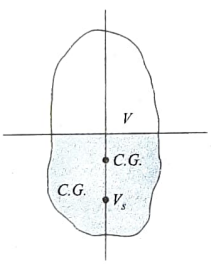
Important Points:

- When a body is in air, the net downward force on it is due to earth's gravity only. This net force = $mg = Vdg$. This force is known as weight of the body.
- When the same body is immersed in a liquid of density σ, the net downward force on it is $mg - B$.
- Buoyant force = $Vdg - v\rho g$
- This downward force is also known as apparent weight of the body in the liquid.
- When a body is immersed in a liquid, it feels lighter due to buoyant force. Hence, the buoyant force is also known as loss in weight (= $v\rho g$).
- The net force on the floating object should be zero, i.e., weight of the object is same as the buoyant force on it.
- **Buoyant force in accelerating fluids:** Suppose a body is dipped inside a liquid of density ρ_L placed in an elevator moving with an acceleration $\vec{a}$. The buoyant force F in this case becomes $F = V\rho_L g_{eff}$. Here, $g_{eff} = |\vec{g} - \vec{a}|$.

Centre of Buoyancy

Centre of gravity of the displaced fluid is called buoyant centre or centre of buoyancy. Buoyant force is applied on buoyant centre.

Let density of the solid be ρ_s, density of the fluid be ρ_l, submersed volume of the floating object be V_s and total volume of the object be V. Then

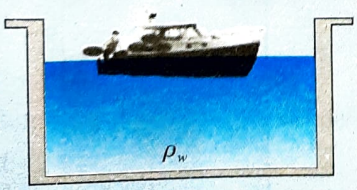


$$W = B \Rightarrow V\rho_s g = V_s \rho_l g \Rightarrow \frac{V_s}{V} = \frac{\rho_s}{\rho_l}$$

The net torque on the floating object should be zero, i.e., centre of gravity of the object and buoyant centre should lie on the same vertical line.

ILLUSTRATION 6.23

A boat containing some piece of material is floating in a pond. What will happen to the level of water in the pond if on unloading the pieces in the pond, the piece **(a)** floats **(b)** sinks ?



Sol. If M is the mass of boat and m of the objects in it, then initially as the system is floating,

$$(V_D)_{\text{before}}\,\rho_w g = (M+m)g$$

The water displaced initially $(V_D)_{\text{before}} = \dfrac{(M+m)}{\rho_w} = \dfrac{M}{\rho_w} + \dfrac{m}{\rho_w}$

(a) Now objects are unloaded and float on the surface of water.

Volume of water displaced by ship: $(V_D)_{\text{ship}} = \dfrac{M}{\rho_w}$

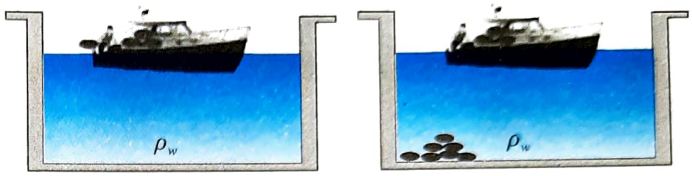
Volume of water displaced by objects: $(V_D)_{\text{objects}} = \dfrac{m}{\rho_w}$

Total volume of water displaced $(V_D)_{\text{after}} = \dfrac{M}{\rho_w} + \dfrac{m}{\rho_w}$

$(V_D)_{\text{after}} = (V)_{\text{before}}$

Water level will remain same.

(b) Now objects are unloaded and objects sink in water,



Volume of water displaced by ship: $(V_D)_{\text{ship}} = \dfrac{M}{\rho_w}$

Volume of water displaced by objects: $(V_D)_{\text{objects}} = \dfrac{m}{\rho_o}$

Total volume of water displaced $(V_D)_{\text{after}} = \dfrac{M}{\rho_w} + \dfrac{m}{\rho_o}$

$(V_D)_{\text{after}} < (V_D)_{\text{before}}$

Water level will fall.

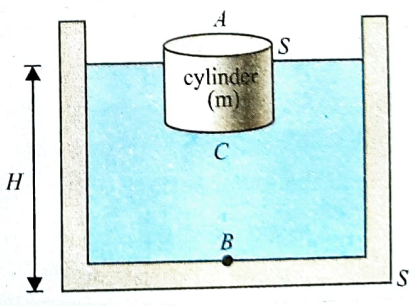
ILLUSTRATION 6.24

A liquid of density ρ is filled in a beaker of cross section S to a height H and then a cylinder of mass m and cross section s is made to float in it as shown in figure. If the atmospheric pressure is p_0, find the pressure:

(a) at the top face A of the cylinder

(b) at the bottom face C of the cylinder and

(c) at the base B of the beaker. Can ever these three pressures be equal?



Sol.

(a) Above the face A, there is external pressure due to atmosphere only. So, p_A = atmospheric pressure = p_0.

(b) At point C the pressure will be due to atmosphere and also due to the weight of the cylinder, i.e.,

$$p_C = p_0 + \frac{mg}{s}$$

(c) $p_B = p_0 + \rho g H$

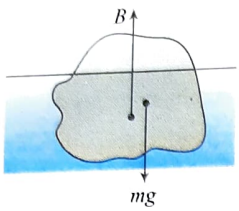
If the system is in a state of free fall (as in a satellite), i.e., $g \to 0$, then $p_A = p_C = p_B = p_0$ [as weight = 0]

ILLUSTRATION 6.25

What fraction of an iceberg lies beneath the surface of the sea? Density of sea water (ρ) = 1.028×10^3 kg/m^3, density of ice (d) = 0.917×10^3 kg/m^3.

Sol. For floating conditions, we know

Weight of the iceberg = Buoyant force



Let V be the total volume of the iceberg and v be the volume inside the sea water. Then

$$Vdg = v\rho g$$

$$\frac{v}{V} = \frac{d}{\rho} \Rightarrow \frac{v}{V} = \frac{0.917 \times 10^3}{1.028 \times 10^3} \Rightarrow \frac{v}{V} = 0.892$$

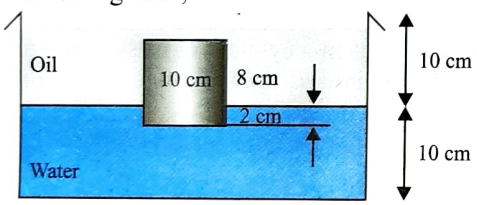
About 89.2% of the iceberg remains inside.

ILLUSTRATION 6.26

A cubical block of wood 10 cm along each side floats at the interface between an oil and water with its lowest surface 2 cm below the interface. If the heights of oil and water columns are 10 cm each and ρ_{oil} = 0.8 g/cc, find **(a)** the mass of the block, **(b)** the liquid pressure at the lower surface of the block.

Sol.

(a) For the floating cube,



Weight of the cube = Buoyant force

$$\Rightarrow mg = B_{oil} + B_{water}$$

$$= 8 \times 10^2 \times \rho_{oil} g + 2 \times 10^2 \rho_w \times g$$

$$\Rightarrow m = 800 \times 0.8 + 200 \times 1$$

$$= 640 + 200 = 840\ g$$

(b) The pressure at the lower surface of the cube is

$$P_{oil} + P_w = \frac{10}{100} \times (0.8 \times 10^3) 9.8 + \frac{2}{100} \times (1 \times 10^3) 9.8$$

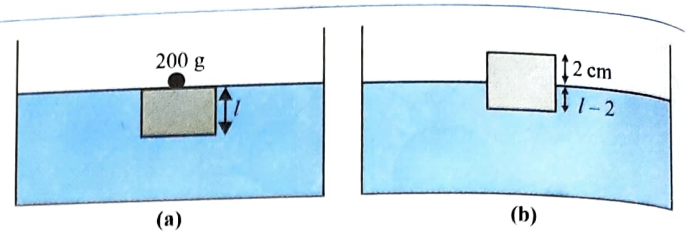
$$= 980\ \text{N/m}^2$$

(Here we have used h_{oil} = 10 cm and h_w = 2 cm above the lower surface.)

ILLUSTRATION 6.27

A cube of wood supporting a 200 g mass just floats in the water. When the mass is removed, the cube rises by 2 cm. What is the size of the cube?

Sol. The cube 'just floats' means that it is fully under water while floating.



Let m be the mass of the cube. Then

$$(m + 200)g = l^3 \rho g \quad \text{...(i)}$$

When the mass is removed,

$$mg = (l - 2)l^2 \rho g \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we have

$$200 = \rho [l^3 - (l - 2) l^2]$$

$\Rightarrow$ l = 10 cm (using ρ = 1 g/cc of water)

ILLUSTRATION 6.28

A cylindrical ice block of height H is floating in water. Its 10% of its total volume is outside the water. Kerosene oil (relative density = 0.8) is poured slowly on top of water in the container. Assume that the oil does not mix with water.

(a) If we pour kerosene on the top of water, how does the volume of ice block above the water level change?

(b) What is the thickness of kerosene layer above the water when 20% of the volume of the ice block is above the water surface?

(c) Find the ratio of volume of ice block in kerosene to its volume in water after the kerosene layer rises above the top surface of ice and the block gets completely submerged. Neglect any melting of ice.

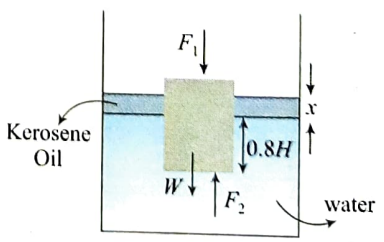
Sol. As 90% of the volume of the ice cylinder is inside water, the relative density of ice should be 0.9.

(a) When we pour kerosene on water surface, the pressure at bottom of the ice cylinder increases. This increased pressure push out the ice cylinder from water. After the ice block gets completely submerged, there is no change in pressure difference at the bottom surface and the top surface. The ice block does not move after this.

(b) Let A = area of cross section of the ice cylinder

H = height of the cylinder

F_1 = force on top surface due to atmospheric pressure (P_0) and F_2 = force on the bottom surface due to water pressure



W = weight of the cylinder

For equilibrium

$$F_2 = F_1 + W \quad \text{...(i)}$$

$$[P_0 + \rho_k gx + \rho_w g\ (0.8H)]A = P_0 A + AH\rho_{ice} g$$

$$\Rightarrow \quad \frac{\rho_k}{\rho_w}x + 0.8H = \frac{\rho_{ice}}{\rho_w}H$$

or $\quad 0.8x + 0.8H = 0.9H \Rightarrow x = \frac{H}{8}$

We can also write Eq. (i) is using Archimedes principle. As the ice cylinder is floating,

Buoyancy force = weight of the cylinder

Weight of water displaced + Weight of kerosene displaced = W

$$0.8HA\rho_w g + xA\rho_k g = HA\rho_{ice}g$$

(c) Using Archimedes principle and condition of floatation

Buoyancy force = weight of the cylinder

$$Ah_1\rho_k g + Ah_2\rho_w g = A(h_1 + h_2)\rho_{ice}g$$

On solving we get, $h_1 = h_2$

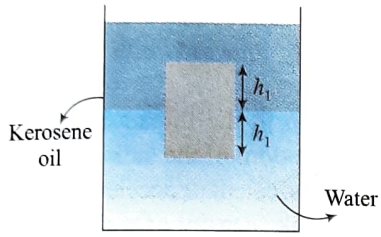
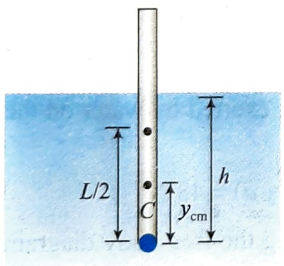


ILLUSTRATION 6.29

A wooden cylindrical of length L, radius R and density ρ has a small metal piece of mass m (of negligible volume) attached to its one end. Find the minimum value for the mass m that would make the rod float vertically in equilibrium in a liquid of density σ.

Sol. For the rod to be in stable equilibrium, center of gravity should be below the center of buoyancy. For minimum m, the two will just coincide.



Let h be the length of immersed portion. For translational equilibrium,

Weight of rod + weight of the particle attached = force of buoyancy

$$(M+m)g = \pi R^2 h\sigma g \qquad \ldots(i)$$

where $M = \pi R^2 L\rho$

The height of center of mass from bottom of the rod

$$y_{cm} = \frac{(M)L/2 + m\times 0}{m+M} = \frac{ML}{2(m+M)} \qquad \ldots(ii)$$

For rotational equilibrium and for minimum m, this should be equal to $h/2$.

Hence from Eqs. (i) and (ii)

$$\therefore \quad \frac{h}{2} = \frac{ML}{2(m+M)} \Rightarrow h = \frac{ML}{(m+M)} \qquad \ldots(iii)$$

Substituting for h in Eq. (i), we get

$$(M+m)g = \pi R^2\sigma g \cdot \frac{ML}{(m+M)}$$

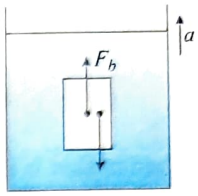
$$(M+m)^2 = \pi R^2\sigma ML$$

$$(M+m) = \sqrt{M\pi R^2\sigma L} = \sqrt{\pi R^2 L\rho \cdot \pi R^2\sigma L}$$

$$m = \pi R^2 L\sqrt{\sigma\rho} - \pi R^2 L\rho = \pi R^2 L\rho\left(\sqrt{\frac{\sigma}{\rho}} - 1\right)$$

BUOYANT FORCE IN ACCELERATING FLUID

Suppose the body is submerged into a liquid of density ρ, which is accelerating upwards. If a is the acceleration of the liquid, then



$$F_b - mg = ma$$

or $\quad F_b = m(g + a)$

Here m is the mass of the displaced liquid, which is equal to $V\rho$. Therefore,

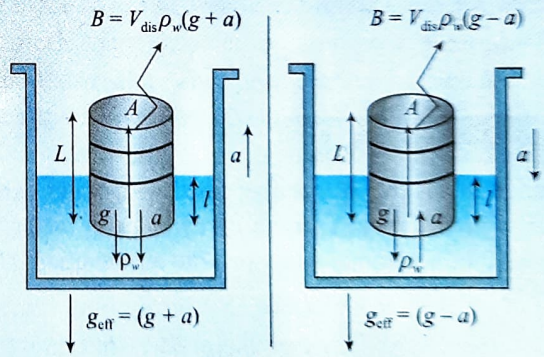
$$F_b = \rho V(g + a)$$

Important Points:

- The expression for the buoyant force for an accelerated fluid.
- Instead of taking gravity 'g' we take the effective gravity g_{eff}

$$F_{buoyant} = V_{displaced}\rho_{liquid}g_{eff}$$

- If floating object is accelerating in vertical direction



- If floating object is accelerating in horizontal direction. The liquid surface in the container will inclined at angle θ with horizontal where, $\tan\theta = \frac{a}{g}$. As up-thrust force (buoyancy force) acts perpendicular to equipressure surface hence, in this case buoyancy force will not be vertical but it will act normal to free surface of the liquid as shown in the figure.

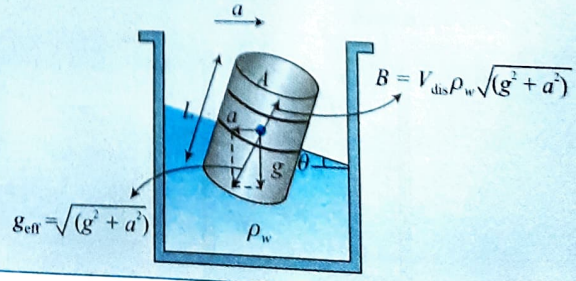
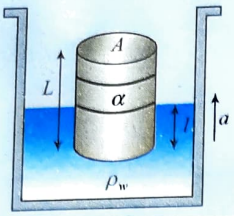


ILLUSTRATION 6.30

A cylinder is floating in a beaker filled with water, what will be change in submerged length of cylinder in water if it starts accelerating upward with acceleration a?

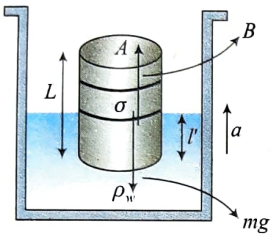


Sol. When beaker is at rest

If cylinder is floating, $B = mg$

$$(lA)\rho_w g = (LA\sigma)g \Rightarrow l = L\left(\frac{\sigma}{\rho_w}\right)$$

When beaker is accelerating



$$B - mg = ma \Rightarrow (l'A)\rho_w g_{\text{eff}} - (LA\sigma)g = (LA\sigma)a$$

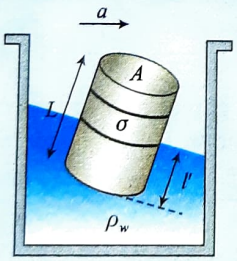
$$l'A\rho_w(g+a) - LA\sigma g = LA\sigma a$$

$$\Rightarrow \quad l'A\rho_w(g+a) = LA\sigma(g+a)$$

$$\Rightarrow \quad l' = L\left(\frac{\sigma}{\rho_w}\right)$$

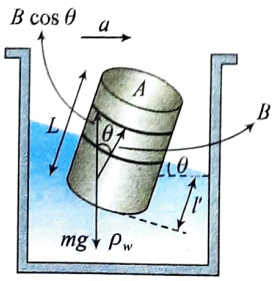
ILLUSTRATION 6.31

A cylinder is floating in a beaker filled with water, what will be change in submerged length of cylinder in water if start accelerating in horizontal direction with acceleration a?



Sol. Submerged length when beaker is at rest $l = L\left(\frac{\sigma}{\rho_w}\right)$

When beaker is accelerating



$$\tan\theta = \frac{a}{g} \Rightarrow \cos\theta = \frac{g}{\sqrt{g^2+a^2}}$$

$$B\cos\theta = mg$$

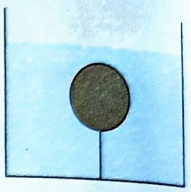
$$(l'A)\rho_w g_{\text{eff}}\cos\theta = (LA\sigma)g$$

$$l'A\rho_w \cdot \sqrt{g^2+a^2} \cdot \frac{g}{\sqrt{g^2+a^2}} = LA\sigma g$$

$$\Rightarrow l'\rho_w = L\sigma \Rightarrow l' = L\left(\frac{\sigma}{\rho_w}\right)$$

ILLUSTRATION 6.32

A solid sphere of mass $m = 2$ kg and specific gravity $s = 0.5$ is held stationary relative to a tank filled with water as shown. The tank is accelerating vertically upward with acceleration $a = 2\ \text{m s}^{-2}$.



(a) Calculate tension in the thread connected between the sphere and the bottom of the tank.

(b) If the thread snaps, calculate acceleration of the sphere with respect to the tank (density of water is $\rho = 1000\ \text{kg m}^{-3}$, $g = 10\ \text{ms}^{-2}$).

Sol. Density of the spherical material = specific gravity × density of water = $s\rho = 0.5 \times 1000\ \text{kg m}^{-3} = 500\ \text{kgm}^{-3}$

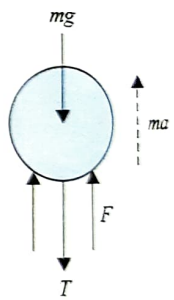
Its volume is $V = \frac{m}{s\rho}$

Hence, mass of water displaced by the sphere is

$$m_\rho = \frac{m}{s\rho}\rho = \frac{m}{s} = \frac{2}{0.5} = 4\ \text{kg}$$

(a) Since, the tank is accelerating upwards with acceleration a; therefore, apparent value of gravitational acceleration is

$$g' = g + a = 12\ \text{ms}^{-2}$$



Hence, upthrust exerted by water on the sphere is

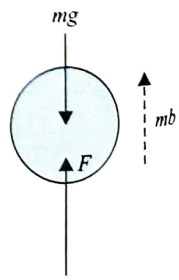
$$F = m_\rho(g+a) = 48\ \text{N}$$

Now considering the free-body diagram of the sphere accelerating with the tank,

$$F - mg - T = ma$$

$$\Rightarrow \quad T = F - mg - ma = 24\ \text{N}$$

(b) When thread snaps, tension T disappears. Let the sphere now starts accelerating upwards with acceleration b. Considering the free-body diagram,



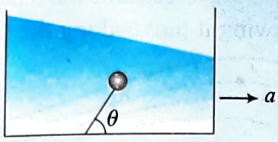
$$F - mg = mb$$

$$\therefore \quad b = 14\ \text{ms}^{-2}$$

This is the absolute acceleration of the sphere. But the tank itself is accelerating upwards with acceleration a. Therefore, upward acceleration of the sphere relative to the tank is $b - a = 12\ \text{ms}^{-2}$.

ILLUSTRATION 6.33

A container partially filled with water is moved horizontally with acceleration $a = \frac{g}{3}$. A small wooden ball of mass m is tied to the bottom of the container using a string. The ball remains inside water with the string inclined at an angle θ to the horizontal. Assuming that the density of ball is half the density of water, find the tension in the string.



Sol. As the container is accelerating the liquid surface will be inclined with horizontal such that

$$\tan\theta = \frac{a}{g} = \frac{g/3}{g} = \frac{1}{3}$$

Hence $\sin\theta = \frac{1}{\sqrt{10}}$ and $\cos\theta = \frac{3}{\sqrt{10}}$

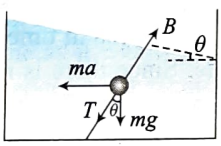
In this case the bouncy force on the ball will be normal to water surface. The magnitude of bouncy force

$$B = V_{dis}\rho_w g_{eff}$$

$$\Rightarrow \quad B = \left(\frac{m}{\rho}\right)\rho_w\sqrt{g^2+a^2} = 2m\sqrt{g^2+\frac{g^2}{9}}$$

which gives $B = \frac{2\sqrt{10}}{3}mg$

The force acting on the ball are shown in FBD (w.r.t. container)



As the ball is in equilibrium with respect to container, then

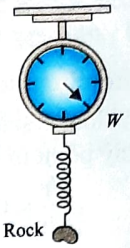
$$B = T + mg\cos\theta + ma\sin\theta$$

$$\frac{2\sqrt{10}}{3}mg = T + mg\cdot\frac{3}{\sqrt{10}} + m\left(\frac{g}{3}\right)\cdot\frac{1}{\sqrt{10}}$$

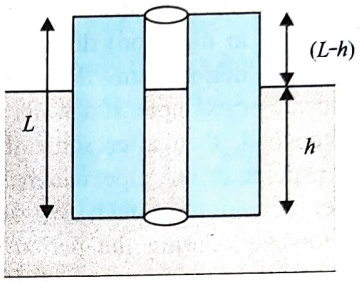
$$\Rightarrow \quad T = \frac{\sqrt{10}}{3}mg$$

CONCEPT APPLICATION EXERCISE 6.4

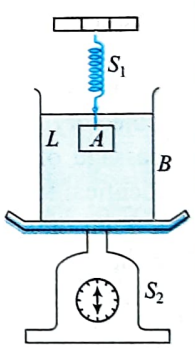
1. A beaker exactly full of water has an ice piece floating on it. As the cube melts what happens to the water level if the cube contains **(a)** an air bubble, **(b)** a lead piece and **(c)** a cork piece.
2. A body of density ρ floats with volume V_1 of its total volume V immersed in a liquid of density ρ_1 and with the remainder of volume V_2 immersed in another liquid of density ρ_2, where $\rho_1 > \rho_2$. Find the volume immersed in two liquids. (V_1 and V_2).
3. A rock is found to weigh w in air when suspended from a spring scale. When completely submerged in water while attached to the spring scale, it weighs $w_{submerge}$ as shown in figure. Find the density of the rock ρ_{rock} in terms of the scale reading and the density of water.



4. A large block of ice 5 m thick has a vertical hole drilled through it and is floating in the middle of a lake. What is the minimum length of a rope required to scoop up a bucket full of water through the hole? Relative density of ice = 0.9.



5. An iron casting has a number of cavities in it. It weighs 6000 N in air and 4000 N in water. Determine the total volume of all the cavities in the casting. The density of iron (without cavities) is 8.0 g/cm^3, ρ_w = 1 g/cm^3.
6. A necklace weighs 50 g in air, but it weighs 46 g in water. Assume that copper is mixed with gold to prepare the necklace. Find how much copper is present in it. (Specific gravity of gold is 20 and that of copper is 10.)
7. A block of ice of area A and thickness 0.5 m is floating in fresh water. In order to just support a man of 100 kg, find the area A. (The specific gravity of ice is 0.917 and density of water is 1000 kg/m^3.)
8. As figure shows, S_1 and S_2 are spring balances. Block A is hanging from spring balance S_1 and immersed in liquid L which is contained in beaker B. The mass of beaker B is 1 kg and mass of liquid L is 1.5 kg. Balances S_1 and S_2 reads 2.5 kg and 7.5 kg, respectively. What will be the readings of S_1 and S_2 when block A is pulled up out of the liquid? Find the reading of S_1 and S_2.



9. A ball floats on the surface of water in a container exposed to the atmosphere. Will the ball remain immersed at its initial depth or will it sink or rise somewhat if the container is shifted to the moon?

ANSWERS

3. $\frac{w\rho_{water}}{w - w_{submerge}}$ **4.** 0.5 m **5.** 0.125 m^3 **6.** 30 g

7. 2.41 m^2 **8.** 7.5 kg, 2.5 kg

HYDRODYNAMICS

TYPES OF FLOW

Steady and Unsteady Flow

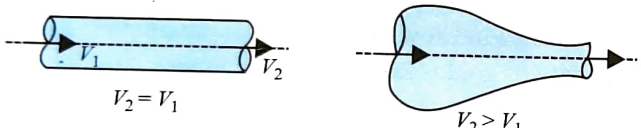
A flow is said to be steady if the velocity, pressure and density at any point in the flow do not change with time, i.e.,

$$\frac{dv}{dt} = 0, \frac{dp}{dt} = 0 \text{ and } \frac{d\rho}{dt} = 0$$

In an unsteady flow, the velocity at a point in the flow varies with time, i.e., $\frac{dv}{dt} \neq 0$.

Uniform and Non-uniform Flow

A flow is said to be **uniform** at any instant of time if the velocity (both in magnitude and in direction) does not vary along the direction of flow. In a non-uniform flow, velocity varies in the direction of flow. For example, if a water tap is kept fully open and left untouched, then after some time the velocity of the flow at any point in the pipe will remain unchanged, provided the other conditions remain the same. Hence, the flow is steady. However, during the period of opening and closing the water tap, the flow is unsteady. If the diameter of a pipe remains constant and also the quantity of liquid flowing through it remains the same, then the flow will be uniform as shown in Fig. (a). However, if the diameter of the pipe varies along its length and the quantity of liquid flowing through it remains the same, then the flow will be non-uniform as shown in Fig. (b).



(a) Uniform flow (b) Non-uniform flow

VISUALIZATION OF FLOW

The fluid flow pattern may be visualized in terms of pathlines and streamlines.

1. **Pathline:** It is a line drawn such that it describes the trajectory of a given fluid particle as it moves with the passage of time. The concept of the pathline in fluids is identical to the trajectory of a solid body. Tangent drawn at a point on a pathline gives the direction of velocity at that point at the time when the particle passes that point.

   Important properties of a pathline:
   - A pathline can intersect itself at different times.
   - Pathlines are the history lines of individual fluid particles over a period of time.

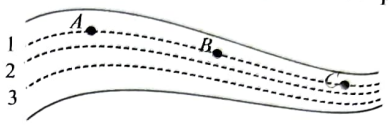
2. **Streamline:** It is the most common graphical concept of flow visualization. It is a line drawn such that a tangent at every point of it is in the direction of velocity at that instant.

   A collection of streamlines drawn in this manner represents the complete flow pattern at an instant. Thus, for an unsteady flow, the streamline pattern may or may not remain the same at the next instant. Note that if the unsteadiness is due to a change in the magnitude of the velocity, then the streamline pattern still remains the same at different instants of time. On the other hand, if the unsteadiness is due to change in the direction of the velocity, the pattern will change from time to time.

Unlike pathlines, a streamline cannot intersect itself; furthermore, two streamlines cannot meet or intersect at a point. This is because instantaneously fluid can have a unique velocity at a point.

Streamline Flow

If every point of a steadily flowing liquid follows exactly the same path that has been followed by the particles preceding it, the flow is said to be streamlined. The path is known as 'streamline'. In figure, the paths 1, 2, 3 are streamlines. If a liquid follows the path ABC, particles following it move along the same path.



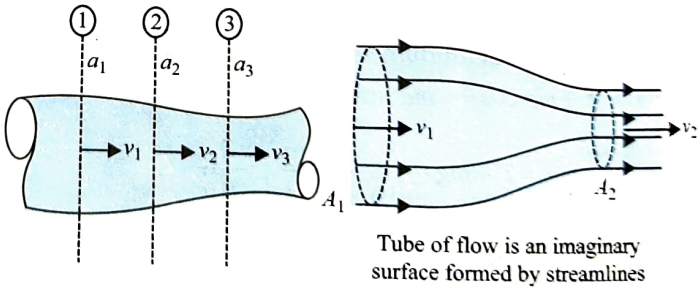
Turbulent Flow

When the velocity exceeds a certain critical value, the nature of flow becomes complicated. Random, irregular, local currents (called vortices) develop throughout the fluids. The resistance to the flow increases tremendously. This type of flow is called 'turbulent flow'.

PRINCIPLE OF CONTINUITY: THE CONTINUITY EQUATION

The continuity equation is a mathematical expression of the law of conservation of mass in fluid dynamics.

It states that 'in a steady flow, the amount (mass) of liquid entering a section is the same as that leaving another section in a given time interval'. If a_1, a_2, a_3, etc., are the cross-sectional areas of a pipe (carrying a liquid in a steady flow) at various sections, and v_1, v_2, v_3, etc., are the corresponding velocities (average), with ρ_1, ρ_2, ρ_3, etc., as the densities, then the mass of liquid flowing past these sections in time Δt would be $\rho_1 v_1 a_1 \Delta t$, $\rho_2 v_2 a_2 \Delta t$, $\rho_3 v_3 a_3 \Delta t$, etc. Since there is no accumulation of the liquid within the pipe, so



Tube of flow is an imaginary surface formed by streamlines

$$\rho_1 v_1 a_1 \Delta t = \rho_2 v_2 a_2 \Delta t = \rho_3 v_3 a_3 \Delta t = \cdots$$

$$\Rightarrow \quad \rho_1 v_1 a_1 = \rho_2 v_2 a_2 = \rho_3 v_3 a_3 = \cdots$$

In case of incompressible liquids, $\rho_1 = \rho_2 = \rho_3 = \cdots$

$$\therefore \quad v_1 a_1 = v_2 a_2 = v_3 a_3 = \cdots$$

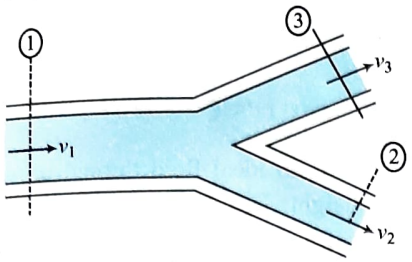
The above equation is known as the 'equation of continuity',

$$A_1 v_1 = A_2 v_2$$

ILLUSTRATION 6.34

A broad pipe having a radius 10 cm branches into two pipes of radii 5 cm and 3 cm. If the velocity of flowing water in the pipe of radius 3 cm is 5 cm/s, determine the velocities of water in the remaining two pipes. Given that the rate of discharge through the main branch is 600π cm^3/s.

Sol. Consider any three sections (1), (2) and (3) in the three pipes of different radii as shown in figure. If v_1 and v_2 be the velocities of water at sections (1) and (2), respectively; then



$a_1 = \pi(10)^2$ cm^2, $a_2 = \pi(5)^2$ cm^2

and $a_3 = \pi(3^2)$ cm^2

$v_1 = ?$, $v_2 = ?$, $v_3 = 5$ cm/s

The rates of discharge through the three pipes are

$a_1v_1 = 100\pi v_1$ cm^3/s

$a_2v_2 = 25\pi v_2$ cm^3/s

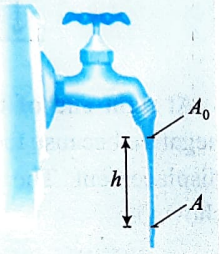
$a_3v_3 = 9\pi \times 5 = 45\pi$ cm^3/s

Now, $Q = a_1v_1 = a_2v_2 + a_3v_3 \Rightarrow 600\pi = 100\pi v_1 = 25\pi v_2 + 45\pi$

Solving we get $v_1 = 6$ cm/s and $v_2 = 22.2$ cm/s

ILLUSTRATION 6.35

Water is flowing straight down from an open tap. A stream of water emerging from a faucet "necks down" as it falls as shown in figure. The horizontal cross-sectional area of the falling stream decrease from A_0 to A. The two levels are separated by a vertical distance h. What is the volume flow rate from the tap?



Sol. The volume flow rate (in cubic meters per second) of the falling water is the same as it was when it left the faucet. This is because no water is added to or taken out of the stream after the water leaves the faucet. In figure, as water falls from a tap, its speed increases. As the volume flow rate must be the same at all horizontal cross sections of the stream, the stream must "neck down" (narrow).

The equation of continuity applies in the form

$$A_0v_0 = Av, \quad \ldots(i)$$

where v_0 and v are the water speeds at the levels corresponding to A_0 and A.

Since the effects of air resistance are being ignored, the water can be treated as a freely-falling object. Thus, the acceleration of the water is that due to gravity. Because the water is falling freely with acceleration g, we use the relation

$$v_2 = v_0^2 + 2gh. \quad \ldots(ii)$$

Eliminating v between Eqs. (i) and (ii) and solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2ghA^2}{A_0^2 - A^2}}$$

From Eq. (ii), the volume flow rate R_V is then

$$R_V = A_0v_0 = A_0\sqrt{\frac{2ghA^2}{A_0^2 - A^2}}$$

APPLICATIONS OF PRINCIPLE OF CONTINUITY

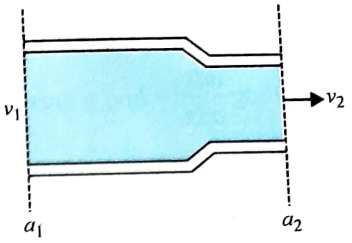
A liquid flowing in a tapering pipe has its velocity increased. If a_1 and a_2 be the cross-sectional areas of the pipe in the broader and narrower regions and v_1 and v_2 be the velocities, then from equation of continuity,

$$a_1v_1 = a_2v_2$$

$$\Rightarrow \frac{v_2}{v_1} = \frac{a_1}{a_2}$$

Now, $\frac{a_1}{a_2} > 1$

$$\Rightarrow \frac{v_2}{v_1} > 1 \Rightarrow v_2 > v_1$$



Thus, in case of a steady flow of liquid, the greater the cross-sectional area of the pipe, the lesser is the velocity and vice versa.

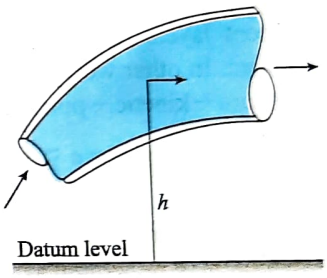
ENERGY ASSOCIATED WITH A MOVING LIQUID

POTENTIAL ENERGY

It is the energy possessed by a flowing liquid, by virtue of its position, i.e., height above a fixed datum level (see figure).

The potential energy possessed by a liquid at a height h above the datum level per unit volume will be

$$\frac{\Delta m}{\Delta V}gh = \rho gh$$



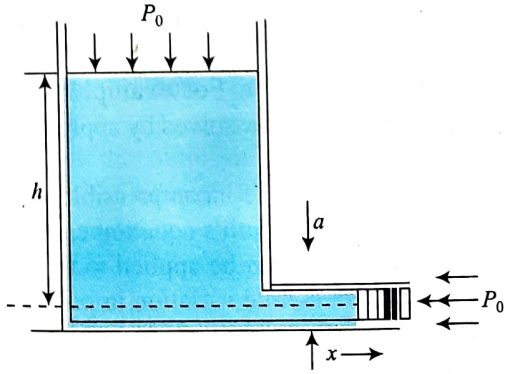
KINETIC ENERGY

It is the energy possessed by a moving liquid due to its motion. If v be the velocity of liquid flow, then the kinetic energy per unit volume of the liquid is given by

$$\frac{1}{\Delta V}\left(\frac{1}{2}\Delta m v^2\right) = \frac{1}{2}\rho v^2$$

PRESSURE ENERGY

As a direct consequence of Pascal's law of transmission of pressure, it is easy to see that a liquid can be made to possess energy (in the form of pressure energy) by applying external pressure. To understand the concept of pressure energy, let us consider a broad vessel, filled with water (or other liquid) up to a height 'h', and fitted with a thin, side horizontal tube at its bottom. Let the water within the side tube be stopped by a frictionless cork.



Let the cork applies an external force, pushing the water in the side tube into the vessel by a distance dx. Work done by the external agency against the hydrostatic pressure is given by

$$\Delta W = \text{Force} \times \text{Displacement}$$
$$= (\text{Pressure} \times \text{Area}) \times \text{Displacement} = h\rho g\, adx$$

Now this work gets stored in the form of energy (pressure energy) within the liquid that has been forced into the vessel. Energy per unit volume of the forced liquid is

$$h\rho g \frac{adx}{adx} = h\rho g = \text{pressure}$$

Thus, the pressure energy per unit volume of a liquid is equal to the hydrostatic pressure.

BERNOULLI'S EQUATION

According to Bernoulli's theorem, the sum total of these three kinds of energies, kinetic energy, potential energy and pressure energy remains constant along a streamline in a steady flow of an ideal fluid. That is,

$$\frac{1}{2}\rho v^2 + \rho g y + p = \text{constant}$$

An ideal fluid is inviscid and incompressible. In other words, there is no dissipation of energy due to internal friction between adjacent layers of the fluid and density of this fluid remains constant. In other words, the sum of total energy per unit volume (pressure + kinetic + potential) is constant for an ideal fluid.

OTHER FORMS OF BERNOULLI'S EQUATION

If the kinetic, potential and pressure energies are expressed in terms of per unit mass or per unit weight, then the Bernoulli's equation may be written as

$$\frac{p}{\rho} + \frac{v^2}{2} + gy = \text{constant} \qquad \text{(per unit mass)}$$

$$\frac{p}{\rho g} + \frac{v^2}{2g} + y = \text{constant} \qquad \text{(per unit weight)}$$

The Bernoulli's equation in the per unit weight form is the most commonly used expression. Each term in this equation has the dimensions of length. The term $v^2/2g$ is known as the velocity head, $p/\rho g$ is known as the pressure head and y is known as the potential head.

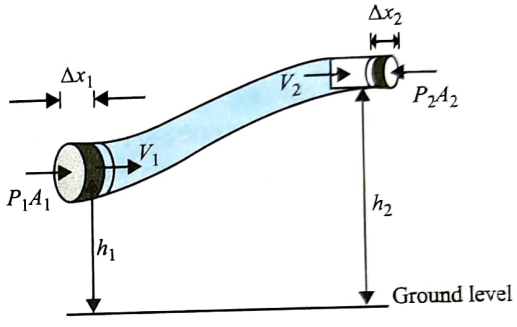
Applications and Limitations

- The flow is assumed to be steady, i.e., there is no change in pressure, velocity and density of the fluid at any point with respect to time.
- However, in the problems of unsteady flow with gradually changing conditions, Bernoulli's equation can be applied without appreciable error. For example, a problem of emptying a large tank can be solved by applying Bernoulli's equation.
- The fluid is assumed to be incompressible. Since liquids are incompressible, Bernoulli's equation can be applied to all liquids. However, it can be applied to the problems of gas flow when there is little variation in pressure, velocity and temperature so that density of gas can be assumed to be constant.
- The fluid is assumed to be irrotational. The irrotationality means the net angular momentum at any point in the fluid flow is zero.
- The fluid is assumed to be ideal, i.e., the energy loss due to friction is assumed to be absent.

PROOF OF BERNOULLI'S EQUATION

Figure shows flow of an ideal fluid through a tube of varying cross section and height. The fluid at the left end exerts a force P_1A_1, which moves the fluid through a distance Δx_1 and therefore does work $W_1 = F_1\Delta x_1 = P_1A_1\Delta x_1$ on the fluid. At the same time an equal volume of fluid at the upper right end is displaced through Δx_2. So the work done on this fluid volume is

$$W_2 = -F_2\Delta x_2 = -P_2A_2\Delta x_2$$



At right end of tube the work done on this fluid element is negative because the force on it is in the opposite direction to the displacement. The total work done on the entire fluid element shown is

$$W = W_1 + W_2 = P_1A_1\Delta x_1 - P_2A_2\Delta x_2$$

Using equation of continuity,

$$A_1\Delta x_1 = A_2\Delta x_2 = \Delta V$$

$$\Rightarrow \quad W = (P_1 - P_2)\Delta V$$

The change in kinetic energy between the right and left end is

$$\Delta K = \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2$$

And the change in potential energy is

$$\Delta U = \Delta m g h_2 - \Delta m g h_1$$

Using energy conservation,

$$\Delta W = \Delta K + \Delta U$$

$$\Rightarrow \quad (P_1 - P_2)\Delta V = \left(\frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2\right) + (\Delta m g h_2 - \Delta m g h_1)$$

Dividing this equation by ΔV and substituting $\rho = \Delta m/\Delta V$, we find

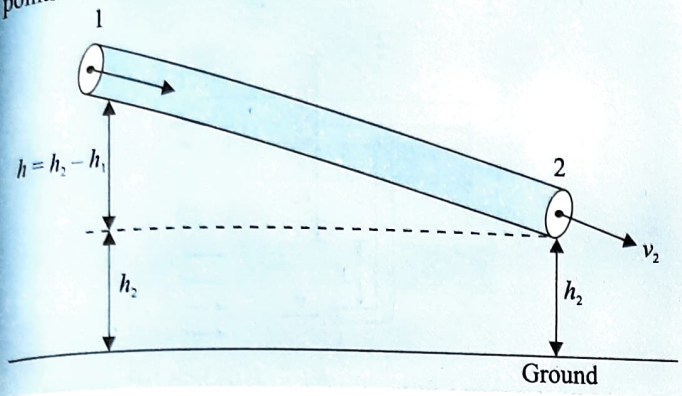
$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

$$\therefore \quad P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant} \qquad \ldots(i)$$

Equation (i) is called Bernoulli's equation. This equation is, in fact, the conservation of energy.

ILLUSTRATION 6.36

A liquid moves in a smooth tube of uniform cross section. Find the pressure P_2 at point 2 of the tube if the pressure at point 1 of the extended tube is P_1 and the vertical distance between points 1 and 2 is h.



Sol. Since the cross section of the tube is uniform, $A_1 = A_2 = A$.

Then, applying equation of continuity at points 1 and 2, we have $v_1 = v_2$.

Substituting $v_1 = v_2$ $(= v)$ in Bernoulli's equation, we have

$$P_1 + \frac{1}{2}\rho v^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v^2 + \rho g h_2$$

This gives $P_1 + \rho g h_1 = P_2 + \rho g h_2$

Since $h_1 - h_2 = h$, we have $P_2 = P_1 + \rho g h$

ILLUSTRATION 6.37

An airplane has an effective wing surface area of 16 m^2 that is generating the lift force. In level flight the air speed over the top of the wings is 62.0 m/s, while the air speed beneath the wings is 54.0 m/s. What is the weight of the plane? Take the density of air 1.29 kg/m^3. This is an approximation, since the density of air decreases with increasing altitude above sea level.

Sol. In level flight the lift force must balance the plane's weight W, so its magnitude is also W. The lift force arises because the pressure P_B beneath the wings is greater than the pressure P_T on top of the wings. The lift force, then, is the pressure difference times the effective wing surface area A, so that $W = (P_B - P_T)A$. The area is given, and we can determine the pressure difference by using Bernoulli's equation.

We have, $P_B + \frac{1}{2}\rho v_B^2 + \rho g y_B = P_T + \frac{1}{2}\rho v_T^2 + \rho g y_T$

Since the flight is level, the height is constant and $y_B = y_T$, where we assume that the wing thickness may be ignored. Then, Bernoulli's equation simplifies and may be rearranged as follows:

$$P_B + \frac{1}{2}\rho v_B^2 = P_T + \frac{1}{2}\rho v_T^2 \quad \text{or} \quad P_B - P_T = \frac{1}{2}\rho v_T^2 - \frac{1}{2}\rho v_B^2$$

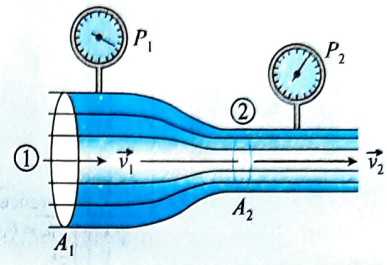
Recognizing that $W = (P_B - P_T)A$, we can substitute for the pressure difference from Bernoulli's equation to show that

$$W = (P_B - P_T)A = \frac{1}{2}\rho(v_T^2 - v_B^2)A$$

$$= \frac{1}{2}(1.29)[(62.0)^2 - (54.0)^2](16) = 9600 \text{ N}$$

ILLUSTRATION 6.38

A horizontal constricted pipe illustrated in figure, is used to measure the flow speed of an incompressible fluid. If the pressure difference between sections '1' and '2' is $P_1 - P_2$ and the flow speed at section '1' is v_1, determine the flow speed at section '2' of figure.



Sol. The Bernoulli's equation shows how the pressure of an ideal fluid decreases as its speed increases. From the design of the tube (areas A_1 and A_2) and measurements of the pressure difference $P_1 - P_2$, we can calculate the speed of the fluid with this equation. Because the problem states that the fluid is incompressible, we can categorize it as one in which we can use the equation of continuity for fluids and Bernoulli's equation.

Applying Bernoulli's equation to points 1 and 2, noting that $y_1 = y_2$ because the pipe is horizontal:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \qquad \ldots(i)$$

Solve the equation of continuity for v_1, $v_1 = \frac{A_2}{A_1}v_2$

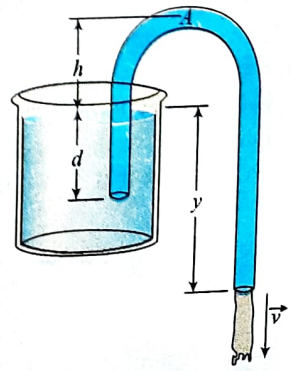
Substitute this expression into Eq. (i),

$$P_1 + \frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Solve for v_2, $v_2 = A\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$

ILLUSTRATION 6.39

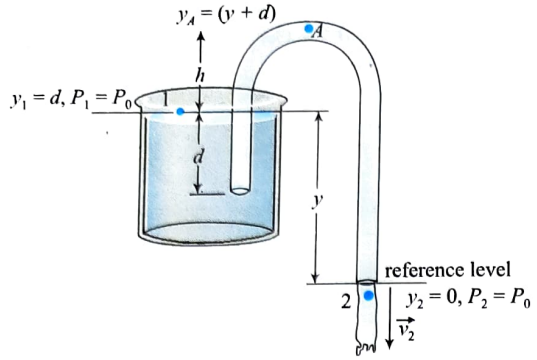
A siphon tube is useful device for draining liquid from a tank. For this purpose ,the siphon tube is first filled with liquid, and then one end is inserted into the tank. Liquid then drains out the other end, as shown in figure. **(a)** Derive an expression for the speed v of the fluid emerging from the tube. This expression should give v in terms of the vertical height y and the acceleration due to gravity g. **(b)** At what value of the vertical distance y will the siphon stop working? **(c)** Derive an expression for the absolute pressure at the highest point in the siphon (point A) in terms of the atmospheric pressure P_0, the fluid density ρ, g, and the heights h and y.



Sol.

(a) If we take the pipe exit defines $y = 0$, as our reference level for measuring elevations (and thus gravitational potential

energy). Noting that the pressure at the top of the tank and at the pipe exit is at the atmospheric pressure p_0 (because both places are exposed to the atmosphere).we write Bernoulli's equation between point 1 and 2, as



$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

$$p_0 + \frac{1}{2}\rho v_2^2 + \rho g(0) = p_0 + \frac{1}{2}\rho(0)^2 + \rho g y \Rightarrow v_2 = \sqrt{2gy}$$

Note that this speed does not depend on the depth d of the tube below the surface of the liquid.

(b) The siphon will stop working when $v_2 = 0$ m/s, or $y = 0$, i.e., when the end of the hose is at the water level in the tank.

(c) The fluid speed at point A is the same as the speed of the fluid emerging from the tube, because the cross-sectional area of the tube is the same everywhere hence $v_A = v_2$, we write Bernoulli's equation between point A and 2, as

$$P_A + \frac{1}{2}\rho v_A^2 + \rho g y_A = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

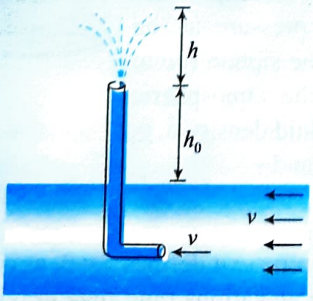
$$\Rightarrow \quad P_A + \frac{1}{2}\rho v_2^2 + \rho g(y+h) = P_0 + \frac{1}{2}\rho v_2^2 + \rho g(0)$$

or $\quad P_A = P_0 - \rho g(y+h)$

We can observe the pressure at highest point of tube is less than atmospheric pressure. This creates a pressure drop throughout the siphon tube, in the same sense that 'sucking' on a straw reduces the pressure along its length all the way to the intake point.

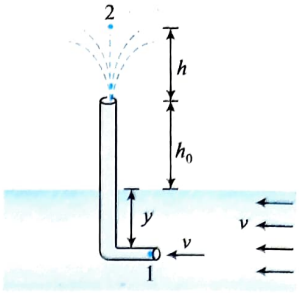
ILLUSTRATION 6.40

Water is flowing in a canal speed v. A bend tube is lowered into the canal as shown in figure. The upper end of the tube is held at a height h_0 above the surface of the water. To what height 'h' above the upper end of the tube, will the water jet spurt? Assume that flow remains ideal.



Sol. Let the depth of the tube entrance be y from top surface of flowing water. Consider point 1 (at the entry) and point 2 (at the maximum height of the fountain) on a streamline. Apply Bernoulli's theorem

$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$



Let us consider point 1 as reference level hence, $h_1 = 0$ and $P_1 = P_0 + \rho g y$, and $P_2 = P_0$ where P_0 = atmospheric pressure

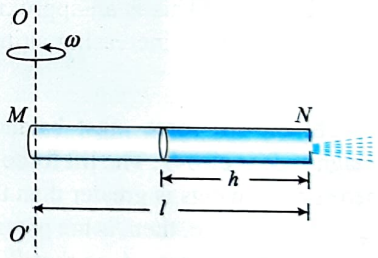
$$(P_0 + \rho g y) + 0 + \frac{1}{2}\rho v^2 = P_0 + \rho g(y + h_0 + h) + 0$$

[Since speed at top point 2 is zero]

$$\therefore \quad \frac{1}{2}\rho v^2 = \rho g(h_0 + h) \quad \therefore \quad h = \frac{v^2}{2g} - h_0$$

ILLUSTRATION 6.41

A horizontal oriented tube MN of length l rotates with a constant angular velocity ω about a stationary vertical axis OO' passing through the end A (see figure). The tube is filled with an ideal fluid. The end M of the tube is open, the closed end N has a very small orifice. Find the velocity of the fluid relative to the tube as a function of the column length h.



Sol. Applying Bernoulli's theorem at '1' and '2' just inside orifice and just outside orifice respectively. We have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \qquad \ldots(i)$$

Here P_2 is equal to atmospheric pressure (P_{atm}). The velocity of liquid at point '1' is much smaller than velocity of liquid at point '2', hence we can neglect v_1. Now Eq. (i) becomes

$$P_1 + 0 = P_{atm} + \frac{1}{2}\rho v^2 \Rightarrow P_1 - P_{atm} = \frac{1}{2}\rho v^2 \qquad \ldots(ii)$$

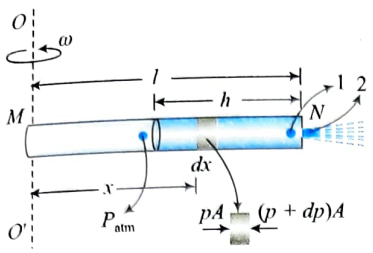
Let us consider a water element of mass 'dm' rotating about axis in radius x. We can write the force equation of circular motion of this elemental mass.

$$dpA = dm\omega^2 x = A dx \rho \omega^2 x$$

or $\quad dp = \rho \omega^2 x\, dx$

Integrating both sides $\int_{P_{atm}}^{P_1} dp = \rho\omega^2 \int_{l-h}^{l} x\,dx$

$$= P_1 - P_{atm} = \frac{1}{2}\rho\omega^2(2hl - h^2) \quad \ldots(iii)$$

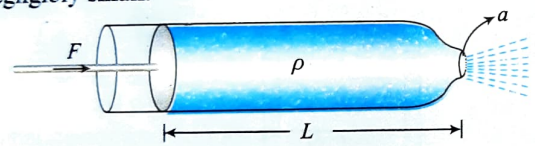


From Eqs. (ii) and (iii), $\frac{1}{2}\rho v^2 = \frac{1}{2}\rho\omega^2(2hl - h^2)$

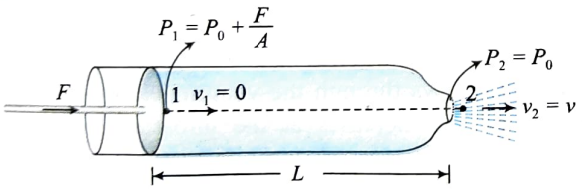
$$\Rightarrow \quad v = \omega h\sqrt{\left(\frac{2l}{h} - 1\right)}$$

ILLUSTRATION 6.42

Calculate how much work should be done in order to squeeze all water from a horizontally located cylinder as shown in figure during the time t by means of a container force acting on the piston? The volume of water in the cylinder is equal to V, the cross sectional area of the orifice is a, with a being considerably less than the piston area. The friction and viscosity are negligibly small.



Sol. Work done to squeeze water $= F \cdot L$



If v be the velocity at point B, then applying Bernoulli's theorem at points '1' and '2', we have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Here $P_1 = P_0 + \frac{F}{A}$ and $P_2 = P_0$

also v_1 is very much smaller than v_2, can be neglected hence we have

$$P_0 + \frac{F}{A} = P_0 + \frac{1}{2}\rho v^2 \quad \ldots(i)$$

Force on piston, $F = P \cdot A = \frac{1}{2}\rho v^2 A \quad \ldots(ii)$

Volume of liquid coming through orifice per unit time $= a\,v$

Volume of liquid coming through during time 't', $V = a\,v\,t$

Velocity of liquid through orifice, $v = \left(\frac{V}{at}\right)$

From Eqs. (i) and (ii) $F = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 A$

Thus work done is $W = FL = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 AL = \frac{1}{2}\rho\left(\frac{V}{at}\right)^2 V$

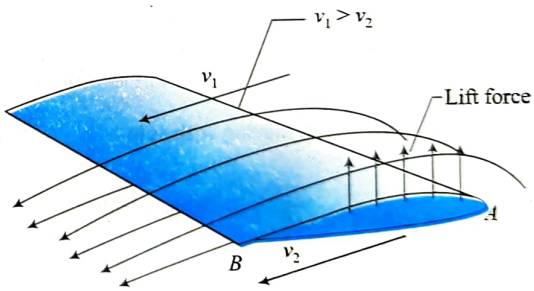
hence $W = \frac{1}{2}\rho\frac{V^3}{a^2t^2}$

APPLICATIONS OF BERNOULLI'S THEOREM

The fluid dynamic lift or simply dynamic lift is a force that acts on a body because of its motion through a fluid. It is a consequence of Bernoulli's theorem according to which pressure is low where speed is high and vice-versa. Further, dynamic lift is different from static lift, which is the buoyant force that acts on a body according to Archimedes' principle. The examples of dynamic lift are to be found in an airplane wing , a spinning ball and many others. Let us consider the first two in some detail.

The Physics of Airplane Wings

In figure, AB represents an aerofoil of an aeroplane. Its shape is such that it is slightly convex upwards and concave downwards. On account of this shape, when the aerofoil moves through the air, the molecules of air which separate from each other at A simultaneously, will meet at B at the same time because the motion is streamlined. The molecules of air above the airfoil have to travel a longer distance than the molecules which are below the aerofoil in the same time.



Air above the wing is moving with a larger velocity than that below it. The pressure at top of wing is less than pressure at bottom of wing.

This means that the air above the wing is moving with a larger velocity than that below it. Since air is moving more rapidly (with velocity v_1) above the wing than below it (with velocity v_2), according to Bernoulli's theorem, the pressure above the wing is less than the pressure below it. It is this difference in pressure which provides a net upward lift.

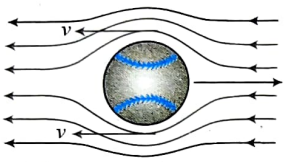
Dynamic Lift on a Spinning Ball

Let us first consider a non-spinning ball (baseball, cricket ball or a tennis ball) as it moves through the air from left to right. If we look at the motion of the ball through the air with reference to the coordinate system fixed on the ball, the flow of air is steady and the streamlines are symmetrical as shown in Fig. (a). As the streamlines approach the ball, half bend upward and half downward. Thus, the velocity of the air flow rightward the ball is

equal to the velocity leftward the ball. According to Bernoulli's principle, the pressure upward of the ball is equal to the pressure downward it. Hence, there is no net upward or downward force. Let us now consider a ball that has no translatory motion but is spinning about an axis perpendicular to the plane of the paper. Due to friction, air is dragged with the ball and the resulting streamlines, as seen from the spinning ball, are as shown in Fig. (b). Note that the tangential velocity at the top is to the right, i.e., v_t, while at the bottom it is to the left, i.e., $(-v_t)$.

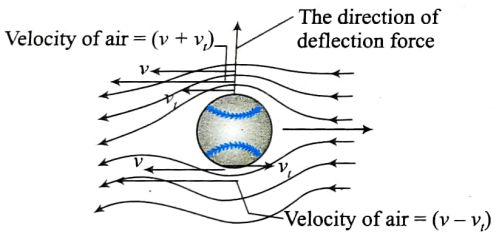
Let us now consider the case in which the ball has both translatory and spinning motions. It is clear that the streamlines are much closer above the ball than below the ball because the velocity of air above the ball is $(v+v_t)$ and below the ball is $(v-v_t)$, both to the left. Thus, according to Bernoulli's principle, the pressure above the ball is less than the pressure below the ball.

This results in a net force F acting upward as shown in Fig. (c). Thus, as the spinning ball moves through the air, there is an upward lift which is called the dynamic lift.

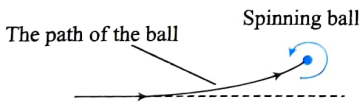


The path of the ball

(a) The ball moving without spin



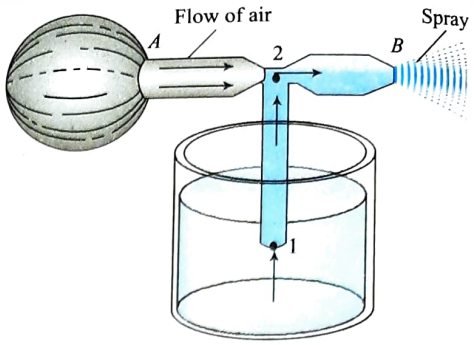
(b) The ball moving with spin



(c) The trajectory with ball

The dynamic lift due to spinning is called Magnus effect. It is on account of dynamic lift that a spinning ball deviates from its parabolic trajectory as it moves through air. For example, a cricket ball bowled with an initial spin or a tennis ball 'cut' (i.e., given a rapid spin along an axis at right angles to its direction of flight by striking at the centre) has a curved flight.

Atomizer or sprayer: An atomiser is usually used in perfume and deodorant bottles. When the rubber balloon is pressed, the air passes with a large velocity over the tube dipping in the liquid to be sprayed. Due to this, pressure over the tube dipping in the liquid decreases. It makes the liquid rise up in the tube. Due to the applied pressure, the air rushing out with a large velocity from the balloon blows away the liquid coming out of the nozzle in the form of a fine spray.

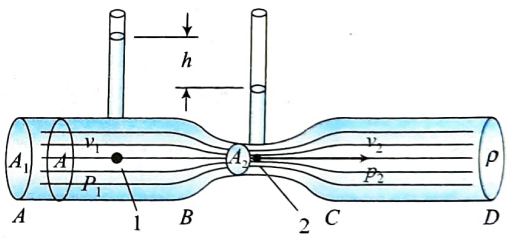


Blowing off of the roofs of houses during a storm: In a storm, cyclone or hurricane, sometimes the light roofs of thatched houses are blown off. This is because, due to the high velocity wind blowing over the roof, the pressure above the roof decreases. As the atmospheric pressure below the roof is greater than the pressure above the roof, the roof gets lifted and is blown away by the wind.

VENTURIMETER

Venturimeter is a device based on Bernoulli's theorem. It is used to measure the rate of flow of a liquid.

As shown in figure, a venturimeter consists of a wide tube having a constriction in the middle. The liquid enters through the broad end AB, called convergent cone. After passing through the narrow short horizontal part BC, called throat, the liquid, then leaves through the other broad end CD, called divergent cone. Two vertical tubes are connected to the venturimeter, one to the convergent cone at the point P and the other to the throat at the point Q.



When liquid flows through the venturimeter, its velocity (or its kinetic energy) increases at the throat (constriction) and hence pressure energy (or pressure) decreases. This decrease in pressure is measured by measuring the difference in levels (h) of the liquid in the two vertical tubes.

Let A_1 and A_2 be the areas of cross-section of the venturimeter at the points P and Q respectively and v_1 and v_2 be the velocities of the liquid, while crossing at these points. According to the equation of continuity,

$$A_1 v_1 = A_2 v_2$$

or $$v_2 = \frac{A_1 v_1}{A_2} \quad \text{...(i)}$$

Since flow of the liquid through the venturimeter is horizontal, the potential energy of the liquid at point Q remains the same as that at point P. If ρ is density of liquid, then according to Bernoulli's theorem

$$\frac{P_1}{\rho}+\frac{1}{2}v_1^2=\frac{P_2}{\rho}+\frac{1}{2}v_2^2$$

Or $P_1+\frac{1}{2}\rho v_1^2 = P_2+\frac{1}{2}\rho\, v_2^2$ ...(ii)

Here, P_1 and P_2 are the values of the pressure of the liquid at points P and Q respectively. Equation (ii) gives

$$P_1-P_2=\frac{1}{2}\rho(v_2^2-v_1^2)$$

From the Eq. (i), substituting for v_2 in the above equation, we have

$$P_1-P_2=\frac{1}{2}\rho\left(\frac{A_1^2\,v_1^2}{A_2^2}-v_1^2\right)=\frac{1}{2}\rho v_1^2\left(\frac{A_1^2}{A_2^2}-1\right)$$

or $$P_1-P_2=\frac{1}{2}\rho\, v_1^2\left(\frac{A_1^2-A_2^2}{A_2^2}\right)$$ ...(iii)

or $$v_1=A_2\sqrt{\frac{2(P_1-P_2)}{\rho(A_1^2-A_2^2)}}$$ ...(iv)

If difference of levels of the liquid in the two vertical tubes is h, then

$$P_1-P_2=h\rho g$$ ...(v)

Therefore, from the Eqs. (iv) and (v), we have

or $$v_1=A_2\sqrt{\frac{2gh}{A_1^2-A_2^2}}$$ ...(vi)

It gives the velocity of flow of the liquid. The volume of liquid flowing per second i.e. rate of flow of the liquid is given by

$$V=A_1v_1$$

or $$V=A_1A_2\sqrt{\frac{2(P_1-P_2)}{\rho(A_1^2-A_2^2)}}=A_1A_2\sqrt{\frac{2gh}{A_1^2-A_2^2}}$$ ...(vii)

If instead of liquid gas is flowing through the pipe, we connect a differential manometer to measure pressure difference. In this case,

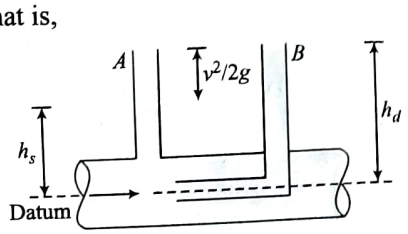
$$P_1-P_2=hdg$$ ...(viii)

here d is the density of the manometer liquid. From (vii) and (viii), we get

$$V=A_1A_2\sqrt{\frac{2(P_1-P_2)}{\rho(A_1^2-A_2^2)}}=A_1A_2\sqrt{\frac{2hdg}{\rho(A_1^2-A_2^2)}}$$

STATIC PRESSURE AND DYNAMIC PRESSURE

The static pressure at a point in a fluid flow may be obtained by connecting a single-tube manometer perpendicular to the direction of the fluid flow as shown by tube A in figure. The height h_s to which the liquid rises in it with respect to the reference level (datum) is a measure of the static pressure in the fluid at a point on the axis. That is,



$$p_s=\rho g h_s \quad \text{or} \quad \frac{p_s}{\rho g}=h_s$$

The height h_s is also called the static pressure head. When a tube bent at right angle is inserted in the fluid flow such that the open end faces against the direction of fluid flow as shown by tube B in figure. At the open end of the tube the velocity of the fluid suddenly reduces to zero and the kinetic energy gets converted into pressure energy. The liquid in tube B rises to a higher level than that in tube A. The difference in the levels of liquid is a measure of velocity of the fluid. Tube B measures the dynamic pressure and the height h_d of the liquid in the tube is called the dynamic head, which is equal to the sum total of the static pressure head and velocity head.

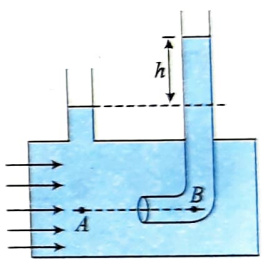
That is, $h_d=h_s+\frac{v^2}{2g}$.

PITOT TUBE

Pitot tube is used to measure the speed of flow of a fluid. The cross section of tube at B is perpendicular to the direction of flow and tube is at rest, so that at B, velocity of fluid becomes zero, i.e., $v_B=0$. Let v be the velocity of fluid at A and P_A and P_B are pressures at A and B. Then $P_B-P_A=h\rho g$,

where ρ is density of liquid in vertical tubes.

For liquids: If points A and B are in the same horizontal line and area of is the same at A and B, gravitational head is the same and the pressure difference is due to difference in velocity. Applying Bernoulli's theorem at A and B, we have



$$P_A+\frac{1}{2}\rho v^2=P_B+0$$

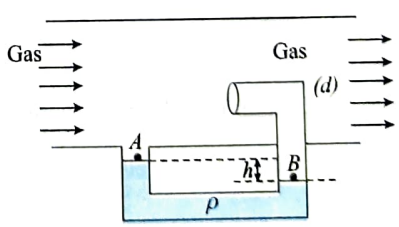
$$\Rightarrow \quad \frac{2(P_B-P_A)}{\rho}=v^2$$

$$\Rightarrow \quad v=\sqrt{\frac{2(P_B-P_A)}{\rho}} \quad \Rightarrow v=\sqrt{2gh}$$

For gases: Let ρ be the density of the liquid in the vertical tube.

$$P_B-P_A=h\rho g$$

Applying Bernoulli's theorem at A and B, we have

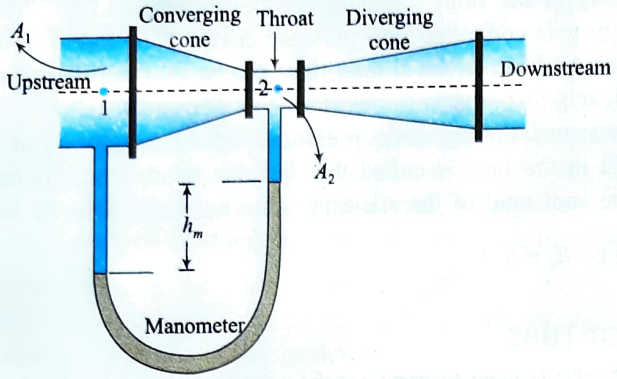


$$P_A+\frac{1}{2}dv^2=P_B \quad \Rightarrow v^2=\frac{2(P_B-P_A)}{d}$$

$$\Rightarrow \quad v^2=\frac{2h\rho g}{d} \quad \Rightarrow v=\sqrt{\frac{2h\rho g}{d}}$$

ILLUSTRATION 6.43

A pitot tube is mounted on an aeroplane wing to measure the speed of the plane. The tube contains alcohol and shows a level difference of 40 cm as shown in figure. What is the speed of the plane relative to air? Given that relative density of alcohol = 0.8 and density of air = 1 kg m^{-3}.

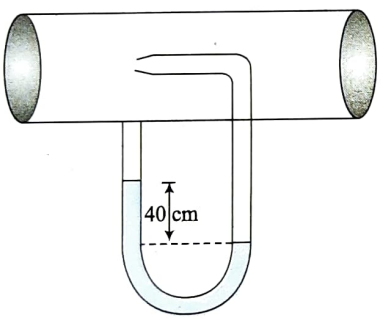


Sol. Here, density of air, $\rho = 1$ kg m^{-3}

Since relative density of alcohol = 0.8,

density of alcohol = 0.8 g cm^{-3} = 800 kg m^{-3}

Let P_1 be pressure of air at the nozzle of the U-tube and P_2, the value of pressure of air at the other end connected to the main tube (refer figure).



Now, $P_1 - P_2$ = 40 cm of alcohol = 0.4 m of alcohol

$= 0.4 \times$ density of alcohol $\times g$

$= 0.4 \times 800 \times 9.8\ \text{Nm}^{-2}$

If v_1 is velocity of air at the nozzle and v_2, at the other end of the tube, then

$$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2$$

or $$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

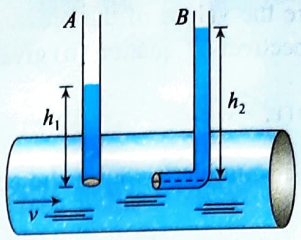
As the nozzle of the tube is parallel to direction of motion of air, $v_1 \approx 0$.

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2$$

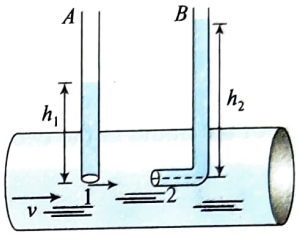
or $$v_2 = \sqrt{\frac{2(P_1 - P_2)}{\rho}} = \sqrt{\frac{2 \times 0.4 \times 800 \times 9.8}{1}} = 79.2\ \text{m s}^{-1}$$

ILLUSTRATION 6.44

A liquid is flowing in a horizontal pipe of uniform cross section with velocity v. Two tubes A and B of small cross-sectional area are inserted into the pipe as shown. Assume the flow to remain streamline inside the pipe. Calculate the difference in height of the liquid in two tubes



Sol. The liquid will not flow through the tubes. There is no speed of the liquid inside tubes. Let us take two points 1 (just outside the tube A) and 2 (just inside the tube B) as shown in figure. The speed of liquid at a point 2 is zero.



Using Bernoulli's equation between point 1 and 2 give

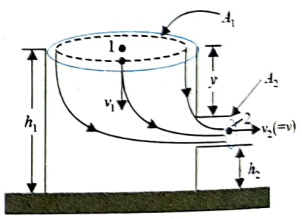
$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + 0$$

$$\therefore\ P_2 - P_1 = \frac{1}{2}\rho v^2 \qquad [v_1 = v \text{ and } v_2 = 0]$$

$$\therefore\ \rho g h_2 - \rho g h_1 = \frac{1}{2}\rho v^2 \Rightarrow\ h_2 - h_1 = \frac{v^2}{2g}$$

VELOCITY OF EFFLUX: TORRICELLI'S THEOREM

The liquid comes out of a hole made at a depth y. The areas of the base and hole are A_1 and A_2, respectively. Let us calculate the velocity of efflux.



Choose two points 1 and 2 at the free surface of liquid and at just outside the hole, respectively. Since the points 1 and 2 are exposed to atmosphere, $P_1 = P_2 = P_{atm}$. We have done this to get a mathematical advantage. However, you can take any other point inside the tube instead of taking at the free surface of the liquid. Equation of continuity gives

$$A_1 v_1 = A_2 v_2 \qquad \text{...(i)}$$

$$P_{atm} + \frac{\rho v_1^2}{2} + \rho g h_1 = P_{atm} + \frac{\rho v_2^2}{2} + \rho g h_2$$

Substituting $h_1 - h_2 = y$, we have

$$v_2 = \sqrt{v_1^2 + 2gy} \qquad \text{...(ii)}$$

Substituting $v_1 = (A_2/A_1)v_2$ from Eq. (i) in Eq. (ii), we have

$$v_2 = \sqrt{\frac{2gy}{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

Note:

- If area of opening is much lesser than area of cross section of tank ($A_2 << A_1$), then velocity of efflux

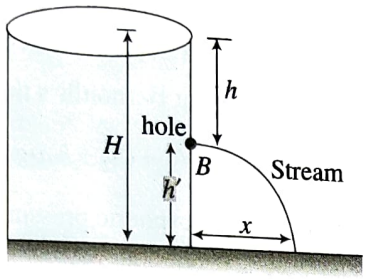
$$v = \sqrt{2gh}$$

which is the same speed that an object would acquire in falling from rest through a distance h under gravity.

- The velocity of efflux is the velocity of escaping liquid relative to the container (but not necessarily relative to ground when the container moves).

HORIZONTAL RANGE OF THE ESCAPING LIQUID

Water stands at a depth H in a tank whose sides are vertical. A hole is made in one of the walls at a depth h below the water surface. Let container remains at rest.



When the liquid stream emerges out of the hole, it goes along a parabolic path. So time taken by water to fall through a height of $H-h$ is given as

$$H-h = 0\times t + 1/2\, gt^2 \Rightarrow t = \sqrt{\frac{2(H-h)}{g}}$$

Horizontal range is given by

$$x = v\times t = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}} = \sqrt{4h(H-h)} = 2\sqrt{hh'}$$

or $x = \sqrt{4h(H-h)}$

$$= \sqrt{4hH - 4h^2 + H^2 - H^2} = \sqrt{H^2 - (H^2 - 4hH + 4h^2)}$$

$$= \sqrt{H^2 - (2h-H)^2}$$

For the range to be maximum, $(2h-H)^2 = 0$

$\Rightarrow$ $2h - H = 0 \Rightarrow h = H/2$

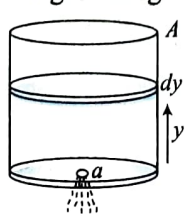
So the maximum range $x = 2\sqrt{hh'}$.

The formula $= 2\sqrt{hh'}$ tells us if we make two holes at equal vertical distances from top and bottom, both liquids jets will strike the same spot (but not simultaneously).

ILLUSTRATION 6.45

A cylindrical vessel of area of cross section A is filled with a liquid up to a height H. A very small hole of area of cross section a is made at the bottom of the vessel. Find the time taken by the vessel to become empty.

Sol. The liquid escapes with a velocity $\left(=\sqrt{2gh}\right)$ which varies with the level of liquid. Hence, we have to use integral. Let y be the height of the liquid at an instant. This height changes by dy in time dt.



Volume of water leaving out per second $= -A\, dy/dt$.

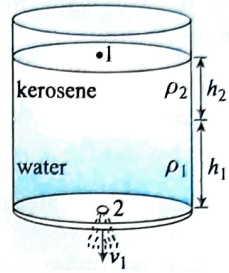
At the hole, volume escaping per second is $av = a\sqrt{2gy}$

$$\therefore\quad a\sqrt{2gy} = -A\frac{dy}{dt} \Rightarrow \int_H^0 \frac{-dy}{\sqrt{y}} = \frac{a\sqrt{2g}}{A}\int_0^t dt$$

$$\Rightarrow\quad t = \frac{A}{a\sqrt{2g}}2\sqrt{H} = \frac{A}{a}\sqrt{\frac{2H}{g}} \quad \text{(assuming } a \ll A\text{)}$$

ILLUSTRATION 6.46

A wide vessel with a small hole in the bottom is filled with water and kerosene. Find the velocity of water flow if the thickness of water layer is $h_1 = 30$ cm and that of kerosene is $h_2 = 20$ cm.



Sol. **Method 1:** Let the densities of water and kerosene be ρ_1 and ρ_2, respectively. Let the area of the vessel be A and area of the hole be a.

Applying Bernoulli's theorem at cross sections 1 and 2, we have

$$\frac{1}{2}\rho_1 V_1^2 + P_{atm} + 0 = \frac{1}{2}\rho_1 V_2^2 + (P_{atm} + h_2\rho_2 g) + h_1\rho_1 g$$

$$av_1 = Av_2$$

$$\Rightarrow\quad \frac{1}{2}\rho_1 V_1^2\left[1 - \frac{a^2}{A^2}\right] = \left[h_1 + \frac{h_2\rho_2}{\rho_1}\right]\rho_1 g$$

$$\Rightarrow\quad v_1 = \sqrt{2g\left(h_1 + \frac{h_2\rho_2}{\rho_1}\right)}$$

Method 2: We can replace kerosene with water column which produces the same pressure as kerosene column at interface of kerosene and water.

Pressure produced by kerosene, $P_{interface} = \rho_2 g h_2$

Let the equivalent height of water column be h'_1.

Hence, $P_{interface} = \rho_1 g h'_1 = \rho_2 g h_2$

$$\Rightarrow\quad h'_1 = \frac{\rho_2 h_2}{\rho_1} \qquad \text{...(i)}$$

Total height of water above orifice, which produces the same pressure as before, is

$$H = h_1 + h'_1 = h_1 + \frac{\rho_2}{\rho_1}h_2$$

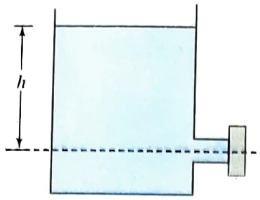
Using Torricelli's formula for velocity of efflux,

$$v = \sqrt{2gH} = \sqrt{2g\left(h_1 + \frac{\rho_2}{\rho_1}h_2\right)}$$

FORCE OF REACTION DUE TO EJECTION OF LIQUID

A cylindrical vessel has an opening of cross-sectional area 'a' near the bottom. A disc is held against the opening to prevent liquid of density ρ from coming out. If the height of the liquid

above opening is h. Let us analyse the force on the disc in this situation.



The disc experiences a hydrostatic pressure from the liquid inside the vessel. Pressure at the level of the disc is

$$P_1 = P_{atm} + \rho gh$$

The air pressure on the outside of disc is $P_2 = P_{atm}$.

The net outward force $= (P_2 - P_1)a = \rho gha$.

Now the disc is moved a short distance away in horizontal direction. The liquid comes out, strikes the disc inelastically and drops vertically downward. The water in this case will impart impulsive (inpact) force on the disc.

When the disc is moved away, the liquid moves out with speed $v = \sqrt{2gh}$. The (mass per second), i.e., the rate of mass coming out of the opening is given by $dm/dt = \rho av = \rho a\sqrt{2gh}$.

Momentum per second imparted by water rightward is given by

$$\frac{dm}{dt}v = (\rho av)v = \rho av^2 = 2\rho gha$$

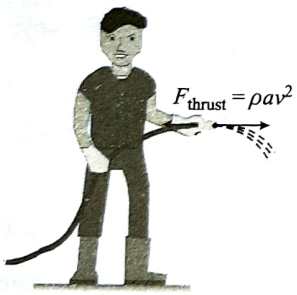
The change in momentum per second after striking the disc is

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

$$\Rightarrow \quad 0 - (\rho av^2) = -\rho av^2 = -2\rho gha$$

Taking the rightward direction as positive, the force on the liquid is towards the left and its reaction with disc is towards the right. The force obtained is twice the hydrostatic force. The force due to atmospheric pressure is cancelled out on the both sides.

Note: If the velocity of a liquid (or gas) of density ρ coming out through an opening of area of cross section a is v, then there will be a thrust force $F = \rho av^2$ due to liquid coming out the opening. The direction of the thrust force will be just opposite to the velocity direction.



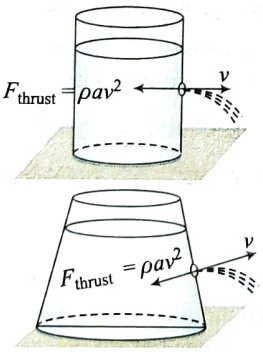


ILLUSTRATION 6.47

On the opposite sides of a wide vertical vessel filled with water, two identical holes are opened, each having cross-sectional area 'a'. The height difference between them is equal to h. Find the resultant force of reaction of water flowing out of vessel.

Sol. Let A and B be two openings, each of area 'a' the velocity of water coming out of opening are V_A and V_B, respectively. Volume of water emerging per second through A is aV_A.

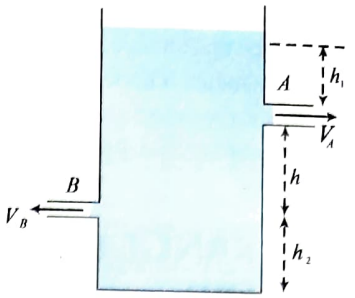
Volume of water emerging per second through B is aV_B.

Force of reaction due to water coming out at A is given by

F_A = Rate of change of momentum per second at A

$= (aV_A\rho)V_A = a\rho V_A^2$

Similarly, force of reaction at B is given by $F_B = a\rho V_B^2$



Therefore, net force on vessel is

$$F = F_B - F_A = a\rho\left(V_B^2 - V_A^2\right) \qquad \text{...(i)}$$

Now, applying Bernoulli's theorem at A and B, we get

$$p_a + \frac{1}{2}\rho V_A^2 + (h_2 + h)\rho g = p_a + \frac{1}{2}\rho V_B^2 + \rho gh_2 \qquad \text{...(ii)}$$

where p_a is atmospheric pressure.

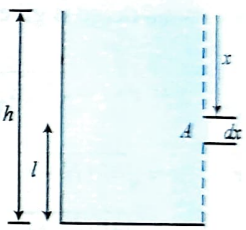
$\therefore \quad V_B^2 - V_A^2 = 2gh$

Substituting this value in Eq. (i), we get $F = a\rho 2gh$

or $F = 2\rho g\, ah$

ILLUSTRATION 6.48

The side wall of a wide vertical vessel of height $h = 75$ cm has a narrow slit (vertical) running all the way down to the bottom of the vessel. The length of the slit is $l = 50$ cm and the width is $b = 1$ mm. With the slit closed, water is filled to the top. Find the resultant reaction force of water coming out as the slit is opened.



Sol. Let us consider an infinitesimal portion of length dx of the slit at a depth x below water level. Reaction force due to this portion is given by

$$dF = Av^2\rho = (bdx)2gx\rho$$

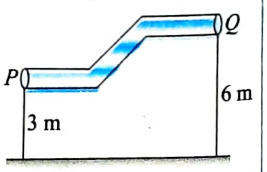
Total reaction force is

$$F = \int dF = \int_{h-l}^{h} 2g\rho b(xdx)$$

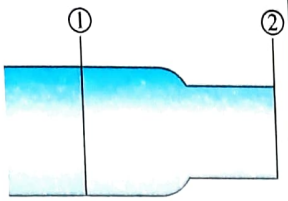
$$= \frac{2g\rho b}{2}\left[h^2 - (h-l)^2\right] = g\rho bl(2h-l)$$

CONCEPT APPLICATION EXERCISE 6.5

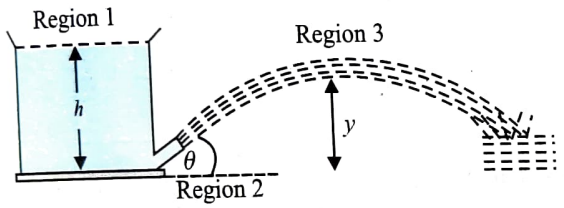
1. A non-viscous liquid of constant density 500 kg/m^3 flows in a variable cross-sectional tube. The area of cross section of the tube at two points P and Q at heights of 3 m and 6 m are 2×10^{-3} m^3 and 4×10^{-3} m^3, respectively. Find the work done per unit volume by the forces of gravity as the fluid flows from point P to Q.



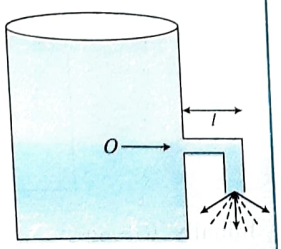
2. A horizontal pipe of varying cross-sectional area carries water flowing at a constant rate. The pressure at sections (1) and (2) are 50 cm and 20 cm of Hg. If the velocity of section (1) is 5 m/s, what is the velocity of section (2). Also determine the ratio of areas at sections (1) and (2) (as shown in figure).



3. The speed of flow past the lower surface of a wing of an aeroplane is 50 ms^{-1}. What speed of flow over the upper surface will give a dynamic lift of 1000 Pa? Density of air = 1.3 kg m^{-3}.
4. A gardener holds a pipe of inside cross-sectional area 3.60 cm^2 at a height of 1.50 m from ground. The opening at the nozzle of pipe has cross-sectional area 0.250 cm^2. Water velocity in the segment of the hose that lies on the ground is 50.0 cm/s.
   (a) What is the velocity of water leaving the nozzle?
   (b) What is the thrust force experienced by the gardener due to flow of water through nozzle?
   (c) What is the water pressure in the hose on the ground?
5. Figure shows a stream of fluid emerging from a tube in the base of an open tank. Obtain an expression for 'y' in terms of θ and 'h'. Can 'y' exceed 'h'?



6. If water flows horizontally through a pipe of varying cross section and the pressure of water equals 10 cm of Hg at a point where the velocity of flow is 40 cm/s, what is the pressure at another point where the velocity of flow is 50 cm/s?
7. A pipe is running full of water. At a certain point A it tapers from 60 cm diameter to 2.0 cm diameter at B. The pressure difference between A and B is 100 cm of water column. Find the rate of flow of water through the pipe.
8. A pitot tube is fixed in a water pipe of diameter 14 cm and difference of pressure by the gauge is 20 cm of water column. Calculate the rate of flow of water.
9. Water is used as the manometric liquid in a pitot tube mounted in an aircraft to measure air speed. If the maximum difference in height between the liquid columns is 0.1 m, what is the maximum air speed that can be measured? Density of air d_{air} = 1.3 kg/m^3.
10. Water flows out of a big tank along a tube bent at right angles; the inside of the tube is of radius r. The length of the horizontal portion is l. The rate of water flow is Q. Find the moment of reaction forces of flowing water, acting on the tube walls relative to point O.



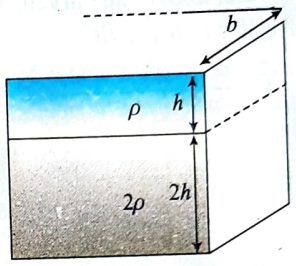
ANSWERS

1. 1.5×10^4 J/m^3 **2.** 41 : 20 **3.** 63.55 m/s
4. (a) 7.20 m/s (b) 1.3 N (c) 1.41×10^5 Pa
6. 9.97 cm of Hg **7.** 140 litre/s **8.** 30.8 litre/s
9. 38.83 m/s **10.** $Q^2\rho l/\pi r^2$

Solved Examples

EXAMPLE 6.1

Two immiscible liquids of densities ρ and 2ρ and thickness (heights) h and $2h$, respectively, push a vertical plate. Find the total side thrust given by the liquids on the vertical plate.



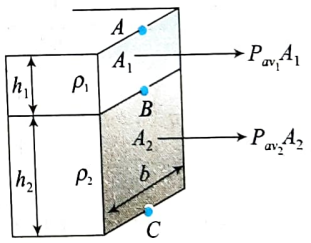
Sol. Let the hydrostatic forces acting on the rectangular patches of areas A_1 and A_2 of the gate be F_1 and F_2. Then, the total hydrostatic force on the gate is

$$F = F_1 + F_2$$

where $F_1 = P_{av_1} A_1$ and $F_2 = P_{av_2} A_2$.

P_{av_1} and P_{av_2} are the average pressures on the areas A_1 and A_2, respectively.

$$F = F_{av_1} A_1 + P_{av_2} A_2$$



Substituting $P_{av_1} = (P_A + P_B)/2, A_1 = bh_1 = bh, P_{av_1} = (P_B + P_C)/2,$

$A_2 = bh_2 = b(2h)$ we have

$$F = \frac{(P_A + P_B)}{2} bh + \frac{(P_B + P_C)}{2} 2bh$$

Substituting $P_A = 0, P_B = \rho gh$

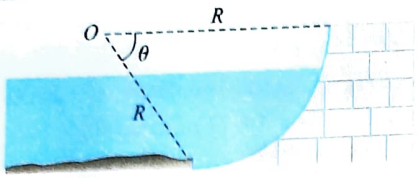
and $P_C = (\rho_1 h_1 + \rho_2 h_2) = [\rho h + (2\rho)(2h)]g = 5\rho gh$

we have $F = \frac{13}{2} \rho g h^2 b$

EXAMPLE 6.2

The cross section of a dam wall is an arc of a circle of radius R = 20 m subtending on angle of $\theta = 60°$ at the centre of the circle. The centre (O) of the circle lies in the water surface. The width of the dam [i.e., dimension perpendicular to the

figure] is $b = 10$ m. Neglect atmospheric pressure in following calculations.



(a) Calculate the vertical component of force (F_v) applied by water on the curved dam wall.

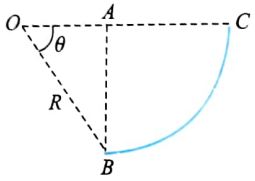
(b) Calculate the horizontal component of force (F_H) applied by water on the curved dam wall.

(c) Calculate the resultant force applied by the water on the curved dam wall.

Sol.

(a) F_V = Weight of water above the curved surface.

= Weight of water in part ABC



$= b$ (Area of sector BOC – Area of ΔBOA) ρg

$$= b\left[\frac{\pi R^2}{6} - \frac{1}{2}\frac{R}{2}\frac{\sqrt{3}R}{2}\right]\rho g = b\rho g R^2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$

$$= 10\times10^3\times10\times20^2\left[\frac{3.14}{6} - \frac{1.732}{8}\right] = 1.2\times10^7\ \text{N}$$

(b) F_H = Force on projection of curved surface on to a vertical plane

= (Average pressure) × Area $= \left(\frac{1}{2}\rho g h\right)(bh)$

$$= \left(\frac{1}{2}\rho g\frac{\sqrt{3}}{2}R\right)\left(b\frac{\sqrt{3}}{2}R\right) \qquad \left[\because h = AB = \frac{\sqrt{3}}{2}R\right]$$

$$= \frac{3}{8}\times10^3\times10\times10\times(20)^2 = 1.5\times10^7\ \text{N}$$

(c) $F = \sqrt{F_H^2 + F_V^2} = 1.92\times10^7$ N

EXAMPLE 6.3

An ice cube of size $l = 50$ cm is floating in a tank (base area $A = 2$ m × 2 m) partially filled with water. Density of water is $\rho_w = 1000$ kg m^{-3} and that of ice is $\rho_{ice} = 900$ kg m^{-3}. Calculate increase in gravitational potential energy when ice melts completely. ($g = 10$ ms^{-2})

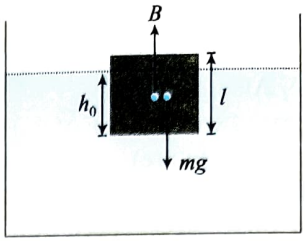
Sol. As the density of the ice (ρ_{ice}) is less than the density of the water (ρ_{water}), hence the ice cube float partially immersed.

In case of floatation,

Buoyancy force = weight of cube

$$B = mg \text{ or } l^2 h_0 \rho_{water} g = l^3 \rho_{ice} g$$

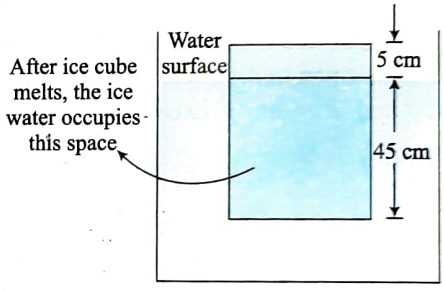
$$\Rightarrow \quad h_0 = l\times\frac{\rho_{ice}}{\rho_{water}} = 50\times\frac{900}{1000} = 45\,\text{cm}$$



Centre of mass of ice cube is at a depth of 10 cm from its upper surface. But the upper surface is at a height of 2 cm from water surface. Therefore depth of the centre of the mass of ice cube from free water surface is $h_1 = 25 - 5 = 20$ cm from free water surface.

When ice melts, water is formed and that water occupies the space which was displaced by the cube. Therefore depth of centre of mass of the water body formed after the melting of ice cube is

$$h_2 = \frac{45}{2} = 22.5\,\text{cm}$$



It means that the centre of mass move down through distance

$$\Delta h = h_2 - h_1 = 22.5 - 20 = 2.5\,\text{cm}$$

It means gravitational potential energy of the system will decrease. Decrease in gravitational potential energy of the system

$$\Delta U = mg\Delta h = (0.5)^3\times1000\times10\times\frac{2.5}{1000} = 31.25\ \text{J}$$

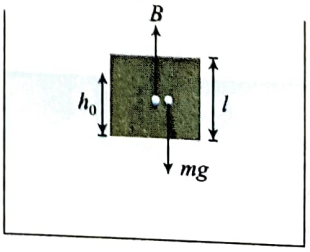
Since, level of free water surface does not rise or fall, hence size of tank is of no concern.

EXAMPLE 6.4

A cubical wooden block has side $a = 30$ cm. It is floating in a rectangular tank partially filled with water and having base area $A = 45$ cm × 60 cm. Calculate the work done to press the block slowly so that it is just immersed in water.

($\rho_{block} = 500$ kg m^{-3}, $\rho_{water} = 1000$ kg m^{-3})

Sol. As the density of wood is less than the density of water, it means the block will float partially immersed.



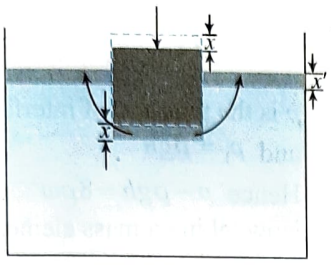
Initially $B = mg$

$$l^2 h\rho_{water} g = l^3 \rho_{block} g$$

$$\Rightarrow h_0 = l \times \frac{\rho_{block}}{\rho_{water}} = l \times \frac{500}{1000} = \frac{l}{2}$$

When the block is pressed down, it further displaces water. Therefore, upthrust exerted by water increases. Hence the external force applied to press the block is variable. This external force acts work to press the block down.

Let at an instant external force applied on block be F and downward displacement of block from its equilibrium position be x relative to ground.



Due to downward displacement of block, water level in the tank rises. To calculate this rise x', we may consider that additional water. Lower shaded portion in figure displaced by the block occupies upper shaded position.

Volume of upper shaded position = volume of lower shaded part

$$(A - l^2)x' = l^2 . x$$

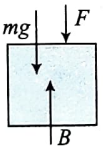
$$\left(1 \times 1 - \frac{1}{2} \times \frac{1}{2}\right)x' = \frac{1}{2} \times \frac{1}{2} \times x \Rightarrow x' = \frac{x}{3}$$

$$h = \frac{l}{2} + x + x' = \frac{l}{2} + x + \frac{x}{3} = \frac{9}{2} + \frac{4x}{3}$$

and finally when $h = l$, then from (i)

$$l = \frac{l}{2} + \frac{4x}{3} \Rightarrow x = \frac{3}{8}l$$

At any time the block is always in equilibrium



$$F + mg = B$$

$$F + l^3 \rho_{block} g = l^2 \rho_{water} gh$$

$$F = l^2 \rho_{water} g\left(\frac{l}{2} + \frac{4}{3}x\right) - l^3 \rho_{block} g$$

Given, $\rho_{water} = 2\rho_{block}$

$$F = l^2 \rho_{water} g\left(\frac{l}{2} + \frac{4}{3}x\right) - l^3 \frac{\rho_{water}}{2} g$$

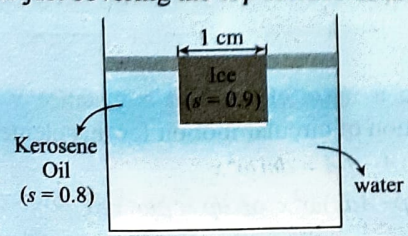
$$\Rightarrow F = \frac{4}{3}\rho_{water} g l^2 x \quad \ldots(ii)$$

$$W = \frac{4}{3}\rho_{water} g l^2 \left[\frac{x^2}{2}\right]_0^{\frac{3}{8}l} = \frac{3}{32}\rho_{water} g l^4$$

$$= \frac{3}{32} \times 100 \times 10 \times \frac{1}{(2)^4} = 14.65\,\text{J}$$

EXAMPLE 6.5

An ice cube of side 1 cm is floating at the interface of kerosene and water in a beaker of base area 10 cm^2. The level of kerosene is just covering the top surface of the ice cube.



(a) Find the depth of submergence in the kerosene and that in the water.

(b) Find the change in the total level of the liquid when the whole ice melts into water.

Sol.

(a) According to the condition of floating,

$$0.8\rho_w g h_k + \rho_w g h_w = 0.9\rho_w gh \quad \ldots(i)$$

where h_k and h_w are the submerged depth of the ice in the kerosene and water, respectively.

$$h_k + h_w = h \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get $h_k = 0.5\,\text{cm}$, $h_w = 0.5\,\text{cm}$.

(b) $1\,\text{cm}^3$ ice $\xrightarrow{\text{melts}}$ $0.9\,\text{cm}^3$ water

After melting the ice, $0.5\,\text{cm}^3$ of volume will be occupied by the kerosene. Therefore, the fall in the level of kerosene is

$$\Delta h_K = \frac{0.5}{A}$$

Out of $0.9\,\text{cm}^3$ of water formed, $0.5\,\text{cm}^3$ of water will be filled in the vacant part of ice and remaining $(0.9 - 0.5) = 0.4\,\text{cm}^3$ of water will be available to increase the level of water.

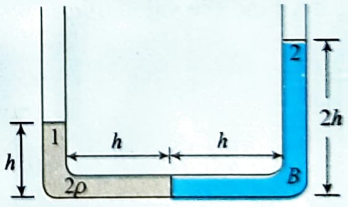
Rise in the level of water is $\Delta h_w = \frac{0.9 - 0.5}{A} = \frac{0.4}{A}$

Hence, the net fall in the overall level is

$$\Delta h = \frac{0.1}{A} = \frac{0.1}{10} = 0.01\,\text{cm} = 0.1\,\text{mm}$$

EXAMPLE 6.6

A vertical U-tube has two liquids 1 and 2. The heights of liquids columns in both the limbs are h and $2h$, as shown in the figure. If the density of the liquid 1 is 2ρ.



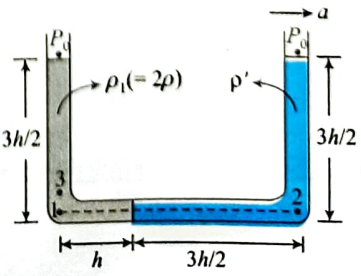
(a) Find the density of liquid 2.

(b) If we accelerate the tube towards right till the heights of liquid columns will be the same, find the acceleration of the tube.

Sol.

(a) First of all, we take two points 1 and 2 at the bottom of the left and right vertical limbs. The pressure at 1 is

$$P_1 = P_0 + (2\rho)gh$$



The pressure at 2 is $P_2 = P_0 + \rho' g\,(2h)$

where ρ' = density of the other liquid

Since $P_1 = P_2$ for stationary liquid, we have

$$P_0 + 2\rho gh = P_0 + \rho' g(2h)$$

This gives $\rho' = \rho$.

(b) If the levels in both the limbs are equal, conserving the volume of the total (two) liquids, the height of each liquid column will be equal to $3h/2$. The lengths of liquids in the horizontal limbs are $h/2$ and $3h/2$, respectively.

The pressure at interface 3 of the liquid is

$$P_3 = P_1 - (2\rho)a\left(\frac{h}{2}\right)$$

where $P_3 = P_0 + (2\rho)a\left(\frac{3h}{2}\right)$

This gives $P_3 = P_0 + (2\rho)g\left(\frac{3h}{2}\right) - (2\rho)a\left(\frac{h}{2}\right)$

$$= P_0 + 3\rho gh - \rho ah \quad \ldots(i)$$

P_3 can also be given as $P_3 = P_2 + \rho a\left(\frac{3h}{2}\right)$

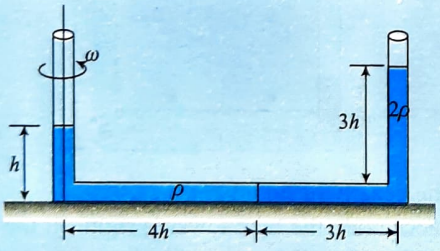
where $P_2 = P_0 + \rho a\left(\frac{3h}{2}\right)$

This gives $P_3 = P_0 + \frac{3}{2}\rho gh + \frac{3}{2}\rho ah \quad \ldots(ii)$

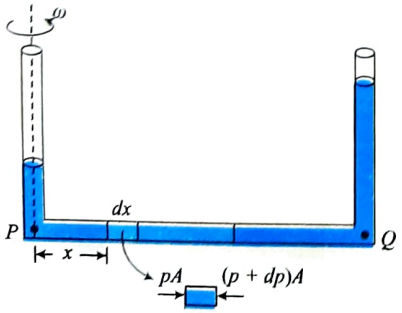
Eliminating P_3 from Eqs. (i) and (ii), we have $a = \frac{3}{5}g$

EXAMPLE 6.7

A vertical U-tube is spinned about the vertical axis passing through the left hand limb, with an angular speed ω such that the liquid columns in the limbs have heights h and $3h$, as shown in the figure. Find the value of ω.



Sol. Let us take a mass element (dm) at a distance x from axis of rotation, in liquid of density ρ. This element is rotating in a circular path of radius x.



Writing equation of motion for this element

$$(p + dp)A - pA = dm\omega^2 x$$

or $dpA = (\rho A dx)\omega^2 x \quad \ldots(i)$

Here A is area of cross section of the tube. Now integrating Eq. (i) both sides with proper limits.

$$\int_{p_1}^{p} dp = \rho\omega^2 \int_0^{4h} x\,dx$$

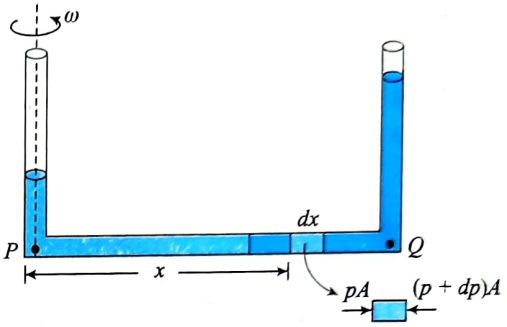
$$p - p_1 = \rho\omega^2\left[\frac{x^2}{2}\right]_0^{4h} = 8\rho\omega^2 h^2$$

p is the pressure of interface of two liquids.

and $p_1 = \rho gh$

Hence $p - \rho gh = 8\rho\omega^2 \quad \ldots(ii)$

Now taking a mass element in the liquid of density 2ρ.



Equation of circular motion for this element

$$(p + dA)A - pA = dm\omega^2 x$$

or $dp \cdot A = (2\rho A\,dx)\omega^2 x$

Now integrating above equation with proper limits

$$\int_{p}^{p_2} dp = 2\rho\omega^2 \int_{4h}^{h} x\,dx$$

or $p_2 - p = 33\rho\omega^2$

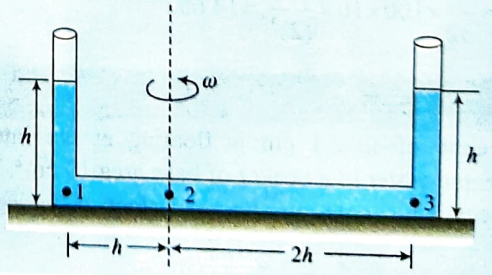
and here $p_2 = (2\rho)g(3h) = 6\rho gh$

Hence $6\rho gh - p = 33\rho\omega^2 \quad \ldots(iii)$

From (ii) and (iii) we get $\omega = \sqrt{\frac{5g}{41h}}$

EXAMPLE 6.8

A vertical U-tube has a liquid up to a height h. If the tube is slowly rotated to an angular speed $\omega = \sqrt{g/h}$, find the **(i)** heights of the liquid column, **(ii)** pressures at the points 1, 2 and 3 in the limbs in steady state.

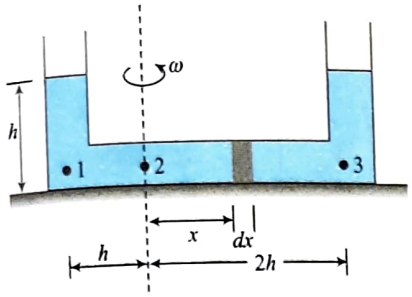


Sol. Taking a mass element at a distance x from axis of rotation. Equation of circular motion for this element

$$(p + dp)A - pA = dm\omega^2 x$$

$$dpA = (\rho \cdot A dx)\omega^2 x \text{ or } dp = \rho\omega^2 x\,dx$$

Integrating both sides, we get $\int_{p_1}^{p_3} dp = \rho\omega^2 \int_{-h}^{2h} x\,dx$



$$p_3 - p_1 = \rho\omega^2\left[\frac{x^2}{2}\right] = \frac{\rho\omega^2}{2}[4h^2 - h^2]$$

$$\rho g(h_3 - h_1) = \frac{3\rho\omega^2 h^2}{2} = \frac{3\rho}{2}h^2\left(\frac{g}{h}\right)$$

$$(h_3 - h_1) = \frac{3}{2}h$$

But $h_3 + h_1 = 2h$

After solving $h_3 = \frac{7h}{4}$ and $h_1 = \frac{h}{4}$

$$p_3 = p_0 + \rho g\left(\frac{7h}{4}\right) \text{ and } p_1 = p_0 + \rho g\left(\frac{h}{4}\right)$$

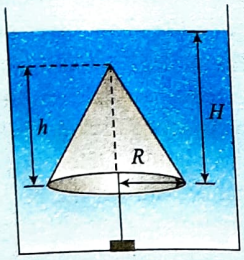
and $$p_3 = p_0 = \frac{\rho\omega^2}{2}[x^2]_0^{2h} = \frac{\rho\omega^2}{2}(4h)^2$$

$$p_3 - p_2 = \frac{\rho}{2}\left(\frac{g}{h}\right)\cdot 4h^2 = 2\rho gh$$

$$p_0 + \rho g\left(\frac{7h}{4}\right) - p_2 = 2\rho gh \quad \Rightarrow p_2 = p_0 - \frac{\rho gh}{4}$$

EXAMPLE 6.9

A solid wooden cone has been supported by a string inside water as shown in the figure. The radius of the circular base of the cone is R and its height is h. In equilibrium the base of the cone is at a depth H below the water surface. Density of wood is d ($< \rho$, density of water).



(a) Find tension in the string.

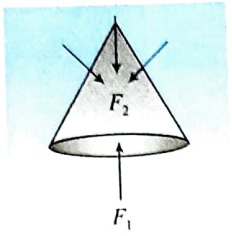
(b) Find the force applied by the water on the slant surface of the cone. Take atmospheric pressure to be P_0.

Sol.

(a) Buoyancy force is $F_B = V_{cone}\rho g$

Weight of the cone $W = V_{cone}dg$

where $V_{cone} = \frac{1}{3}\pi R^2 h$



$\therefore$ Tension, $T = W - F_B = \frac{1}{3}\pi R^2 h g(d - \rho)$

(b) $F_B = \frac{1}{3}\pi R^2 h\rho g$

Buoyancy is resultant of vertically upward force (F_1) applied by water pressure on the base of the cone and the vertically downward force (F_2) applied by water on the slant surface. The horizontal component of sums up to zero due to symmetry.

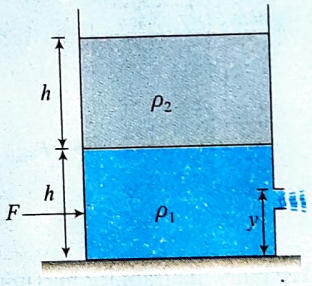
$\therefore$ $F_1 - F_2 = F_B$

$$\pi R^2[\rho_0 + \rho gH] - F_2 = \frac{1}{3}\pi R^2 h\rho g$$

$\therefore$ $$F_2 = \pi R^2[\rho_0 + \rho gH] - \frac{1}{3}\pi R^2 h\rho g$$

EXAMPLE 6.10

A cylindrical vessel having cross-sectional area $A = 0.5\ m^2$ is filled with two liquids of density $\rho_1 = 500\ kg\ m^{-3}$ and $\rho_2 = 1000\ kg\ ms^{-3}$, to a height $h = 50$ cm each as shown in figure.



A small hole having area $a = 5\ cm^2$ is made in right vertical wall at a height $y = 10$ cm from the bottom. Calculate:

(i) Velocity of efflux.

(ii) Horizontal force F to keep the cylinder in static equilibrium, if it is placed on a smooth horizontal plane.

(iii) Minimum and maximum values of F to keep the cylinder in static equilibrium, if coefficient of friction between the cylinder and the plane is $\mu = 0.02$. ($g = 10\ ms^{-2}$).

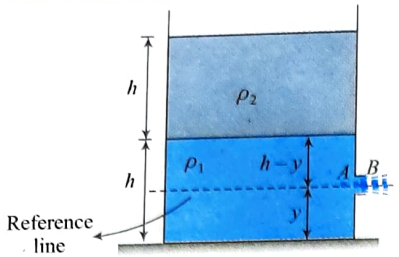
Sol. Since area a of hole is very small in comparison to base area A of the cylinder, therefore, velocity of liquid inside the cylinder is negligible.

(i) Let velocity of efflux be v and atmospheric pressure P_0.

Consider two points A (inside the cylinder) and B (just outside the hole) in the same horizontal line as shown in figure.

Pressure at A, $P_A = P_0 + h\rho_2 g + (h - y)\rho_1 g$

Pressure at B, $P_B = P_0$



According to Bernoulli's theorem,
Pressure energy at A = Pressure energy at B + Kinetic energy at B

$$P_A = P_B + \frac{1}{2}\rho_1 v^2 \Rightarrow v = 3\,\text{ms}^{-1}$$

(ii) When cylinder is on smooth horizontal plane, force F required to keep cylinder stationary equals horizontal thrust exerted by water jet. But that is equal to mass flowing per second × change in velocity of this mass.

$$F_{\text{thrust}} = v_{\text{rel}}\frac{dm}{dt} = (v-0)\times(av\rho) = a\rho v^2$$

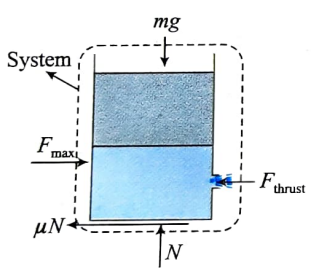
Hence $F_{\text{thrust}} = 5\times10^{-4}\times1000\times(3)^2 = 4.5\,\text{N}$

(iii) Total mass of the liquid in the cylinder is,

$$m = Ah\rho_1 + Ah\rho_2 = 750\,\text{kg}$$

Limiting friction = μmg = 150 N

F < Limiting friction, therefore, minimum force required is zero.

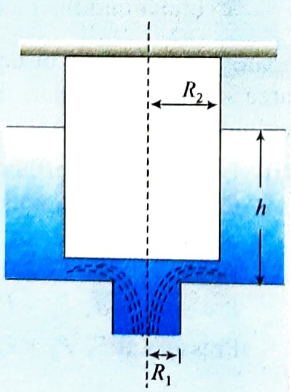


Considering free body diagram for maximum value of force.
Considering vertical force, $N = mg$
Now considering horizontal force, $F_{\text{max}} = F_{\text{thrust}} + \mu N$

$$F_{\text{max}} = 4.5 + 150 = 154.5\,\text{N}$$

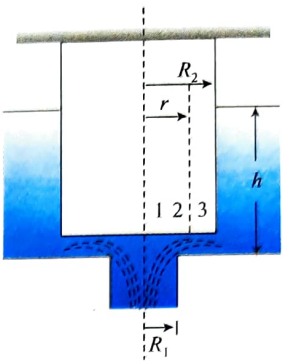
EXAMPLE 6.11

The horizontal bottom of a wide vessel with an ideal fluid has a round orifice of radius R_1 over which a round closed cylinder is mounted, whose radius $R_2 > R_1$ figure. The clearance between the cylinder and the bottom of the vessel is very small, the fluid density is ρ. Find the static pressure of the fluid in the clearance as a function of the distance r from the axis of the orifice (and the cylinder), if the height of the fluid is equal to h.



Sol. In the figure shown, we consider the three section 1,2 and 3 at radius R_1, r and R_2 respectively. Let D be the width of the gap. Applying continuity equation, we have

$$2\pi R_1 D v_1 = 2\pi R_2 D v_2 = 2\pi r D v$$



where v is the water flow velocity at a distance r from axis. Here, we get

$$v_1 R_1 = v_2 R_2 = v r \qquad \text{...(i)}$$

Applying Bernoulli's equation between these points, we have

$$P + \frac{1}{2}\rho v^2 = P_a + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

where P_a = atmospheric pressure

Now $$P + \frac{1}{2}\rho v^2 = P_a + \frac{1}{2}\rho v_1^2$$

or $$P = P_a + \frac{1}{2}\rho v_1^2 - \frac{1}{2}\rho v^2$$

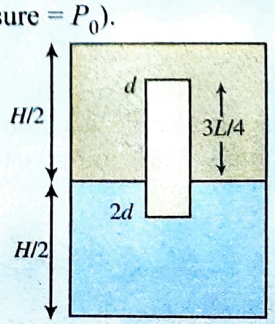
or $$P = P_a + \frac{1}{2}\rho v_1^2\left[1-\left(\frac{v}{v_1}\right)^2\right]$$

or $$P = P_a + \frac{1}{2}\rho v_1^2\left[1-\left(\frac{R_1}{r}\right)^2\right] \quad \left[\text{As } \frac{v}{v_1} = \frac{R_1}{r}\right]$$

$$P = P_a + \rho g h\left[1-\frac{R_1^2}{r^2}\right] \quad \left[\text{As } v_1 = \sqrt{2gh}\right] \qquad \text{...(ii)}$$

EXAMPLE 6.12

A container of a large uniform cross-sectional area A resting on horizontal surface holds two immiscible, non-viscous and incompressible liquids of densities d and $2d$, each of height $H/2$ as shown in the figure. The lower density liquid is open to atmosphere. A homogeneous solid cylinder of length L ($L < H/2$), cross-sectional area $A/5$ is immersed such that it floats with its axis vertical of the liquid–liquid interface with length $L/4$ in the denser liquid. Determine **(a)** density D of the solid and **(b)** the total pressure at the bottom of the container (atmospheric pressure = P_0).



Sol.

(a) For equilibrium of the cylinder,

Weight of the cylinder = Weight of the liquid displaced

$$\Rightarrow L\left(\frac{A}{5}\right)Dg = \frac{3L}{4}\left(\frac{A}{5}\right)dg + \frac{L}{4}\left(\frac{A}{5}\right)2dg$$

This gives $D = \frac{3}{4}d + \frac{d}{2}$ or $D = \frac{5}{4}d$

(b) Total pressure at the bottom of cylinder is

$$P = \frac{\text{weight of liquids} + \text{weight of cylinder}}{A} + P_0$$

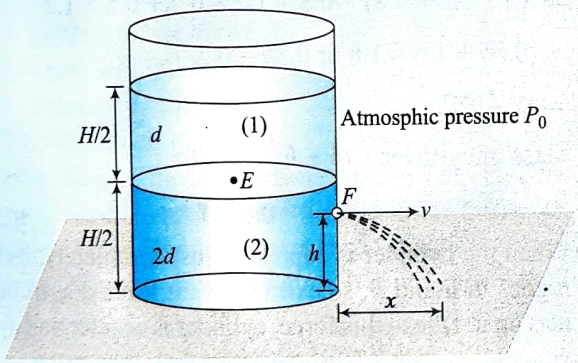
$$= \frac{\left[Ad\left(\frac{H}{2}\right)g + A2d\left(\frac{H}{2}\right)g\right] + \frac{A}{5}\left(\frac{5}{4}d\right)Lg}{A} + P_0$$

$$= \left(\frac{6H + L}{4}\right)dg + P_0$$

Here we are asked to find the total pressure at the bottom of the container and not only the hydrostatic pressure; hence, we need to consider the total weight of liquid and cylinder in the container.

EXAMPLE 6.13

A container of large uniform cross-sectional area A, resting on horizontal surface, holds two immiscible non-viscous and incompressible liquids of density d and $2d$, each of height $H/2$ as shown in the figure. The lower density liquid is open to the atmosphere having pressure P_0. A tiny hole of area $s (s \ll A)$ is punched on the vertical side of the container at a height h ($h < H/2$). Determine



(a) the initial speed of efflux of the liquid at the hole

(b) the horizontal distance x travelled by the liquid initially

(c) the height h_m at which the hole should be punched so that the liquid travels the maximum distance x_m initially. Also calculate x_m (neglect air resistance in the calculations)

Sol.

(a) **Method 1:** Applying Bernoulli's theorem at points E (a point on the interface of the liquids and on the top surface of the liquid of density $2d$) and F, i.e., pressure energy + kinetic energy + potential energy = constant

$$\Rightarrow P_0 + \frac{H}{2}dg + \frac{1}{2}(2d)\times 0 + \frac{H}{2}2dg = P_0 + \frac{1}{2}(2d)v^2 + 2hdg$$

$$\Rightarrow \frac{3}{2}Hdg = dv^2 + 2hdg$$

$$\Rightarrow v^2 = \frac{3}{2}Hg - 2hg = \frac{(3H-4h)g}{2}$$

$$\therefore v = \sqrt{\frac{(3H-4h)}{2}g}$$

Method 2: Let the equivalent depth of liquid (1) of density d in terms of density $2d$ which gives the same pressure as liquid (1) at interface, i.e.,

$$\frac{H}{2}dg = H'(2d)g \text{ which gives } H' = \frac{H}{4}$$

Hence, depth of the liquid above opening (in terms of depth of liquid of density $2d$),

$$H' + \left(\frac{H}{2} - h\right) = \frac{H}{4} + \left(\frac{H}{2} - h\right)$$

$$H_{net} = \left(\frac{3H-4h}{4}\right)$$

And velocity of efflux is given by

$$v = \sqrt{2g\,H_{net}}$$

$$v = \sqrt{\left(\frac{3H-4h}{2}\right)g}$$

(b) Time taken by the liquid to fall through a vertical height h is given by

$$H = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$$

Therefore, horizontal distance traversed by liquid is

$$x = vt = \sqrt{\frac{(3H-4h)g}{2}} \times \sqrt{\frac{2h}{g}}$$

$$= \sqrt{(3H-4h)h}$$

(c) For the maximum horizontal distance x,

$$\frac{dx}{dh} = 0 \Rightarrow \frac{d}{dh}(3Hh - 4h^2)^{1/2} = 0$$

or $$\frac{1}{2}(3Hh - 4h^2)^{-1/2} \times (3H - 8h) = 0$$

This gives $3H - 8h = 0 \Rightarrow h = \frac{3H}{8}$

This is the length of the liquid for which the range is maximum, i.e.,

$$h \to h_m = \frac{3H}{8}$$

The maximum horizontal distance is given by

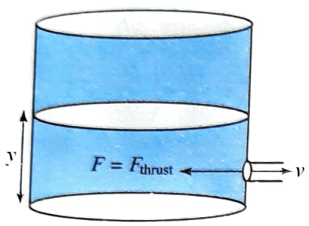
$$x_m = \sqrt{(3H - 4h_m)h_m}$$

$$= \sqrt{\left\{\left(3H - 4\frac{3H}{8}\right)\frac{3H}{8}\right\}} = \frac{3}{4}H$$

EXAMPLE 6.14

A large open top container of negligible mass and uniform cross-sectional area A has a small hole of cross-sectional area $A/100$ in its side wall near the bottom. The container is kept on a smooth horizontal floor and contains a liquid of density ρ and mass M_0. Assuming that the liquid starts flowing out horizontally through the hole at $t = 0$, calculate **(a)** the acceleration of the container and **(b)** its velocity when 75% of the liquid has drained out.

Sol. Liquid is escaping from the container. Let y be the height of the liquid at any instant t after start.



(a) Then velocity of escape is given by $v = \sqrt{2gy}$

Mass of the liquid flowing in time $dt = av\rho dt$, where a = area of the hole $= A/100$.

Force is the rate of change of momentum. Hence,

$$F = \frac{dp}{dt} = \left(\frac{av\rho dt}{dt}\right)v = av^2\rho = a\,(2gy)\,\rho$$

As F = mass of container × acceleration = Ma'

$$\therefore \quad a' = \frac{F}{M}$$

But M = mass of container = $Ay\rho$

$$\therefore \quad a' = \frac{a(2gy)\rho}{M} = \frac{\left(\frac{A}{100}\right)2gy\rho}{Ay\rho} = \frac{g}{50}$$

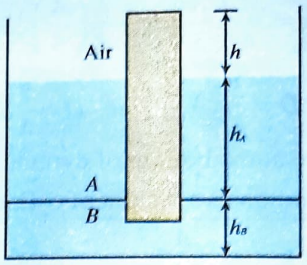
(b) When 75% of the liquid has drained out, the height of the liquid y is given by

$$Ay\rho = \frac{25}{100}M_0 \Rightarrow y = \frac{M_0}{4A\rho}$$

$$\text{But} \quad v = \sqrt{2gy} = \sqrt{2g\left(\frac{M_0}{4A\rho}\right)} = \sqrt{\frac{M_0 g}{2A\rho}}$$

EXAMPLE 6.15

A uniform solid cylinder of density 0.8 g/cm³ floats in equilibrium in a combination of two non-mixing liquids A and B with its axis vertical. The densities of liquids A and B are 0.7 g/cm³ and 1.2 g/cm³, respectively. The height of liquid A is $h_A = 1.2$ cm. The length of the part of the cylinder with liquid B is $h_B = 0.8$ cm.



(a) Find the total force exerted by liquid A on the cylinder.

(b) Find h, the length of part of the cylinder in air.

(c) The cylinder is depressed in such a way that its top surface is just below the upper surface of liquid A and is then released. Find the acceleration of the cylinder immediately just after it is released.

Sol.

(a) Liquid A exerts horizontal forces on all area elements from all directions. By symmetry, the resultant of these horizontal forces is zero. So the total force exerted by liquid A on the cylinder is zero.

(b) Let A be the cross-sectional area of the cylinder.

For floating equilibrium,

Weight of floating cylinder = Buoyant force = weight of liquid displaced.

$$(h + h_A + h_B) \times A \times 0.8g = (h_A A\rho_A g + h_B A\rho_B g)$$

or $(h + h_A + h_B) \times 0.8 = h_A\rho_A + h_B\rho_B$

$\Rightarrow (h + 1.2 + 0.8) \times 0.8 = 1.2 \times 0.7 + 0.8 \times 1.2$

$\Rightarrow 0.8h + 1.6 = 1.8$ or $0.8h = 0.2$

$\Rightarrow h = 0.25$ cm

(c) Mass of cylinder $= (h + h_A + h_B)A\rho$

$= (0.2 + 1.2 + 0.8)A \times 0.8 = 1.8A$

When the cylinder is depressed just completely, its height h goes in liquid B, so extra buoyant force due to liquid B acts on it. Due to this force, cylinder is accelerated upward.

Extra buoyant force $F = (h_A)\rho_B g$

$$= 0.25A \times 1.2g = 0.3Ag$$

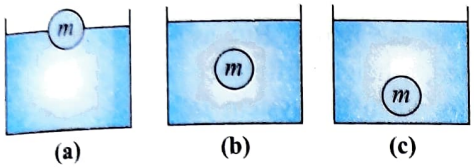
Acceleration is given by $a = \frac{F}{m} = \frac{0.3Ag}{1.8A} = \frac{g}{6}$

$$= \frac{10}{6} = \frac{5}{3}\ \text{m/s}^2$$

Exercises

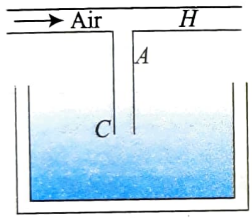
Single Correct Answer Type

1. Three identical vessels A, B and C contain same quantity of liquid. In each vessel balls of different densities but same masses are placed. In vessel A, the ball is partly submerged; in vessel B, the ball is completely submerged but floating and in vessel C, the ball has sunk to the base. If F_A, F_B and F_C are the total forces acting on the base of vessels A, B and C, respectively, then



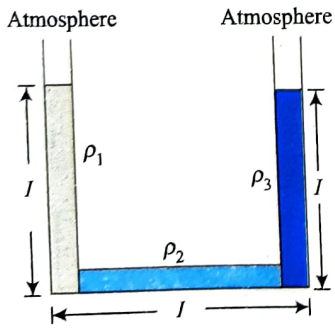
(1) $F_A = F_B = F_C$ (2) $F_A < F_B < F_C$
(3) $F_A = F_B < F_C$ (4) $F_A < F_C < F_B$

2. Figure shows a capillary tube C dipped in a liquid that wets it. The liquid rises to a point A. If we blow air through the horizontal tube H, what will happen to the liquid column in the capillary tube?



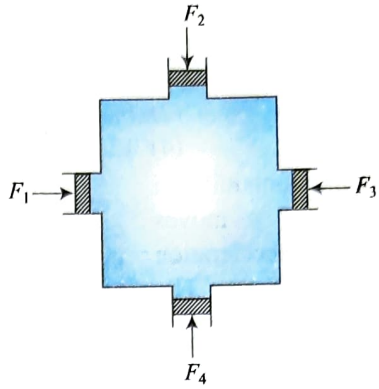
(1) Level will rise above A
(2) Level will fall below A
(3) Level will remain at A
(4) It is difficult to predict

3. Three liquids having densities ρ_1, ρ_2 and ρ_3 are filled in a U-tube. Length of each liquid column is equal to I, $\rho_1 > \rho_2 > \rho_3$ and liquids remain at rest (relative to the tube) in the position shown in the figure. It is possible that



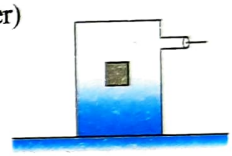
(1) U-tube is accelerating leftwards
(2) U-tube is accelerating upwards with acceleration g
(3) U-tube is moving with a constant velocity
(4) none of the above

4. Figure shows water filled in a symmetrical container. Four pistons of equal area A are used at the four openings to keep the water in equilibrium. Now an additional force F is applied at each piston. The increase in the pressure at the centre of the container due to this addition is



(1) $\frac{F}{A}$ (2) $\frac{2F}{A}$
(3) $\frac{4F}{A}$ (4) 0

5. A block of wood is floating in water in a closed vessel as shown in the figure. The vessel is connected to an air pump. When more air is pushed into the vessel, the block of wood floats with (neglect compressibility of water)



(1) larger part in the water
(2) smaller part in the water
(3) same part in the water
(4) at some instant it will sink

6. A wooden rod of a uniform cross section and of length 120 cm is hinged at the bottom of the tank which is filled with water to a height of 40 cm. In the equilibrium position, the rod makes an angle of 60° with the vertical. The centre of buoyancy is located on the rod at a distance (from the hinge) of
(1) 20 cm (2) 40 cm
(3) 60 cm (4) 75 cm

7. An iceberg is floating partially immersed in sea water. The density of sea water is 1.03 g cm^{-3} and that of ice is 0.92 g cm^{-3}. The approximate percentage of total volume of iceberg above the level of sea water is
(1) 8 (2) 11
(3) 34 (4) 89

8. Equal masses of three liquids are kept in three identical cylindrical vessels A, B and C. Their densities are ρ_A, ρ_B and ρ_C with $\rho_A < \rho_B < \rho_C$. The force on the base will be
(1) maximum in vessel A
(2) maximum in vessel B
(3) maximum in vessel C
(4) same in all the three vessels

9. A lawn sprinkler has 20 holes, each of cross-sectional area 2.0×10^{-2} cm^2, and is connected to a hose pipe of cross-sectional area 2.4 cm^2. If the speed of water in the hose pipe is 1.5 m s^{-1}, the speed of water as it emerges from the holes is
(1) 2.25 m s^{-1} (2) 4.5 m s^{-1}
(3) 9 m s^{-1} (4) 18 m s^{-1}

10. A block of silver of mass 4 kg hanging from a string is immersed in a liquid of relative density 0.72. If relative density of silver is 10, then tension in the string will be

(1) 37.12 N (2) 42 N

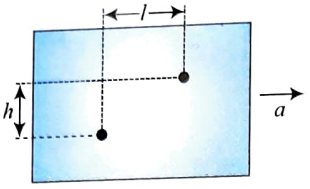
(3) 73 N (4) 21 N

11. A vessel contains oil (density = 0.8 g/cm^3) over mercury (density = 13.6 g/cm^3). A uniform sphere floats with half its volume immersed in mercury and the other half in oil. The density of the material of sphere in g/cm^3 is

(1) 3.3 (2) 6.4

(3) 7.2 (4) 12.8

12. A sealed tank containing a liquid of density ρ moves with horizontal acceleration a as shown in the figure. The difference in pressure between two points A and B will be



(1) $h\rho g$ (2) $l\rho g$

(3) $h\rho g - l\rho a$ (4) $h\rho g + l\rho a$

13. An empty balloon weighs W_1. If air equal in weight to W is pumped into the balloon, the weight of the balloon becomes W_2. Suppose that the density of air inside and outside the balloon is the same. Then

(1) $W_2 = W_1 + W$ (2) $W_2 = \sqrt{W_1 W}$

(3) $W_2 = W_1$ (4) $W_2 = W_1 - W$

14. A wooden block is floating in a liquid. About 50% of its volume is inside the liquid when the vessel is stationary. Percentage volume immersed when the vessel moves upwards with acceleration $a = g/2$ is

(1) 75% (2) 25%

(3) 50% (4) 33.33%

15. A cylindrical block is floating (partially submerged) in a vessel containing water. Initially, the platform on which the vessel is mounted is at rest. Now the platform along with the vessel is allowed to fall freely under gravity. As a result, the buoyancy force

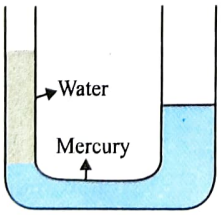
(1) becomes zero

(2) decreases

(3) increases

(4) information is insufficient

16. In a U-tube in which the cross-sectional area of the limb on the left is one-quarter, the limb on the right contains mercury (density 13.6 g/cm^3), and the level of mercury in the narrow limb is at a distance of 36 cm from the upper end of the tube. What will be the rise in the level of mercury in the right limb if the left limb is filled to the top with water?



(1) 1.2 cm (2) 2.35 cm

(3) 0.56 cm (4) 0.8 cm

17. We have two different liquids A and B whose relative densities are 0.75 and 1.0, respectively. If we dip solid objects P and Q having relative densities 0.6 and 0.9 in these liquids, then

(1) P floats in A and Q sinks in B

(2) P sinks in A and Q floats in B

(3) P floats in B and Q sinks in A

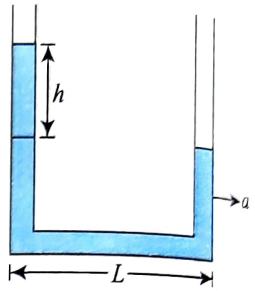
(4) P sinks in B and Q floats in A

18. A cubical block of wood of side a and density ρ floats in water of density 2ρ. The lower surface of the cube just touches the free end of a massless spring of force constant K fixed at the bottom of the vessel. The weight W put over the block so that it is completely immersed in water without wetting the weight is

(1) $a(a^2\rho g + K)$ (2) $a(a\rho g + 2K)$

(3) $a\left(\frac{a\rho g}{2} + 2K\right)$ (4) $a\left(a^2\rho g + \frac{K}{2}\right)$

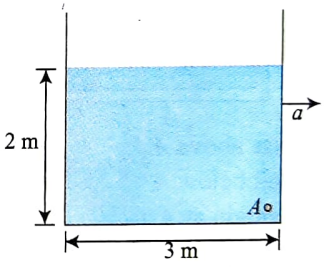
19. When at rest, a liquid stands at the same level in the tubes as shown in the figure. But as indicated, a height difference h occurs when the system is given an acceleration a towards the right. Then h is equal to



(1) $\frac{aL}{2g}$ (2) $\frac{gL}{2a}$

(3) $\frac{gL}{a}$ (4) $\frac{aL}{g}$

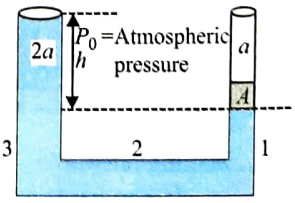
20. The minimum horizontal acceleration of the container so that the pressure at point A of the container becomes atmospheric is (the tank is of sufficient height)



(1) $\frac{3}{2}g$ (2) $\frac{4}{3}g$

(3) $\frac{4}{2}g$ (4) $\frac{3}{4}g$

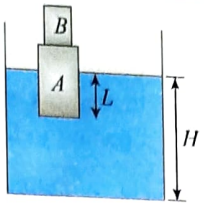
21. In the figure, the cross-sectional area of the smaller tube is a and that of the larger tube is $2a$. A block of mass m is kept in the smaller tube having the same base area a, as that of the tube. The difference between water levels of the two tubes is



(1) $\frac{P_0}{\rho g} + \frac{m}{a\rho}$ (2) $\frac{P_0}{\rho g} + \frac{m}{2a\rho}$

(3) $\frac{m}{a\rho}$ (4) $\frac{m}{2\rho a}$

22. Two wooden blocks A and B float in a liquid of density ρ_L as shown. The distance L and H are shown. After some time, block B falls into the liquid, so that L decreases and H increases. If density of block B is ρ_B, find the correct option.



(1) $\rho_L = \rho_B$ (2) $\rho_L > \rho_B$

(3) $\rho_L < \rho_B$ (4) Unpredictable

23. Water from a tap emerges vertically downwards with an initial velocity v_0. Assuming pressure is constant throughout the stream of water and the flow is steady, find the distance from the tap at which cross-sectional area of stream is half of the cross-sectional area of stream at the tap.

(1) $v_0^2/2g$ (2) $3v_0^2/2g$

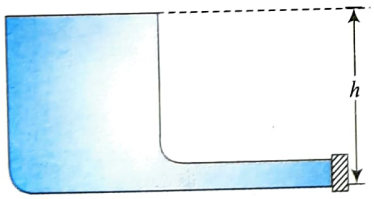
(3) $2v_0^2/2g$ (4) $5v_0^2/2g$

24. A hole is made at the bottom of a tank filled with water (density = 10^3 kg/m^3). If the total pressure at the bottom of the tank is 3 atm (1 atm = 10^5 N/m^2), then the velocity of efflux is

(1) $\sqrt{400}$ m/s (2) $\sqrt{200}$ m/s

(3) $\sqrt{600}$ m/s (4) $\sqrt{500}$ m/s

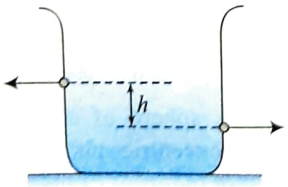
25. The opening near the bottom of the vessel shown in the figure has an area A. A disc is held against the opening to keep the liquid from running out. Let F_1 be the net force on the disc applied by liquid and air in this case. Now the disc is moved away from the opening a short distance. The liquid comes out and strikes the disc inelastically. Let F_2 be the force exerted by the liquid in this condition. Then F_1/F_2 is



(1) $\frac{1}{2}$ (2) 1

(3) $\frac{2}{1}$ (4) $\frac{1}{4}$

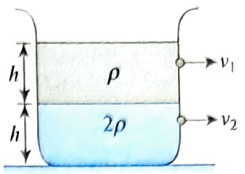
26. There are two identical small holes of area of cross section a on the opposite sides of a tank containing a liquid of density ρ. The difference in height between the holes is h. The tank is resting on a smooth horizontal surface. The horizontal force which will have to be applied on the tank to keep it in equilibrium is



(1) $ghpa$ (2) $\frac{2gh}{\rho a}$

(3) $2\rho agh$ (4) $\frac{\rho gh}{a}$

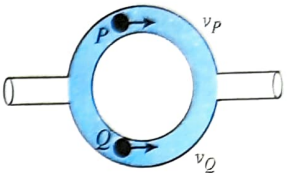
27. Equal volume of two immiscible liquids of densities ρ and 2ρ are filled in a vessel as shown in the figure. Two small holes are punched at depths $h/2$ and $3h/2$ from the surface of lighter liquid. If v_1 and v_2 are the velocities of efflux at these two holes, then v_1/v_2 is



(1) $\frac{1}{2\sqrt{2}}$ (2) $\frac{1}{2}$

(3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{2}}$

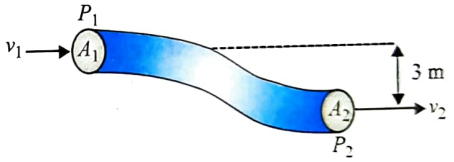
28. Figure shows a liquid flowing through a tube at the rate of 0.1 m^3/s. The tube is branched into two semicircular tubes of cross-sectional area $A/3$ and $2A/3$. The velocity of liquid at Q is (the cross section of the main tube is $A = 10^{-2}$ m^2 and v_P = 20 m/s)



(1) 5 m/s (2) 30 m/s

(3) 35 m/s (4) none of these

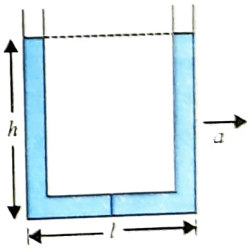
29. An ideal fluid flows in the pipe as shown in the figure. The pressure in the fluid at the bottom P_2 is the same as it is at the top P_1. If the velocity of the top $v_1 = 2$ m/s. Then the ratio of areas $A_1 : A_2$ is



(1) 2 : 1 (2) 4 : 1

(3) 8 : 1 (4) 4 : 3

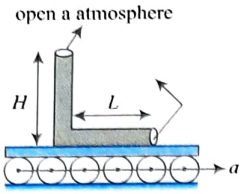
30. A U-tube of base length 'l' filled with the same volume of two liquids of densities ρ and 2ρ is moving with an acceleration 'a' on the horizontal plane. If the height difference between the two surfaces (open to atmosphere) becomes zero, then the height h is given by



(1) $\frac{a}{2g}l$ (2) $\frac{3a}{2g}l$

(3) $\frac{a}{g}l$ (4) $\frac{2a}{3g}l$

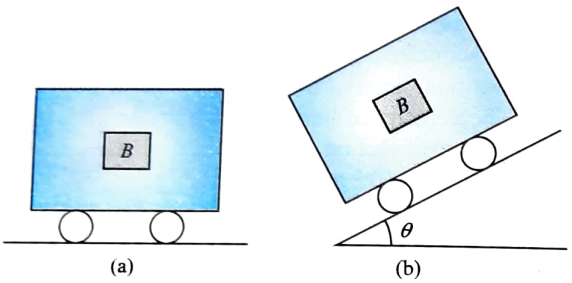
31. A narrow tube completely filled with a liquid is lying on a series of cylinders as shown in the figure. Assuming no sliding between any surfaces, the value of acceleration of the cylinders for which liquid will not come out of the tube from anywhere is given by



(1) $\frac{gH}{2L}$ (2) $\frac{gH}{L}$

(3) $\frac{2gH}{L}$ (4) $\frac{gH}{\sqrt{2}L}$

32. A body B is capable of remaining stationary inside a liquid at the position shown in Fig. (a). If the whole system is gently placed on smooth inclined plane (Fig. (b)) and is allowed to slide down, then $(0 < \theta < 90°)$

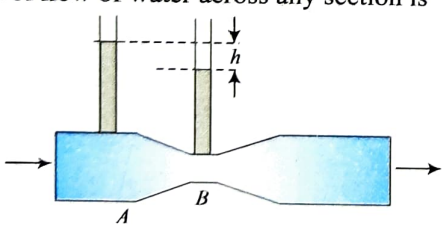


(1) The body will move up (relative to liquid)
(2) The body will move down (relative to liquid)
(3) The body will remain stationary (relative to liquid)
(4) The body will move up for some inclination θ and will move down for another inclination θ

33. An ornament weighing 36 g in air weighs only 34 g in water. Assuming that some copper is mixed with gold to prepare the ornament, find the amount of copper in it. Specific gravity of gold is 19.3 and that of copper is 8.9.

(1) 2.2 g (2) 4.4 g
(3) 1.1 g (4) 3.6 g

34. Water flows through a horizontal tube as shown in the figure. If the difference of heights of water column in the vertical tubes is $h = 0.02$ m, and the areas of cross section at A and B are 4×10^{-4} m^2 and 2×10^{-4} m^2, respectively, then the rate of flow of water across any section is



(1) 1.30×10^{-4} m^3/s (2) 1.46×10^{-4} m^3/s
(3) 1.60×10^{-4} m^3/s (4) 1.70×10^{-4} m^3/s

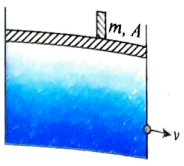
35. A beaker containing water is placed on the platform of a spring balance. The balance reads 1.5 kg. A stone of mass 0.5 kg and density 10^4 kg/m^3 is immersed in water without touching the walls of the beaker. What will be the balance reading now?

(1) 2 kg (2) 2.5 kg
(3) 1 kg (4) 3 kg

36. A tank is filled with water of density 10^3 kg/m^3 and oil of density 9×10^3 kg/m^3. The height of water layer is 1 m and that of the oil layer is 4 m. The velocity of efflux from an opening in the bottom of the tank is

(1) $\sqrt{85}$ m/s (2) $\sqrt{88}$ m/s
(4) $\sqrt{92}$ m/s (4) $\sqrt{98}$ m/s

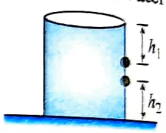
37. A cylindrical vessel contains a liquid of density ρ up to height h. The liquid is closed by a piston of mass m and area of cross section A. There is a small hole at the bottom of the vessel. The speed v with which the liquid comes out of the hole is



(1) $\sqrt{2gh}$ (2) $\sqrt{2\left(gh + \frac{mg}{\rho A}\right)}$

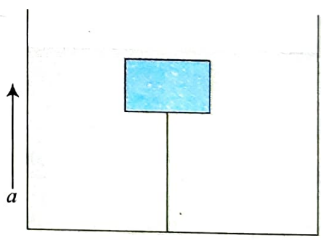
(3) $\sqrt{2\left(gh + \frac{mg}{A}\right)}$ (4) $\sqrt{2gh + \frac{mg}{A}}$

38. In a cylindrical water tank there are two small holes Q and P on the wall at a depth of h_1 from the upper level of water and at a height of h_2 from the lower end of the tank, respectively, as shown in the figure. Water coming out from both the holes strike the ground at the same point. The ratio of h_1 and h_2 is



(1) 1 (2) 2
(3) >1 (4) <1

39. The tension in a string holding a solid block below the surface of a liquid (where $\rho_{liquid} > \rho_{block}$) as in shown in the figure is T when the system is at rest.



Then what will be the tension in the string if the system has upward acceleration a?

(1) $T\left(1 - \frac{a}{g}\right)$ (2) $T\left(1 + \frac{a}{g}\right)$

(3) $T\left(\frac{a}{g} - 1\right)$ (4) $\frac{a}{g}T$

40. A sphere of solid material of specific gravity 8 has a concentric spherical cavity and just sinks in water. The ratio of radius of cavity to that of outer radius of the sphere must be

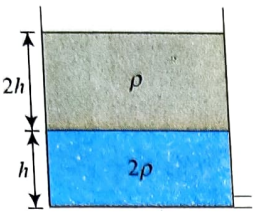
(1) $\frac{7^{1/3}}{2}$ (2) $\frac{5^{1/3}}{2}$

(3) $\frac{9^{1/3}}{2}$ (4) $\frac{3^{1/3}}{2}$

41. A liquid is kept in a cylindrical vessel which is rotated along its axis. The liquid rises at the side. If the radius of the vessel is 5 cm and the speed of rotation is 4 rev/s, then the difference in the height of the liquid at the centre of the vessel and its sides is

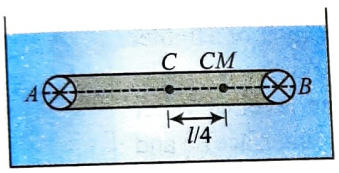
(1) 8 cm (2) 2 cm
(3) 40 cm (4) 4 cm

42. The velocity of the liquid coming out of a small hole of a vessel containing two different liquids of densities 2ρ and ρ as shown in the figure is



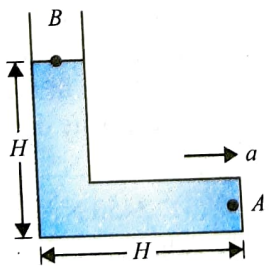
(1) $\sqrt{6gh}$
(2) $2\sqrt{gh}$
(3) $2\sqrt{2gh}$ (4) $\sqrt{gh}$

43. A non-uniform cylinder of mass m, length l and radius r is having its centre of mass at a distance $l/4$ from the centre and lying on the axis of the cylinder. The cylinder is kept in a liquid of uniform density ρ. The moment of inertia of the rod about the centre of mass is I. The angular acceleration of point A relative to point B just after the rod is released from the position as shown in the figure is



(1) $\frac{\pi\rho g l^2 r^2}{I}$ (2) $\frac{\pi\rho g l^2 r^2}{4I}$
(3) $\frac{\pi\rho g l^2 r^2}{2I}$ (4) $\frac{3\pi\rho g l^2 r^2}{4I}$

44. For the L-shaped vessel shown in the figure, determine the value of acceleration a so that pressure at point A becomes equal to $p_0/2$? (p_0 is the atmospheric pressure)

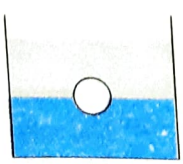


(1) g (2) $\frac{g}{2}+\frac{p_0}{2\rho H}$
(3) $\frac{p_0}{2\rho H}+g$ (4) $\frac{3p_0}{2\rho H}+g$

45. A U-tube is partially filled with water. Oil which does not mix with water is next poured into one side until water rises by 25 cm on the other side. If the density of oil is 0.8, the oil level will stand higher than the water level by

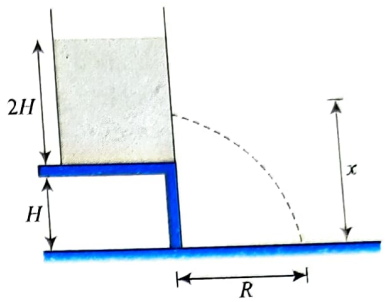
(1) 6.25 cm (2) 12.50 cm
(3) 31.75 cm (4) 62.50 cm

46. A solid uniform ball of volume V floats on the interface of two immiscible liquids (see the figure). The specific gravity of the upper liquid is ρ_1 and that of lower one is ρ_2 and the specific gravity of ball is ρ ($\rho_1 < \rho < \rho_2$). The fraction of the volume of the ball in the upper liquid is



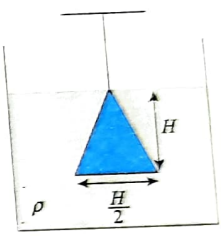
(1) $\frac{\rho_2}{\rho_1}$ (2) $\frac{\rho_2-\rho}{\rho_2-\rho_1}$
(3) $\frac{\rho-\rho_1}{\rho_2-\rho_1}$ (4) $\frac{\rho_1}{\rho_2}$

47. A tank is filled up to a height $2H$ with a liquid and is placed on a platform of height H from the ground. The distance x from the ground where a small hole is punched to get the maximum range R is



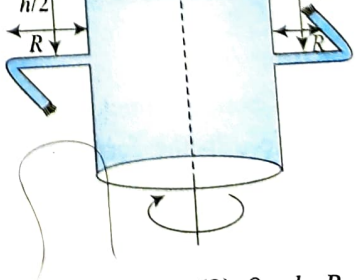
(1) H (2) $1.25H$
(3) $1.5H$ (4) $2H$

48. A solid cone of height H and base radius $H/2$ floats in a liquid of density ρ. It is hanging from the ceiling with the help of a string. The force by the fluid on the curved surface of the cone is (P_0 = atmospheric pressure)



(1) $\pi H^2\left(\frac{P_0}{4}+\frac{\rho gH}{3}\right)$ (2) $\pi H^2\left(\frac{P_0}{4}+\frac{\rho gH}{6}\right)$
(3) $\frac{\pi H^2}{4}\left(\frac{P_0}{4}+\rho gH\right)$ (4) $\frac{\pi H^2}{4}(P_0+\rho gH)$

49. A cylindrical container of radius 'R' and height 'h' completely filled with a liquid. Two horizontal L-shape pipes of small cross-sectional area 'a' are connected the cylinder as shown in the figure. Now the two pipes a opened and fluid starts coming out of the pipes horizonta in opposite directions. Then the torque due to ejected liq on the system is



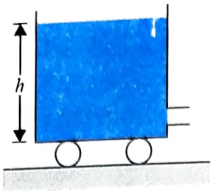
(1) $4agh\rho R$ (2) $8agh\rho R$
(3) $2agh\rho R$ (4) none of these

50. A hemispherical bowl just floats without sinking in a of density 1.2×10^3 kg/m^3. If the outer diameter a

density of the bowl are 1 m and 2×10^4 kg/m^3, respectively, then the inner diameter of the bowl will be

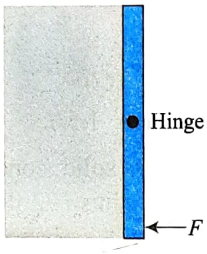
(1) 0.94 m (2) 0.97 m
(3) 0.98 m (4) 0.99 m

51. A container filled with liquid up to height h is placed on a smooth horizontal surface. The container is having a small hole at the bottom. As the liquid comes out from the hole, the container moves in a backward direction with acceleration a and finally, when all the liquid is drained out, it acquires a velocity v. Neglect mass of the container. In this case



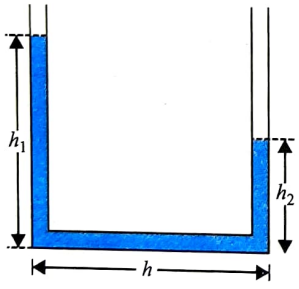
(1) both a and v depend on h
(2) only a depends on h
(3) only v depends on h
(4) neither a nor v depends on h

52. A square of size 1 m × 1 m is hinged at its midpoint. A fluid of density ρ fills the space to the left of the gate. The force F required to hold the gas stationary is



(1) $\frac{\rho g}{3}$ (2) $\frac{\rho g}{2}$
(3) $\frac{\rho g}{6}$ (4) $\frac{\rho g}{8}$

53. The U-tube shown in the figure has a uniform cross section. A liquid is filled up to heights h_1 and h_2 in the two arms and then they are allowed to move. Neglecting viscosity and surface tension, find the state of the liquid when the levels equalize in the two arms. Find the correct statement.



(1) The liquid will be at rest.
(2) The liquid will be moving with acceleration.
(3) The liquid will be moving with velocity

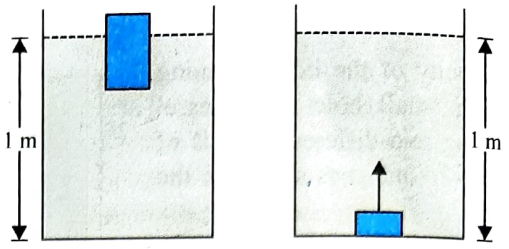
$$(h_1 - h_2)\sqrt{\frac{g}{2(h_1 + h_2 + h)}}.$$

(4) The liquid will exert a force to the right on the tube.

54. A large wooden piece in the form of a cylinder floats on water with two-thirds of its length immersed. When a man stands on its upper surface, a further one-sixth of its length is immersed. The ratio between the masses of the man and the wooden piece is

(1) 1 : 2 (2) 1 : 3
(3) 1 : 4 (4) 1 : 5

55. A wooden block in the form of a uniform cylinder floats with one-third length above the water surface. A small chip of this block is held at rest at the bottom of a container containing water to a height of 1 m and is then released. The time in which it will rise to the surface of water is ($g = 10$ m/s^2)

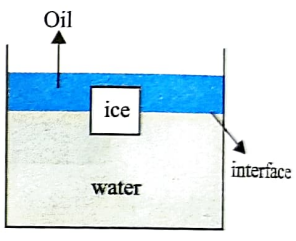


(1) 0.84 s (2) 0.63 s
(3) 0.5 s (4) 1 s

56. A goldsmith desires to test the purity of a gold ornament suspected to the mixed with copper. The ornament weighs 0.25 kg in air and is observed to displace 0.015 litre of water when immersed in it. Densities of gold and copper with respect to water are, respectively, 19.3 and 8.9. The approximate percentage of copper in the ornament is

(1) 5% (2) 10%
(3) 15% (4) 25%

57. An ice cube is floating in water above which a layer of a lighter oil is poured. As the ice melts completely, the level of water surface (or interface) and upper level of oil will



(1) rise and fall
(2) fall and rise
(3) not change and not change
(4) not change and fall

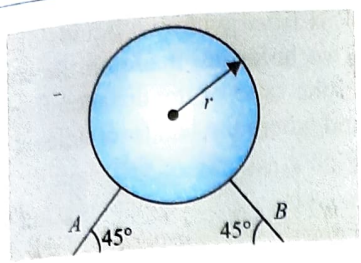
58. A flat plate moves normally towards a discharging jet of water at the rate of 3 m/s. The jet discharges the water at the rate of 0.1 m^3/s and at the speed of 18 m/s. The force exerted on the plate due to the jet is

(1) 1800 N (2) 2100 N
(3) 2450 N (4) 1560 N

59. A hollow cylindrical container floats in water with half its length immersed. A liquid of specific gravity σ is slowly poured into the container until the levels of the liquid inside the container and of water outside are the same. It is observed that the container is submerged to two-thirds its length. Then the value of σ is

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$
(3) $\frac{1}{4}$ (4) $\frac{1}{5}$

60. A hollow sphere of mass $M = 50$ kg and radius $r = \left(\frac{3}{40\pi}\right)^{1/3}$ m is immersed in a tank of water (density $\rho_w = 10^3$ kg/m^3). The sphere is tied to the bottom of a tank by two wires A and B as shown. Tension in wire A is ($g = 10$ m/s^2)



(1) $125\sqrt{2}$ N (2) 125 N

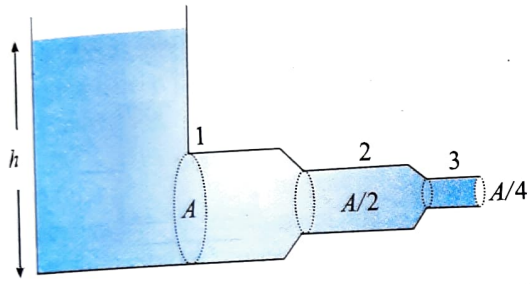
(3) $250\sqrt{2}$ N (4) 250 N

61. In a horizontal pipeline of uniform cross section the pressure falls by 8 N/m² between two points separated by 1 km. If oil of density 800 kg/m³ flows through the pipe, find the change in KE per kg of oil at these points.

(1) 10^{-2} J/kg (2) 10^{-3} J/kg

(3) 10^{-4} J/kg (4) 10^{-1} J/kg

62. In the figure shown the velocity and pressure of the liquid at the cross section (2) are given by (If P_0 is the atmospheric pressure).



(1) $\sqrt{2hg}, P_0 + \frac{\rho hg}{2}$ (2) $\sqrt{hg}, P_0 + \frac{\rho hg}{2}$

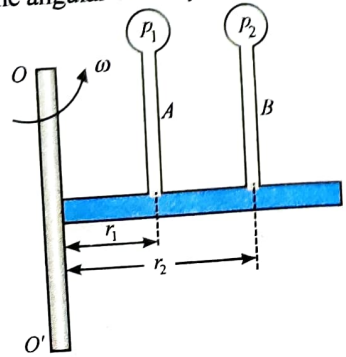
(3) $\sqrt{\frac{hg}{2}}, P_0 + \frac{3\rho hg}{4}$ (4) $\frac{\sqrt{hg}}{2}, P_0 + \frac{3\rho hg}{4}$

63. A liquid is kept in a cylindrical vessel which is rotated along its axis. The liquid rises at the sides. If the radius of the vessel is r and the speed of revolution is n rotations/second, find the difference in height of the liquid at the centre of vessel and its sides.

(1) $h = \frac{2\pi^2 n^2 r^2}{g}$ (2) $h = \frac{4\pi^2 n^2 r^2}{g}$

(3) $h = \frac{\pi^2 r^2 n^2}{g}$ (4) $h = \frac{\pi^2 r^2 n^2}{2g}$

64. A tube filled with water and closed at both ends uniformly rotates in a horizontal plane about the OO' axis. The manometers A and B fixed in the tube at distances r_1 and r_2 from rotational axis indicate pressures p_1 and p_2, respectively. Determine the angular velocity ω of rotation of the tube.



(1) $\omega = \sqrt{\frac{2(p_2 - p_1)}{\rho(r_2^2 - r_1^2)}}$ (2) $\omega = \sqrt{\frac{2(p_2 - p_1)}{\rho(r_2^2 + r_1^2)}}$

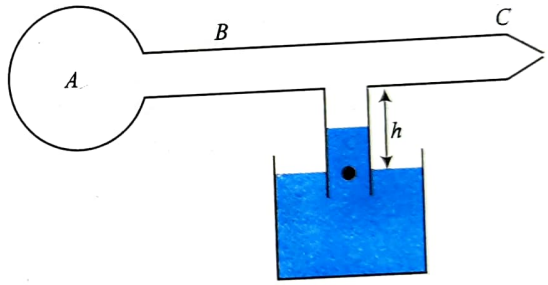
(3) $\omega = \sqrt{\frac{2(r_2^2 - r_1^2)}{p_1 - p_2}}$ (4) $\omega = \frac{\sqrt{2(p_2 - p_1)}}{\rho r_1 r_2}$

65. A horizontal tube has different cross sections at points A and B. The areas of cross section are a_1 and a_2, respectively, and pressures at these points are $p_1 = \rho g h_1$ and $p_2 = \rho g h_2$, where ρ is the density of liquid flowing in the tube and h_1 and h_2 are heights of liquid columns in vertical tubes connected at A and B. If $h_1 - h_2 = h$, then the flow rate of the liquid in the horizontal tube is

(1) $a_1 a_2 \sqrt{\frac{2gh}{a_1^2 - a_2^2}}$ (2) $a_1 a_2 \sqrt{\frac{2g}{h(a_1^2 - a_2^2)}}$

(3) $a_1 a_2 \sqrt{\frac{(a_1^2 + a_2^2)h}{2g(a_1^2 - a_2^2)}}$ (4) $\frac{2a_1 a_2 gh}{\sqrt{a_1^2 - a_2^2}}$

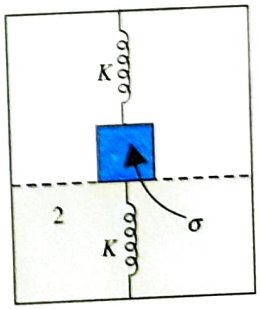
66. Figure shows a crude type of atomizer. When bulb A is compressed, air flows swiftly through tube BC causing a reduced pressure in the particles of the vertical tube. Liquid rises in the tube, enters BC and is sprayed out. If the pressure in the bulb is $P_a + P$, where P is the gauge pressure and P_a is the atmospheric pressure, v is the speed of air in BC, find how large would v need to be to cause the liquid to rise to BC. Density of air = 1.3 kg/m³.



(1) $\frac{P + \rho gh}{\sqrt{0.65}}$ (2) $\frac{Ph\rho g}{\sqrt{0.65}}$

(3) $\frac{P\sqrt{h\rho g}}{0.65}$ (4) $\frac{P}{h\rho g\sqrt{0.65}}$

67. A cubic block of side a is connected with two similar vertical springs as shown. Initially, bottom surface of the block of density σ touches the surface of the fluid of density 2σ while floating. A weight is placed on the block so that it is immersed half in the fluid, find the weight.



(1) $a\left(\frac{K}{2} + a^2\sigma g\right)$ (2) $a(K + a^2\sigma g)$

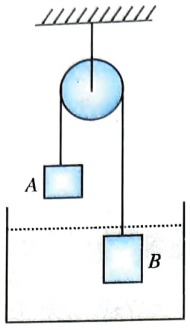
(3) $a\left(K + \frac{a^2}{2}\sigma g\right)$ (4) $\frac{a}{2}(K + a^2\sigma g)$

68. Two unequal blocks placed over each other of different densities σ_1 and σ_2 are immersed in a fluid of density of σ.

The block of density σ_1 is fully submerged and the block of density σ_2 is partly submerged so that ratio of their masses is 1/2 and $\sigma/\sigma_1 = 2$ and $\sigma/\sigma_2 = 0.5$. Find the degree of submergence of the upper block of density σ_2.

(1) 50% submerged (2) 25% submerged
(3) 75% submerged (4) Fully submerged

69. In the arrangement as shown, $m_B = 3m$, density of liquid is ρ and density of block B is 2ρ. The system is released from rest so that block B moves up when in liquid and moves down when out of liquid with the same acceleration. Find the mass of block A.

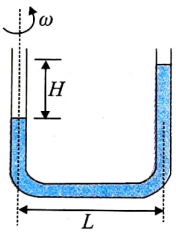


(1) $\frac{7}{4}m$ (2) $2m$
(3) $\frac{9}{2}m$ (4) $\frac{9}{4}m$

70. A large open tank has two holes in the wall. One is a square hole of side L at a depth y from the top and the other is a circular hole of radius R at a depth $4y$ from the top. When the tank is completely filled with water, the quantities of water flowing out per second from both the holes are the same. Then, R is equal to

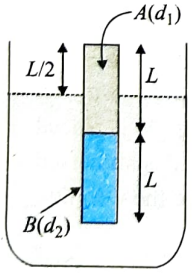
(1) $L/\sqrt{2\pi}$ (2) $2\pi/L$
(3) L (4) $L/2\pi$

71. A U-shaped tube contains a liquid of density ρ and it is rotated about the left dotted line as shown in the figure. Find the difference in the levels of the liquid column.



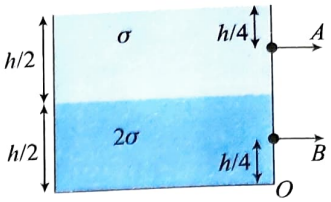
(1) $\frac{\omega^2 L^2}{2\sqrt{2}g}$ (2) $\frac{\omega^2 L^2}{2g}$
(3) $\frac{2\omega^2 L^2}{g}$ (4) $\frac{2\sqrt{2}\omega^2 L^2}{g}$

72. In the arrangement as shown, A and B are two cylinders each of length L and density d_1 and d_2 $(d_1 < d_2)$, respectively. They are joined together and submerged in the liquid of density d with length $L/2$ of A outside the liquid. If the difference in densities of two cylinders is equal to density of the liquid, find d_1.



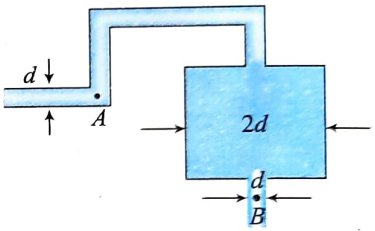
(1) $\frac{5}{4}d$ (2) $\frac{d}{4}$
(3) $\frac{3}{4}d$ (4) $\frac{d}{2}$

73. A large tank is filled with two liquids of specific gravities 2σ and σ. Two holes are made on the wall of the tank as shown. Find the ratio of the distance from O of the points on the ground where the jets from holes A & B strike.



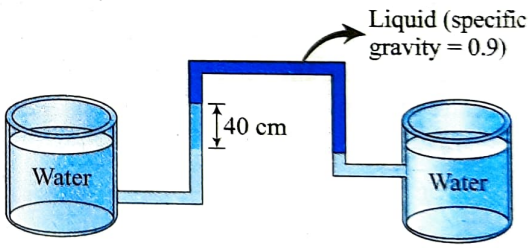
(1) $\sqrt{3}:\sqrt{2}$ (2) $\sqrt{5}:\sqrt{2}$
(3) $\sqrt{5}:\sqrt{3}$ (4) $\sqrt{3}:1$

74. An ideal fluid is flowing through the given tubes which is placed on a horizontal surface. If the liquid has velocities v_A and v_B, and pressures P_A and P_B at points A and B respectively, then the correct relation is (A and B are at same height from ground level, the figure shown is as if the system is seen from the top).



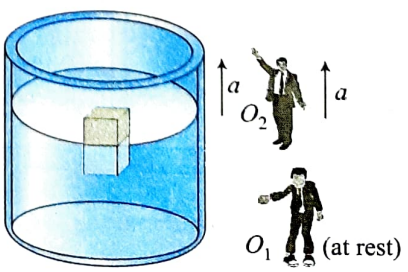
(1) $v_A > v_B$, $P_A < P_B$ (2) $v_A < v_B$, $P_A > P_B$
(3) $v_A = v_B, P_A = P_B$ (4) $v_A > v_B, P_A = P_B$

75. The manometer shown below is used to measure the difference in water level between the two tanks. Calculate this difference for the conditions indicated.



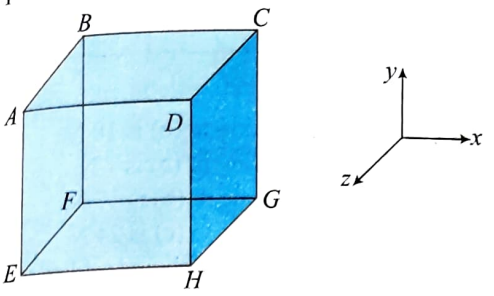
(1) 2 cm (2) 4 cm
(3) 6 cm (4) 8 cm

76. A block is partially immersed in a liquid and the vessel is accelerating upwards with an acceleration "a". The block is observed by two observers O_1 and O_2 one at rest and the other accelerating with an acceleration "a" upward. The total buoyant force on the block is



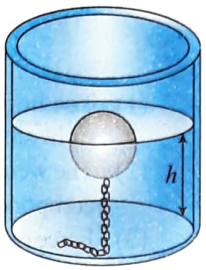
(1) same for O_1 and O_2 (2) greater for O_1 than O_2
(3) greater for O_2 than O_1 (4) data is not sufficient

77. The cubical container $ABCDEFGH$ which is completely filled with an ideal (non-viscous and incompressible) fluid, moves in a gravity free space with a acceleration of $a = a_0(\hat{i} - \hat{j} + \hat{k})$ where a_0 is a positive constant. Then the only point in the container where pressure can be zero is



(1) B (2) C
(3) E (4) H

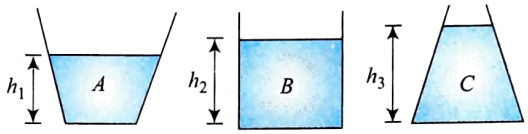
78. One end of a long iron chain of linear mass density λ is fixed to a sphere of mass m and specific density 1/3 while the other end is free. The sphere along with the chain is immersed in a deep lake. If specific density of iron is 7, the height h above the bed of the lake at which the sphere will float in equilibrium is (Assume that the part of the chain lying on the bottom of the lake exerts negligible force on the upper part of the chain.)



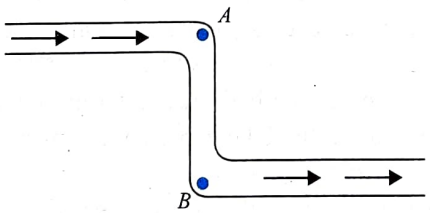
(1) $\frac{16m}{7\lambda}$ (2) $\frac{7m}{3\lambda}$
(3) $\frac{5m}{2\lambda}$ (4) $\frac{8m}{3\lambda}$

Multiple Correct Answers Type

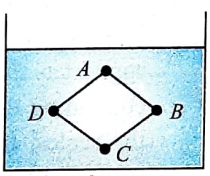
1. A wooden block is floating in a water tank. The block is pressed to the bottom. During the process, a work is done. Which of the following statements are false?
   (1) Work done is equal to work done against upthrust exerted by the water.
   (2) Work done is equal to work done against upthrust plus loss of gravitational potential energy of the block.
   (3) Work done is equal to work done against upthrust minus loss of gravitational potential energy of the block.
   (4) Work done is equal to loss of gravitational potential energy of the block.
2. Choose the correct statements from the following:
   (1) A body will sink in a liquid if its weight is equal to or greater than the weight of the liquid displaced by it.
   (2) A body will float in a liquid if its weight is equal to or less than the weight of liquid displaced by it.
   (3) When a body floats in a liquid, the portion of the body above the surface of the liquid is independent of the density of the body relative to that of the liquid.
   (4) In still air, a hydrogen filled balloon rises up to a certain height and then stops rising.
3. A body is floating in a liquid in a beaker. If the whole system falls freely under gravity, the upthrust on the body due to the liquid is
   (1) zero
   (2) equal to the weight of the liquid displaced
   (3) equal to the weight of the body in air
   (4) equal to the weight of the immersed portion of the body
4. A spring balance reads W_1 when a ball is suspended from it. A weighing machine reads W_2 when a tank of liquid is kept on it. When the ball is immersed in the liquid, the spring balance reads W_3 and the weighing machine reads W_4. Then, which of the following are not correct?
   (1) $W_1 < W_3$ (2) $W_1 > W_3$
   (3) $W_2 < W_4$ (4) $W_2 > W_4$
5. Equal volumes of a liquid are poured in the three vessels A, B and C ($h_1 < h_2 < h_3$). All the vessels have the same base area. Select the correct alternatives.



   (1) The force on the base will be maximum in vessel A.
   (2) The force on the base will be maximum in vessel C.
   (3) Net force exerted by the liquid in all the three vessels is equal.
   (4) Net force exerted by the liquid in vessel A is maximum.
6. If the system is not in free fall, which of the following statements are true about hydrostatic pressure?
   (1) In a liquid, points at different depths can never be at the same pressure.
   (2) In a liquid, points at different depths may be at the same pressure.
   (3) In different liquids, points at different depths can be at the same pressure.
   (4) In different liquids, points at the same depth can never be at same pressure.
7. In the figure, an ideal liquid flows through the tube, which is of uniform cross section. The liquid has velocities v_A and v_B, and pressures P_A and P_B at the points A and B, respectively. Then



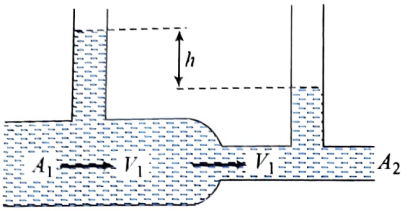
   (1) $v_A = v_B$ (2) $v_A > v_B$
   (3) $P_A = P_B$ (4) $P_B > P_A$
8. Figure shows a container filled with a liquid of density ρ. Four points A, B, C and D lie on the vertices of a vertical square. Points A and C lie on a vertical line and points B and D lies on a horizontal line. Choose the correct statement(s) about the pressure at the four points.



(1) $P_D = P_B$ (2) $P_A < P_B = P_D < P_C$

(3) $P_D = P_B = \frac{P_C - P_A}{2}$ (4) $P_D = P_B = \frac{P_C + P_A}{2}$

9. A liquid flows through a horizontal tube. The velocities of the liquid in the two sections, which have areas of cross section A_1 and A_2, are v_1 and v_2, respectively. The difference in the levels of the liquid in the two vertical tubes is h. Then



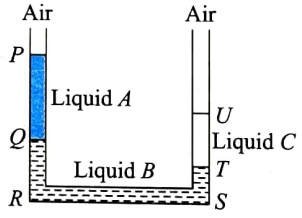
(1) the volume of the liquid flowing through the tube in unit time is $A_1 v_1$

(2) $v_2 - v_1 = \sqrt{2gh}$

(3) $v_2^2 - v_1^2 = 2gh$

(4) the energy per unit mass of the liquid is the same in both sections of the tube

10. A U-tube is held in a vertical plane such that its two limbs are vertical with middle portion horizontal as shown in the figure. Three liquids A, B and C are poured into it and in steady state their interfaces are as shown in the figure. Which of the following statements are correct?



(1) Height of column of liquid A is greater than that of C.

(2) Density of liquid A is less than that of liquid C.

(3) Density of liquid B is maximum among these three liquids.

(4) Pressure at R is greater than that at S.

11. A small body of density ρ' is dropped from rest at a height h into a lake of density ρ, where $\rho > \rho'$. Which of the following statements is/are correct if all dissipative effects are neglected (neglect viscosity)?

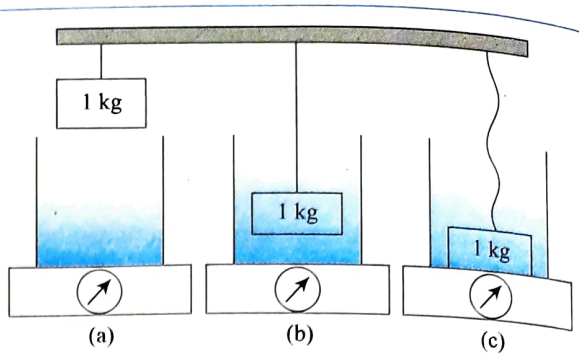
(1) The speed of the body just entering the lake is $\sqrt{2gh}$.

(2) The body in the lake experiences upward acceleration equal to $\{(\rho/\rho') - 1\}\, g$.

(3) The maximum depth to which the body sinks in the lake is $h\rho'/(\rho - \rho')$.

(4) The body does not come back to the surface of the lake.

12. Three identical blocks each of mass $m = 1$ kg and volume 3×10^{-4} m^3 are suspended by massless strings from a support as shown. Underneath are three identical containers containing the same amount of water that are placed over the scales. In Fig. (a), the block is completely out of the water; in Fig. (b), the block is completely submerged but not touching the beaker and in Fig. (c), the block rests on the bottom of the beaker. The scale in Fig. (a) reads 14 N.



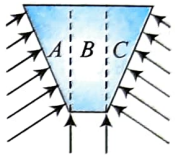
(1) The tension in the string in (b) is 10 N.

(2) The tension in the string in (b) is 7 N.

(3) The reading of the scale in (b) is 17 N.

(4) The reading of the scale in (c) is 24 N.

13. In the following figure shown, a liquid is filled into the vessel up to a height h and the forces exerted against the liquid by the walls of the vessel are shown by arrows, the force being everywhere perpendicular to the walls of the vessel. For this situation, mark out the correct statement(s).



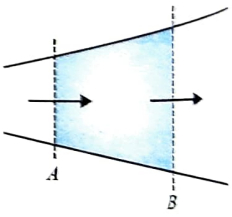
(1) The horizontal component of the forces on the left and right side inclined faces will cancel each other.

(2) The weight of the section lettered A is supported by vertical component of forces acting on the inclined faces.

(3) Force exerted by the liquid on the base of the vessel is greater than $(p_0 + \rho gh)A$ where ρ is the density of the liquid and A is base area of the vessel.

(4) Net force exerted by liquid on the base of the vessel is in the vertical downward direction.

14. A non-viscous liquid is flowing through a non-uniform pipe from section A to section B as shown in the figure. Cross sectional area of the pipe at A is less than that at B. Which of the following statements are correct?



(1) The pressure at A is greater than that at B.

(2) Velocity at A is greater than that at B.

(3) Total energy per unit volume of the liquid is greater at A than at B.

(4) Axis of the pipe cannot be horizontal.

15. A tube is placed on a horizontal surface such that its axis is horizontal, water is flowing in streamline motion through the tube. Consider two points A and B in the tube at the same horizontal level:

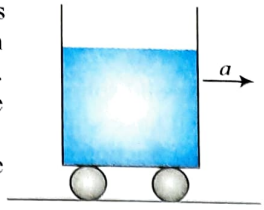
(1) The pressure are equal if the tube has a uniform cross-section

(2) The pressure may be equal even if the tube has a non-uniform cross-section.

(3) The pressure at A and B are equal for any shape of the tube

(4) The pressure can never be equal

16. An open vessel containing liquid is moving with constant acceleration a on a levelled horizontal surface. For this situation mark out the correct statement(s).



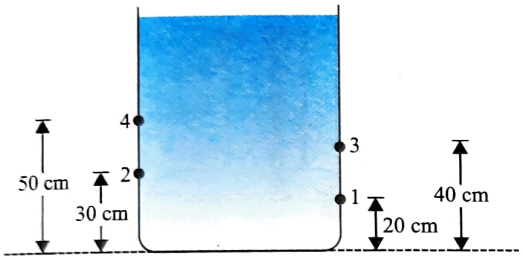
(1) The maximum pressure is at the leftmost bottom corner.

(2) Along a horizontal line within the liquid as we move from left to right the pressure decreases.

(3) The pressure at all points on a line making an angle of $\tan^{-1}\left(\frac{a}{g}\right)$ with horizontal would be same.

(4) Along a horizontal line within the liquid as we move from left to right, the pressure remains same.

17. Figure shows a cylindrical vessel of height 90 cm filled up to the brim. There are four holes in the vessel as shown. The liquid falling at maximum horizontal distance from the vessel would come from

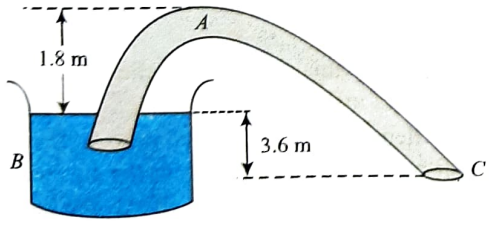


(1) hole 1 (2) hole 2
(3) hole 3 (4) hole 4

18. Water is floating smoothly through a closed-pipe system. At one point A, the speed of the water is 3.0 m/s while at another point B, 1.0 m higher, the speed is 4.0 m/s. The pressure at A is 20 kPa when the water is flowing and 18 kPa when the water flow stops. Then

(1) the pressure at B when water is flowing is 6.5 kPa
(2) the pressure at B when water is flowing is 8.0 kPa
(3) the pressure at B when water stops flowing is 10 kPa
(4) the pressure at B when water stops flowing is 8.0 kPa

19. A siphon has a uniform circular base of diameter $8/\sqrt{\pi}$ cm with its crest A, 1.8 m above the water level vessel B is of large cross section (g = 10 m/s^2 and atmospheric pressure $P_0 = 10^5$ N/m^2).

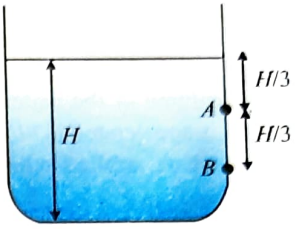


(1) Velocity of flow through pipe is $6\sqrt{2}$ m/s .
(2) Discharge rate of flow through pipe is $96\sqrt{2}\times10^{-4}$ m^3/s.
(3) Velocity of flow through pipe is 6 m/s.
(4) Pressure of A is 0.46×10^{-5} N/m^2.

20. The difference of pressure between two points along a horizontal pipe through which water is flowing is 1.4 cm of Hg. If due to non-uniform cross section the speed of flow at a point of greater cross section is 60 cm/s, the speed of flow at the other point is

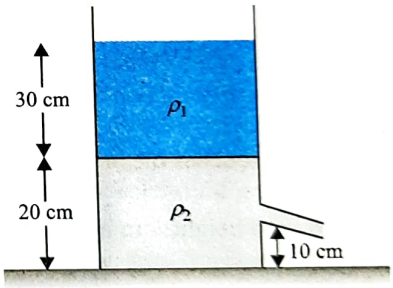
(1) 2 m/s
(2) less than 60 cm/s
(3) not affected by non-uniform cross section
(4) greater than 60 cm/s

21. The area of two holes A and B are $2a$ and a, respectively. The holes are at height (H/3) and (2H/3) from the surface of water. Find the correct option(s):



(1) The velocity of efflux at hole B is 2 times the velocity of efflux at hole A.
(2) The velocity of efflux at hole B is $\sqrt{2}$ time the velocity of efflux at hole A.
(3) The discharge is same through both the holes.
(4) The discharge through hole A is $\sqrt{2}$ time the discharge through hole B.

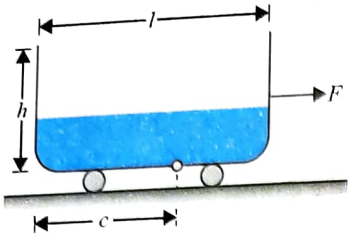
22. A cylindrical vessel of a very large cross-sectional area is containing two immiscible liquids of density ρ_1 = 600 kg/m^3 and ρ_2 = 1200 kg/m^3 as shown in the figure.



A small hole having cross-sectional area 5 cm^2 is made in right side vertical wall as shown. Take atmospheric pressure as $p_0 = 10^5$ N/m^2, g = 10 m/s^2. For this situation, mark out the correct statement(s). (Take cross-sectional area of the cylindrical vessel as 1000 cm^2. Neglect the mass of the vessel.)

(1) If the surface on which the vessel is placed is smooth, then a rightward force of magnitude 3 N is to be applied on the vessel to maintain its static equilibrium.
(2) If the surface on which the vessel is placed is smooth, then no force is needed to maintain its static equilibrium.
(3) If the surface on which the vessel is placed is rough (μ = 0.04), then the minimum force (horizontal) needed to be applied on the vessel to maintain its static equilibrium is zero.
(4) If the surface on which the vessel is placed is rough (= 0.04), then the maximum force (horizontal) needed to be applied on the vessel to maintain its static equilibrium is 19.8 N.

23. A rectangular vessel of dimension ($l \times b \times h$) and mass M contains a liquid of density ρ. The vessel has an orifice at its bottom at a distance c from the rear wall as shown in the figure.



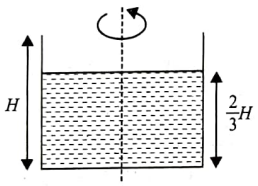
(1) The maximum volume of the water that can be stored when the vessel is accelerated is $hcb/2$.

(2) The maximum volume of the water that can be stored when the vessel is accelerated is $hlb/2$.

(3) Force F that must be applied when maximum water stored is $\left[M + \frac{hcb\rho}{2}\right]\frac{hg}{c}$.

(4) Force F that must be applied when maximum water stored is $\left[M + \frac{hcb\rho}{2}\right]\frac{lg}{c}$.

24. A circular cylinder of radius R and height H is filled with water to a height $(2/3)H$. It starts rotating about its axis with constantly increasing angular speed. Choose the correct alternatives.



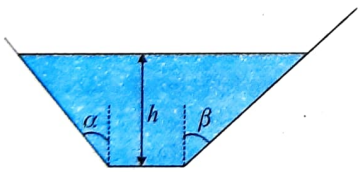
(1) At all speeds, shape of the free surface is parabolloid.

(2) The free surface touches first the brim of the cylinder and then the base of the cylinder.

(3) The free surface cannot touch the base without spilling water.

(4) The free surface touches the brim as well as the base at the same instant.

25. A liquid is filled upto height h in a vessel, as shown. Find correct option(s):



(1) If $\alpha = \beta$, horizontal component of forces on left and right side of inclined faces will be equal and opposite.

(2) If $\alpha \neq \beta$, horizontal component of forces on left and right side of inclined faces will be equal and opposite.

(3) If A is the area of the base of the vessel, then force exerted by liquid on walls of the vessel is greater than $(P_{atm} + \rho gh)A$.

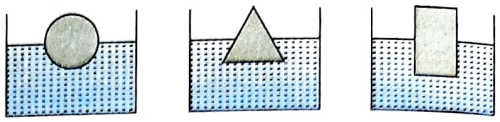
(4) As above, the force exerted by liquid on walls is equal to $(P_{atm} + \rho gh)A$.

26. An object is weighted at the North Pole by a beam balance and a spring balance, giving readings of W_B and W_S respectively. It is again weighed in the same manner at the equator, giving reading of W_B' and W_s' respectively. Assume that the acceleration due to gravity is the same everywhere and that the balances are ante sensitive:

(1) $W_B = W_S$ (2) $W_B' = W_S'$

(3) $W_B = W_B'$ (4) $W_S' < W_S$

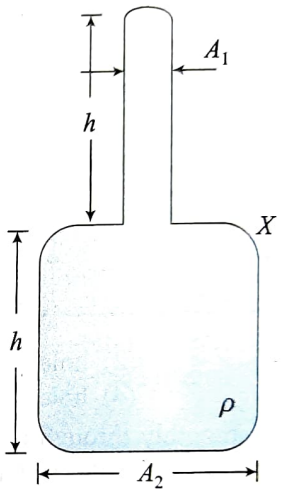
27. A solid sphere, a cone and a cylinder are floating in water. All have same mass, density and radius. Let f_1, f_2 and f_3 are the fraction of their volumes inside the water and h_1, h_2 and h_3 are the depths inside water. Then



(1) $f_1 = f_2 = f_3$ (2) $f_3 > f_2 > f_1$

(3) $h_3 < h_1$ (4) $h_3 < h_2$

28. The vessel shown in the figure has two sections of area of cross-section A_1 and A_2. A liquid of density ρ fills both the sections, up to a height h in each. Neglect atmospheric pressure:



(1) The pressure at the base of the vessel is $2h\rho g$

(2) The force exerted by the liquid on the base of the vessel is $2h\rho gA_2$

(3) The weight of the liquid is $<2h\rho gA_2$

(4) The walls of the vessel at the level X exert a downward force $h\rho g\,(A_2 - A_1)$ on the liquid

29. A beaker exactly full of water has an ice piece floating in it. As the cube melts, what happens to water level?

(1) it remains unchanged if the cube contains an air bubble

(2) the level falls if the cube contains some lead pieces inside it

(3) the level will rise if the cube contains some cork pieces inside

(4) the level remains the same if the cube contains some cork pieces inside it

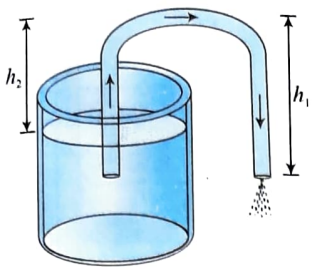
30. A ball of density ρ is dropped onto a horizontal solid surface. It bounces elastically from the surface and returns to its original position in time t_1. Next, the ball is released from the same height, but this time it strikes the surface of a liquid of density ρ_L ($> \rho$ but less than 2ρ), and takes t_2 second to come back to its original height

(1) $t_2 > t_1$ (2) $t_1 > t_2$

(3) the motion of the ball is not simple harmonic

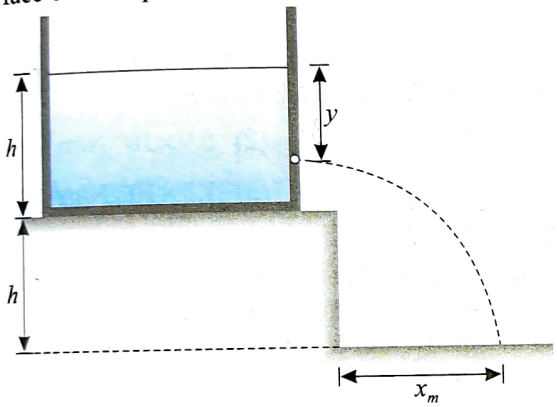
(4) If $\rho = \rho_L$, then the speed of the ball inside the liquid will be independent of its depth

31. Siphon is a device to transfer liquid from a higher level to a lower level. The condition of working of a siphon is



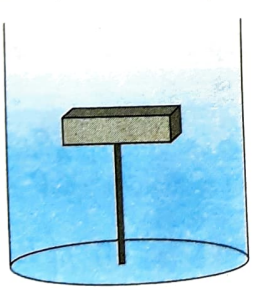
(1) $h_2 > h_1$
(2) $h_2 = 2h_1$
(3) h_1 should be less than the height of corresponding liquid barometer
(4) h_1 should be greater than the height of corresponding liquid barometer

32. A tank is filled upto a height h with a liquid and is placed on a platform of height h from the ground. To get maximum range x_m a small hole is punched at a distance y from the free surface of the liquid. Then



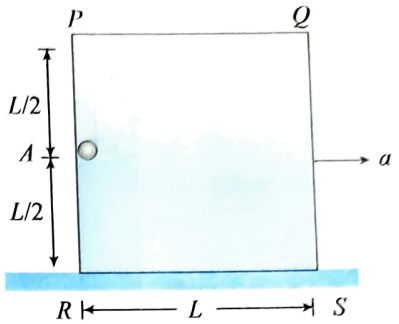
(1) $x_m = 2h$ (2) $x_m = 1.5h$
(3) $y = h$ (4) $y = 0.75h$

33. A string is holding a solid block below the surface of the liquid as shown in figure. Now if the system is given an upward constant acceleration a, then as compared to previous state.



(1) Tension is string will be $\left(1+\frac{a}{g}\right)$ times
(2) Tension in string will be $\left(1-\frac{a}{g}\right)$ times
(3) Upthrust force on block become $\left(1+\frac{a}{g}\right)$ times
(4) Upthrust force on block becomes $\left(1-\frac{a}{g}\right)$ times

34. A small solid ball of density ρ is held inside at point A in a cubical container of side L, filled with an ideal liquid of density 4ρ as shown in the figure. Now, if the container starts moving with constant acceleration a horizontally and the ball is released from point A simultaneously, then choose the correct option(s).



(1) For ball to hit the top of container at end Q, $a = 3g$
(2) For ball to hit the top of container at end Q, $a = 2g$
(3) Ball hits the top of container at end Q after a time $t = \sqrt{\frac{L}{3g}}$
(4) Ball hits the top of container at end Q after a time $t = \sqrt{\frac{2L}{3g}}$

Linked Comprehension Type

For Problems 1–5

If the container filled with liquid gets accelerated horizontally or vertically, pressure in liquid gets changed. In liquid for calculation of pressure, effective g is used. A closed box with horizontal base 6 m by 6 m and a height 2 m is half filled with liquid. It is given a constant horizontal acceleration $g/2$ and vertical downward acceleration $g/2$.

1. The angle of the free surface with the horizontal is equal to
(1) 30° (2) $\tan^{-1}\frac{2}{3}$
(3) $\tan^{-1}\frac{1}{3}$ (4) 45°

2. Length of exposed portion of top of box is equal to
(1) 2 m (2) 3 m
(3) 4 m (4) 2.5 m

3. Water pressure at the bottom of centre of the box is equal to (atmospheric pressure = 10^5 N/m^2, density of water = 1000 kg/m^3, g = 10 m/s^2)
(1) 1.1 MPa (2) 0.11 MPa
(3) 0.101 MPa (4) 0.011 MPa

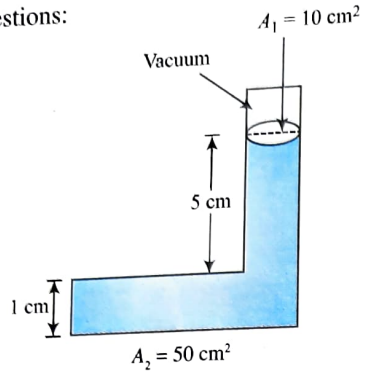
4. The maximum value of water pressure in the box is equal to
(1) 1.4 MPa (2) 0.14 MPa
(3) 0.104 MPa (4) 0.014 MPa

5. What is the value of vertical acceleration of box for given horizontal acceleration ($g/2$), so that no part of the bottom of the box is exposed?

(1) $g/2$ upwards (2) $g/4$ downwards
(3) $g/4$ upwards (4) not possible

For Problems 6–7

The vessel shown in the figure contains water, (take ρ_{water} = 1000 kg/m^3, g = 10 m/s^2). For this arrangement, answer the following questions:

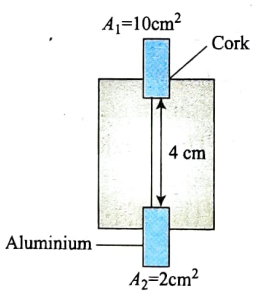


6. The net force acting on the base of the vessel is
(1) 1 N (2) 2 N
(3) 3 N (4) 5 N

7. Mark out the correct statement(s).
(1) Net force acting on the base of the vessel > weight of the liquid inside the vessel.
(2) Net force acting on the base of the vessel = weight of the liquid inside the vessel.
(3) Net pressure force acting on the liquid = weight of the vessel
(4) Both (1) and (3) are correct.

For Problems 8–9

A cylindrical object of cork of mass 15 g and cross-sectional area A_1 = 10 cm^2 floats in a pan of water as shown in the figure. An aluminium cylinder of mass 25 g and cross-sectional area A_2 = 2 cm^2 is attached 4 cm below the cork and slides through a watertight frictionless hole in the bottom of the pan. Take density of the cork, ρ = 0.2 g/cm^3, $\rho_{aluminium}$ = 2.7 g/cm^3, g = 10 m/s^2. Based on the above information, answer the following questions.

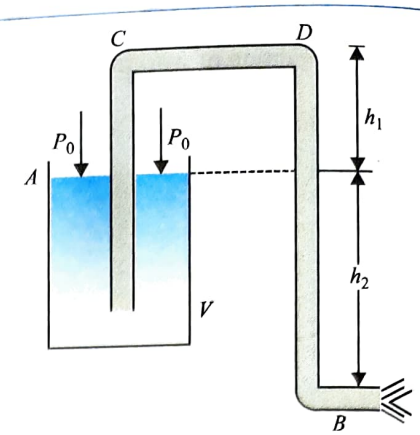


8. The length of the cork cylinder inside the water in equilibrium is
(1) 6 cm (2) 4 cm
(3) 8 cm (4) 3 cm

9. The tension in the string joining the cork and aluminium cylinder is
(1) 2 N (2) 1 N
(3) 0.45 N (4) 0.6 N

For Problems 10–12

A tube of uniform cross section is used to siphon water from a vessel V as shown in the figure. The pressure over the open end of water in the vessel is atmospheric pressure (P_0). The heights of the tube above and below the water level in the vessel are h_1 and h_2, respectively.



10. Determine the velocity v_B of the water issuing out at B.
(1) $v_B = \sqrt{2gh_2}$ (2) $v_B = \sqrt{2g(h_1 + h_2)}$
(3) $v_B = \sqrt{2g(h_2 - h_1)}$ (4) $v_B = \sqrt{gh_2}$

11. If h_2 = 3.0 m, the maximum value of h_1, for which the siphon will work will be
(1) 3.0 m (2) 6.0 m
(3) 7.2 m (4) 4.8 m

12. Given $h_1 = h_2$ = 3.0 m, the gauge pressure of water in the highest level CD of the tube will be
(1) 3.0×10^4 N/m^2 (2) 5.9×10^4 N/m^2
(4) 6.1×10^4 N/m^2 (3) 1.5×10^4 N/m^2

For Problems 13–15

An object of density $\sigma(<\rho)$ is dropped from rest at a height h into a liquid of density ρ kept in a tall vertical cylindrical container. Neglect all dissipative effects and assume that there is no spilling of the liquid.

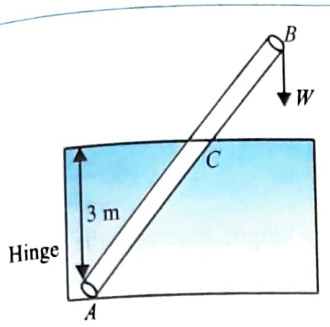
13. The acceleration of the object while it is inside the liquid will be
(1) $\frac{\sigma g}{\rho}$ upwards (2) $\frac{\sigma g}{\rho}$ downwards
(3) $g\left(\frac{\sigma}{\rho}-1\right)$ upwards (4) $g\left(\frac{\rho}{\sigma}-1\right)$ upwards

14. The maximum depth to which the body sinks before it returns to surface is
(1) $\left(\frac{h\sigma}{\rho-\sigma}\right)$ (2) $h\frac{\sigma}{\rho}$
(3) $h\left(\frac{\rho}{\rho-\sigma}\right)$ (4) $h\left(\frac{\rho}{\sigma}-1\right)$

15. The time taken by the object to reach the starting point (from which it is dropped) is ($\rho = 2\sigma$)
(1) $2\sqrt{\frac{2h}{g}}$ (2) $4\sqrt{\frac{2h}{g}}$
(3) $3\sqrt{\frac{2h}{g}}$ (4) it will never reach starting point

For Problems 16–17

A rod of length 6 m has a mass 12 kg. It is hinged at one end A at a distance of 3 m below the water surface. The specific gravity of the material of rod is 0.5.



16. What weight must be attached to the other end B so that 5 m of the rod is immersed in water?

(1) 7 kgf (2) $\frac{17}{3}$ kgf

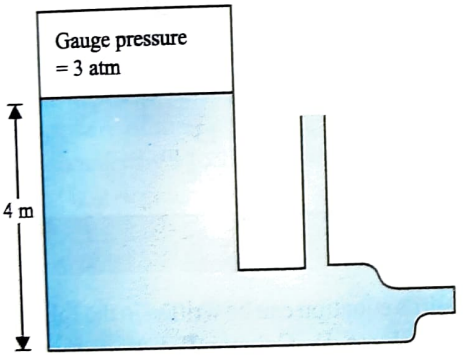
(3) $\frac{7}{5}$ kgf (4) $\frac{7}{2}$ kgf

17. Find the magnitude and direction of the force exerted by the hinge on the rod.

(1) 17/3 kgf in the downward direction
(2) 8 kgf in the downward direction
(3) 4 kgf in the downward direction
(4) 5 kgf in the downward direction

For Problems 18–20

A liquid having density 6000 kg/m^3 stands to a height of 4 m in a sealed tank as shown in the figure. The tank contains compressed air at a gauge pressure of 3 atm. The horizontal outlet pipe has a cross-sectional area of 6 cm^2 and 3 cm^2 at larger and smaller sections. Atmospheric pressure = 1 atm, $g = 10$ m/s^2, 1 atm = 10^5 N/m^2.



Assume that depth of water in the tank remains constant due to s very large base and air pressure above it remains constant. Based on the above information, answer the following questions.

18. The discharge rate from the outlet is

(1) 4.02×10^{-3} m^3/s (2) 6×10^{-5} m^3/s

(3) 7.56×10^{-3} m^3/s (4) 4.02×10^{-5} m^3/s

19. The height at which liquid will stand in the open end of the pipe is

(1) 4 m (2) 3.5 m

(3) 4.5 m (4) 5.6 m

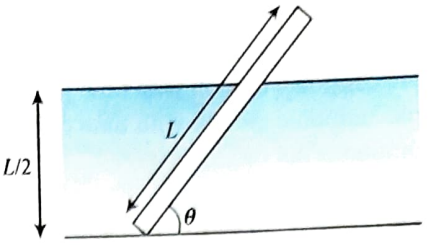
20. If a hole is created in the tank at its top end, then the height of liquid level in the open end of the pipe is

(1) 4 m (2) 3 m

(3) 3.5 m (4) 3.8 m

For Problems 21–22

A uniform rod of length L pivoted at the bottom of a pond of depth $L/2$ stays in stable equilibrium as shown in the figure.



21. Find the angle θ if the density of the material of the rod is half of the density of water.

(1) 60° (2) 30°

(3) 45° (4) 37°

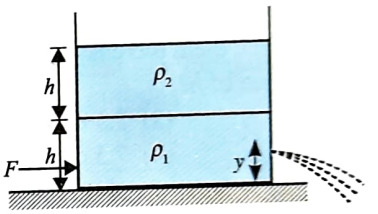
22. Find the force acting at the pivoted end of the rod in terms of mass m of the rod.

(1) $(\sqrt{2}-1)mg$ (2) $(\sqrt{2}+1)mg$

(3) $(\sqrt{3}+1)mg$ (4) $(\sqrt{2}+3)mg$

For Problems 23–25

A cylindrical tank having cross-sectional area $A = 0.5$ m^2 is filled with two liquids of density $\rho_1 = 900$ kg m^{-3} and $\rho_2 = 600$ kg m^{-3}, to a height $h = 60$ cm each as shown in the figure. A small hole having area $a = 5$ cm^2 is made in right vertical wall at a height $y = 20$ cm from the bottom. A horizontal force F is applied on the tank to keep it in static equilibrium. The tank is lying on a horizontal surface. Neglect mass of the cylindrical tank in comparison to mass of the liquids (take $g = 10$ m s^{-2}).



23. The velocity of efflux is

(1) 10 m s^{-1} (2) 20 m s^{-1}

(3) 4 m s^{-1} (4) 35 m s^{-1}

24. Horizontal force F to keep the cylinder in static equilibrium, if it is placed on a smooth horizontal plane, is

(1) 7.2 N (2) 10 N

(3) 15.5 N (4) 20.4 N

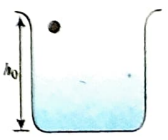
25. The minimum and maximum values of F to keep the cylinder in static equilibrium just after the water starts to spill through the hole, if the coefficient of static friction between contact surfaces is 0.01, are

(1) 0, 40 N (2) 5.4 N, 52.2 N

(3) 0, 70 N (4) 0, 52.2 N

For Problems 26–28

A small spherical ball of radius r is released from its completely submerged position (as shown in the figure) in a liquid whose density varies with height h (measured from the bottom) as $\rho_L = \rho_0[4 - (3h/h_0)]$. The density of the ball is $(5/2)\rho_0$. The height of the vessel is $h_0 = 12/\pi^2$. Consider $r << h_0$ and $g = 10$ m/s^2.



26. Where will the ball come in the equilibrium?

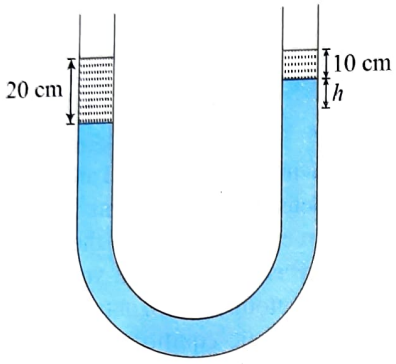
(1) At a depth $h_0/2$ from top
(2) At the bottom of the vessel
(3) At a depth $3h_0/4$ from top
(4) The ball will never be at equilibrium

27. The motion of the ball in the vessel is
(1) oscillatory but not simple harmonic motion
(2) simple harmonic motion with time period 2 s
(3) simple harmonic motion with time period 1 s
(4) the motion is not oscillatory

28. The ball will reach at the bottom of the vessel
(1) in 1 s (2) in 2 s
(3) in 0.5 s
(4) the ball will not reach at the bottom of the vessel

For Problem 29–30

In a U–tube, if different liquids are filled then we can say that pressure at same level of same liquid is same.

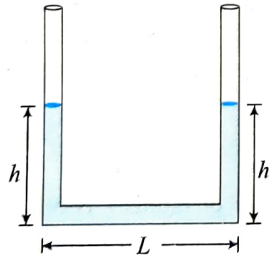


29. In a U–tube 20 cm of a liquid of density ρ is on left hand side and 10 cm of another liquid of density 1.5 ρ is on right hand side. In between them there is a third liquid of density 2ρ. What is the value of h?
(1) 5 cm (2) 2.5 cm
(3) 2 cm (4) 7.5 cm

30. If small but equal lengths of liquid –1 and liquid –2 are increased in their corresponding sides then h will.
(1) remain same (2) increase
(3) decrease (4) may increase or decrease

For Problems 31–32

A U-tube of uniform cross section contains a liquid of density d. Diameter of the tube is D. Height of liquid in the left arm and also in the right arm is h and length of horizontal portion of the tube and hence the horizontal column of liquid is L. Consider the following situations.



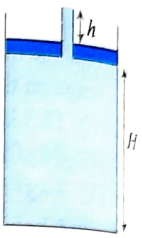
(i) The tube is given a uniform acceleration a towards left.
(ii) The tube is mounted on a horizontal table that is made to rotate with a uniform angular speed ω with one of the arms (left or right) on the axis of rotation.
Take M as the total mass of the liquid and g as the acceleration due to gravity.

31. Difference in height between the liquid columns in the vertical arms (left and right) in situation (i) will be
(1) $\sqrt{\frac{La}{g}} \times h$ (2) $\frac{L^2h^2da}{Mg}$
(3) $\frac{La}{g}$ (4) $\frac{LaD}{gh}$

32. Difference in height between the liquid columns in the vertical arms in situation (ii) will be
(1) $\frac{\omega^2L^2}{2g}$ (2) $\frac{Lh^2\omega^2}{Dg}$
(3) $\frac{h^2(L+2h)^2\omega^2d}{HgM}$ (4) $\sqrt{\frac{Lh^2\omega^2}{g}}$

For Problems 33–34

A piston of mass $M = 3$ kg and radius $R = 4$ cm has a hole into which a thin pipe of radius $r = 1$ cm is inserted. The piston can enter a cylinder tightly and without friction, and initially it is at the bottom of the cylinder. 750 g of water is now poured into the pipe so that the piston and pipe are lifted up as shown.



33. The height h of water in the pipe is
(1) $\frac{3}{2\pi}$m (2) $\frac{3}{\pi}$m
(3) $\frac{1}{\pi}$m (4) $\frac{2}{\pi}$m

34. The height H of water in the cylinder
(1) $\frac{11}{36\pi}$m (2) $\frac{13}{22\pi}$m
(3) $\frac{11}{32\pi}$m (4) $\frac{13}{32\pi}$m

Matrix Match Type

1. Bernoulli's equation can be written in the following different forms (Column I). Column II lists certain units each of which pertains to one of the possible forms of the equation. Match the unit associated with each of the equations.

Column I	Column II
i. $\frac{v^2}{2g}+\frac{p}{\rho g}+z$ = constant	a. Total energy per unit mass
ii. $\frac{\rho v^2}{2}+P+\rho gz$ = constant	b. Total energy per unit weight
iii. $\frac{v^2}{2}+\frac{P}{\rho}+gz$ = constant	c. Total energy per unit volume

2.

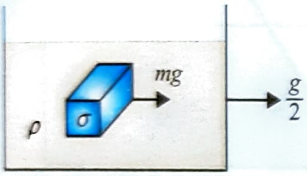
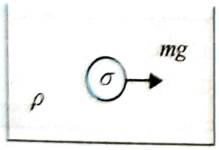
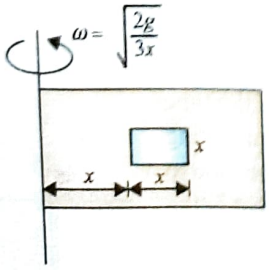
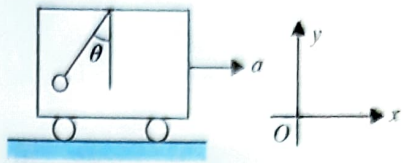
Column I	Column II
i. Bernoulli's theorem	a. Narrower pipes have less pressure
ii. Ball moving with spin	b. Paint gun

iii. Artificial high pressure	c. Non-viscous fluids
iv. Streamlined flow	d. Conservation of energy
	e. Uplift due to pressure difference

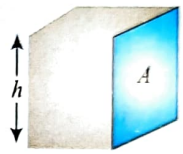
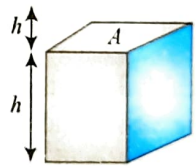
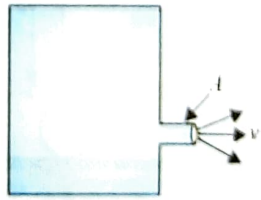
3.

Column I	Column II
i. Magnus energy	a. Pascal's law
ii. Loss of energy	b. Archimedes' principle
iii. Pressure is same at the same level in a liquid	c. Viscous force
iv. Hydraulic machines	d. Lifting of asbestos roofs

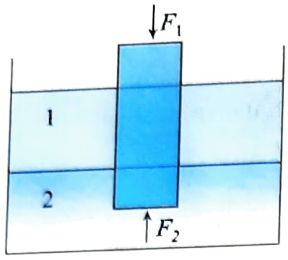
4.

Column I	Column II
i. A cube of mass m and density σ is pulled by a force $\vec{F}_x = mg\hat{i}$ in an accelerating liquid of density ρ. The value of a_x/a_y is 	a. $\frac{\rho}{\sigma}$
ii. The sphere of mass m is pulled horizontally by a force $F_x = mg$ in a non-accelerating liquid. The value of a_x/a_y is 	b. 1:1
iii. A closed tube of length x is placed in a rotating liquid. The value of a_x/g, where a_x is the horizontal acceleration of the cube, is 	c. $\frac{\sigma}{\rho - \sigma}$
iv. The string of a pendulum makes an angle θ with vertical when the cart moves with an acceleration a. If the cart is filled with liquid of density $\rho < \sigma$ (density of the bob), the value of the angle of inclination with the vertical becomes θ'. Then $\frac{\tan\theta}{\tan\theta'}$ is 	d. $\frac{2\sigma + \rho}{2(\sigma - \rho)}$

5.

Column I	Column II
i. Hydrostatic force on the side wall of the cubical vessel, ρ = density of liquid 	a. $A\rho v^2$
ii. Buoyant force on the cube, ρ = density of liquid 	b. $\frac{1}{2}A\rho v^2$
iii. Impact (reaction) force on the vessel by the liquid coming out of the vessel, ρ = density of liquid 	c. ρghA
iv. Aerodynamic force acting on the flat roof surface of area A, ρ = density of air	d. $\frac{1}{2}\rho ghA$

6. A cylinder of weight W is floating in two liquids as shown in figure. Net force on cylinder from top is F_1 and force on cylinder from the bottom is F_2. Match the following two columns.

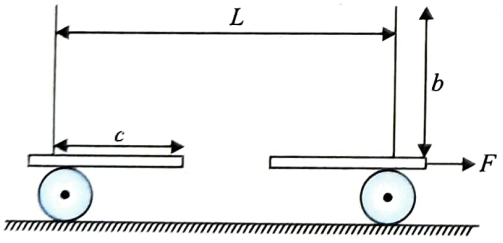


Column I	Column II
i. Net force on cylinder from liquid-1	a. Zero
ii. $F_2 - F_1$	b. W
iii. Net force on cylinder from liquid-2	c. Net upthrust
iv. Net force on cylinder from atmosphere	d. None of these

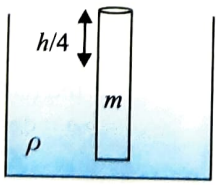
(1) i. → d.; ii. → c.; iii. → a.; iv. → b.
(2) i. → a.; ii. → b., c.; iii. → d.; iv. → d.
(3) i. → b.; ii. → c.; iii. → a., iv. → d.
(4) i. → c.; ii. → b., c.; iii. → a., iv. → d.

Numerical Value Type

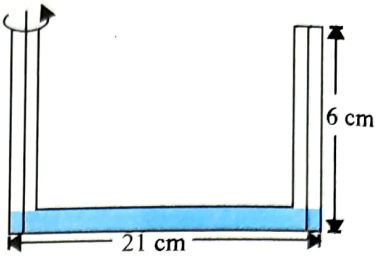
1. A vessel with a symmetrical hole in its bottom is fastened on a cart. The mass of the vessel and the cart is 1.5 kg. With what force F (in $\times 10^2$ N) should the cart be pulled so that the maximum amount of water remains in the vessel. The dimensions of the vessel are as shown in the figure. Given that $b = 50$ cm, $c = 10$ cm, area of base $A = 40$ cm^2, $L = 20$ cm, $g = 10$ m/s^2.



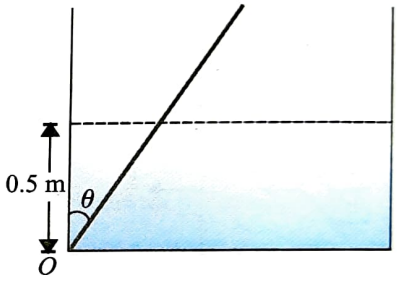
2. A solid cylinder of height h and mass m is floating in a liquid of density ρ as shown in the figure. Find the acceleration of the vessel (in m/s^2) containing liquid for which the relative downward acceleration of the completely immersed cylinder w.r.t. vessel becomes equal to one-third of that of the vessel. (Take $g = 10$ m/s^2)



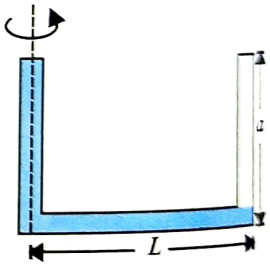
3. Length of horizontal arm of a uniform cross section U-tube is $l = 21$ cm and ends of both of the vertical arms are open to surrounding of pressure 10×500 N/m^2. A liquid of density $\rho = 10^3$ kg/m^3 is poured into the tube such that liquid just fills the horizontal part of the tube. Now, one of the open ends is sealed and the tube is then rotated about a vertical axis passing through the other vertical arm with angular velocity $\omega_0 = 10$ rad/s. If length of each vertical arm is $a = 6$ cm, calculate the length of air column in the sealed arm. (in cm)



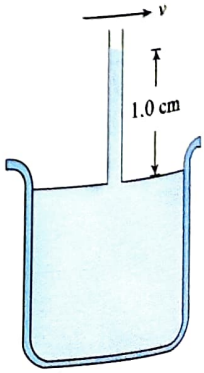
4. A wooden plank of length 1 m and uniform cross section is hinged at one end to the bottom of a tank as shown in the figure. The tank is filled with water up to a height of 0.5 m. The specific gravity of the plank is 0.5. If the angle θ by the inclinaiton of that the plank makes with the vertical in the equilibrium position (exclude the case $\theta = 0$). Find the value of $\frac{1}{\cos^2\theta}$.



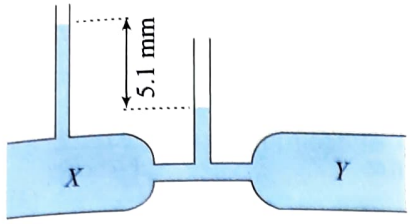
5. Length of a horizontal arm of a U-tube is $L = 1$ m and ends of both the vertical arms are open to atmospheric pressure $P_0 = 10^5$ N/m^2. A liquid of density $\rho = 10^3$ kg /m^3 is poured in the tube such that liquid just fills the horizontal part of the tube as shown in figure. Now one end of the open ends is sealed and the tube is then rotated about a vertical axis passing through the other vertical arm with angular speed $\omega_0 = \sqrt{40/3}$ rad/s. If length of each vertical arm is $a = 1$ m and in the sealed end liquid rises to a height $y = 1/2$ m, find pressure in the sealed tube during rotation. (in $\times 10^5$ N/m^2)



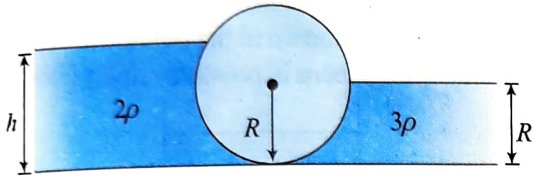
6. When air of density 1.3 kg/m^3 flows across the top of the tube shown in the accompanying figure, water rises in the tube to a height of 1.0 cm. What is the speed of the air (in m/s) ?



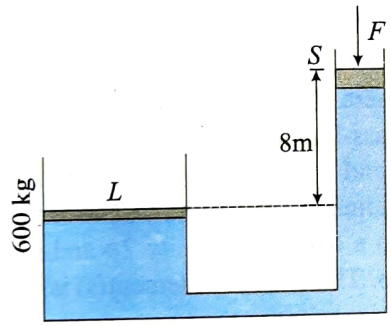
7. The diagram shows venturi meter through which water is flowing. The speed of water at X is 2 cm/s. Find the speed of water at Y in cm/s.(taking $g = 1000$ cm/s^2).



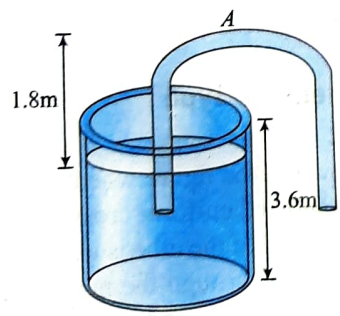
8. In the figure shown, the heavy cylinder (radius $\sqrt{\frac{8}{3}}$ m) kept on a smooth surface separates two liquids of densities 2ρ and 3ρ. The height h (in meter) for the translational equilibrium of the cylinder must be



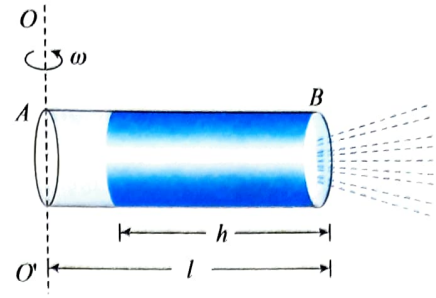
9. For the system shown in the figure, the cylinder on the left at L has a mass of 600 kg and a cross sectional area of 800 cm^2. The piston on the right, at S, has cross sectional area 25 cm^2 and negligible weight. If the apparatus is filled with oil ($\rho = 0.75$ g/cm^3) Find the force F (in N) required to hold the system in equilibrium. Take $g = 10$ m/s^2.



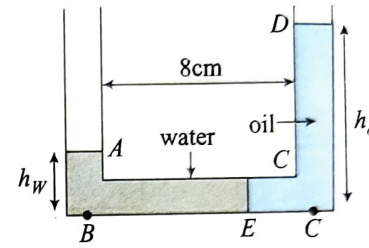
10. A siphon has a uniform circular base of diameter $\frac{8}{\sqrt{\pi}}$ cm with its crests 1.8 m above water level as shown in figure. Find absolute pressure at the crest level A (in kPa). [Use $P_0 = 10^5$ N/m^2 and $g = 10$m/s^2]



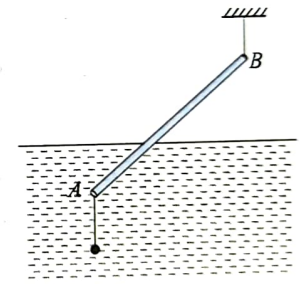
11. A horizontal oriented tube AB of length $l = 2.5$ m rotates with a constant angular velocity $\omega = 5$ rad/s about a stationary vertical axis OO' passing through the end A. Initially the tube is filled with an ideal fluid. The end A of the tube is open, the closed end B has a very small orifice. Find the velocity of the fluid (in m/s) relative to the tube when the liquid column length in tube reduce to $h = 1$ m.



12. A tube of uniform cross–section has two vertical portions connected with a horizontal thin tube 8 cm long at their lower ends. Enough water to occupy 22 cm of the tube is poured into one branch and enough oil of specific gravity 0.8 to occupy 22 cm is poured into the other. Find the distance(in cm) of the common surface E of the two liquids from point B.

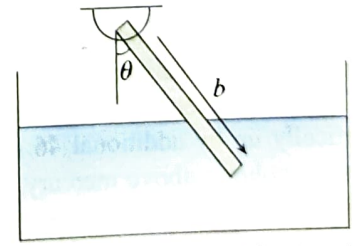


13. A uniform rod AB, 12 m long weighing 24 kg, is supported at end B by a flexible light string and a lead weight (of very small size) of 12 kg attached at end A.



The rod floats in water with one-half of its length submerged. Find the volume of the rod (in $\times 10^{-3}$m^3)
[Take $g = 10$ m/s^2, density of water = 1000 kg/m^3]

14. A uniform rod of length $b = 90$ cm capable of turning about its end which is out of water, rests inclined to the vertical. If its specific gravity is 5/9, find the length immersed in water (in cm).



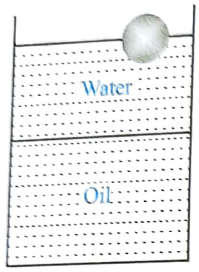
15. A solid block of volume $V = 10^{-3}$ m^3 and density $d = 800$ kg/m^3 is tied to one end of a string, the other end of which is tied to the bottom of the vessel. The vessel contains two immiscible liquids of densities $\rho_1 = 1000$ kg/m^3 and $\rho_2 = 1500$ kg/m^3. The solid block is immersed with 2/5th of its volume in the liquid of higher density and 3/5th in the liquid of lower density. The vessel is placed in an elevator which is moving up with an acceleration of $a = g/2$. Find the tension in the string (in N). [$g = 10$ m/s^2]

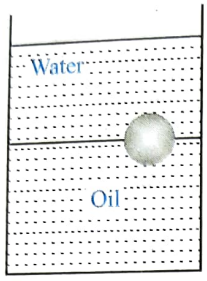
Archives

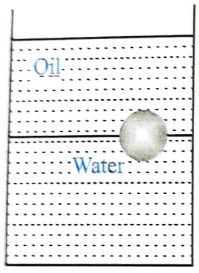
JEE MAIN

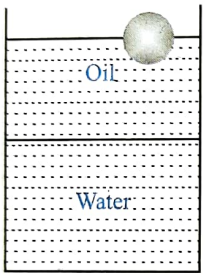
Single Correct Answer Type

1. A ball is made of a material of density ρ, where $\rho_{oil} < \rho < \rho_{water}$ with ρ_{oil} and ρ_{water} representing the densities of oil and water, respectively. The oil and water are immiscible. If the above ball is in equilibrium in a mixture of this oil and water, which of the following pictures represents its equilibrium position?

(1) 

(2) 

(3) 

(4) 

(AIEEE 2010)

2. A uniform cylinder of length L and mass M having cross-sectional area A is suspended, with its length vertical, from a fixed point by a massless spring, such that it is half submerged in a liquid of density σ at equilibrium position. The extension x_0 of the spring when it is in equilibrium is

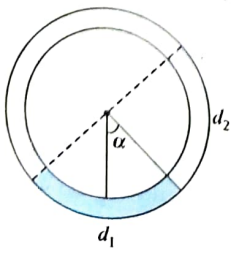
(1) $\frac{Mg}{k}\left(1-\frac{LA\sigma}{M}\right)$ (2) $\frac{Mg}{k}\left(1-\frac{LA\sigma}{2M}\right)$

(3) $\frac{Mg}{k}\left(1+\frac{LA\sigma}{M}\right)$ (4) $\frac{Mg}{k}$ **(JEE Main 2013)**

3. An open glass tube is immersed in mercury in such a way that a length of 8 cm extends above the mercury level. The open end of the tube is then closed and sealed and the tube is raised vertically up by additional 46 cm. What will be length of the air column above mercury in the tube now? (Atmospheric pressure = 76 cm of Hg)

(1) 38 cm (2) 6 cm
(3) 16 cm (4) 22 cm **(JEE Main 2014)**

4. There is a circular tube in a vertical plane. Two liquids which do not mix and of densities d_1 and d_2 are filled in the tube. Each liquid subtends 90° angle at centre. Radius joining their interface makes an angle α with vertical. Ratio d_1/d_2 is



(1) $\frac{1+\tan\alpha}{1-\tan\alpha}$ (2) $\frac{1+\sin\alpha}{1-\cos\alpha}$

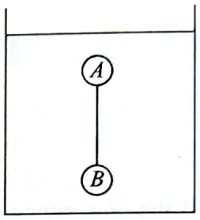
(3) $\frac{1+\sin\alpha}{1-\sin\alpha}$ (4) $\frac{1+\cos\alpha}{1-\cos\alpha}$

(JEE Main 2014)

JEE ADVANCED

Multiple Correct Answers Type

1. Two solid spheres and of equal volumes but of different densities and are connected by a string. They are fully immersed in a fluid of density. They get arranged into an equilibrium state as shown in the figure with a tension in the string. The arrangement is possible only if



(1) $d_A < d_F$ (2) $d_B > d_F$
(3) $d_A > d_F$ (4) $d_A + d_B = 2d_F$

(IIT-JEE 2011)

2. A solid sphere of radius R and density ρ is attached to one end of a mass-less spring of force constant k. The other end of the spring is connected to another solid sphere of radius R and density 3ρ. The complete arrangement is placed in a liquid of density 2ρ and is allowed to reach equilibrium. The correct statement(s) is (are)

(1) the net elongation of the spring is $\frac{4\pi R^3\rho g}{3k}$

(2) the net elongation of the spring is $\frac{8\pi R^3\rho g}{3k}$

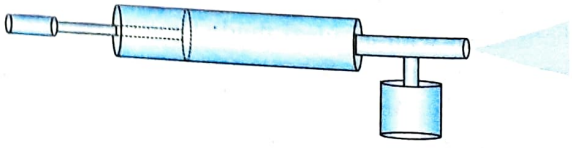
(3) the light sphere is partially submerged.
(4) the light sphere is completely submerged.

(JEE Advanced 2013)

Linked Comprehension Type

For Problems 1–2

A spray gun is shown in the figure where a piston pushes air out of a nozzle. A thin tube of uniform cross section is connected to the nozzle. The other end of the tube is in a small liquid container. As the piston pushes air through the nozzle, the liquid from the container rises into the nozzle and is sprayed out. For the spray gun shown, the radii of the piston and the nozzle are 20 mm and 1 mm respectively. The upper end of the container is open to the atmosphere.



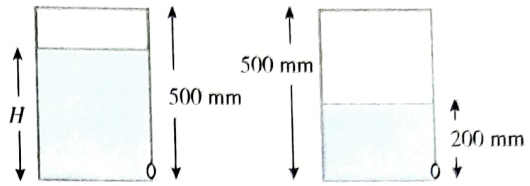
(JEE Advanced 2014)

1. If the piston is pushed at a speed of 5 mms^{-1}, the air comes out of the nozzle with a speed of
(1) 0.1 ms^{-1} (2) 1 ms^{-1}
(3) 2 ms^{-1} (4) 8 ms^{-1}

2. If the density of air is ρ_a and that of the liquid ρ_l, then for a given piston speed the rate (volume per unit time) at which the liquid is sprayed will be proportional to
(1) $\sqrt{\frac{\rho_a}{\rho_l}}$ (2) $\sqrt{\rho_a \rho_l}$
(3) $\sqrt{\frac{\rho_l}{\rho_a}}$ (4) ρ_l

Numerical Value Type

1. A cylindrical vessel of height 500 mm has an orifice (small hole) at its bottom. The orifice is initially closed and water is filled in it up to height H. Now the top is completely sealed with a cap and the orifice at the bottom is opened. Some water comes out from the orifice and the water level in the vessel becomes steady with height of water column being 200 mm. Find the fall in height (in mm) of water level due to opening of the orifice. (Take atmospheric pressure = 1.0×10^5 N/m^2, density of water = 1000 kg/m^3 and g = 10 m/s^2. Neglect any effect of surface tension.)



(IIT-JEE 2009)

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (1)	3. (4)	4. (1)	5. (3)
6. (2)	7. (2)	8. (4)	9. (3)	10. (1)
11. (3)	12. (4)	13. (3)	14. (3)	15. (1, 2)
16. (3)	17. (3)	18. (4)	19. (4)	20. (2)
21. (3)	22. (2)	23. (2)	24. (1)	25. (1)
26. (3)	27. (4)	28. (1)	29. (2)	30. (2)
31. (1)	32. (4)	33. (1)	34. (2)	35. (1)
36. (3)	37. (2)	38. (1)	39. (2)	40. (1)
41. (1)	42. (2)	43. (2)	44. (3)	45. (2)
46. (2)	47. (3)	48. (2)	49. (1)	50. (3)
51. (3)	52. (3)	53. (3)	54. (3)	55. (2)
56. (4)	57. (1)	58. (3)	59. (3)	60. (3)
61. (1)	62. (3)	63. (1)	64. (1)	65. (1)
66. (1)	67. (2)	68. (4)	69. (4)	70. (1)
71. (2)	72. (2)	73. (1)	74. (3)	75. (2)
76. (1)	77. (4)	78. (2)		

Multiple Correct Answers Type

1. (1),(2),(4)	2. (2),(4)	3. (1),(2),(3),(4)
4. (1),(2),(4)	5. (2),(3)	6. (1),(3),(4)
7. (1),(4)	8. (1),(2),(4)	9. (1),(3),(4)
10. (2),(3)	11. (1),(2),(3)	12. (2),(3),(4)
13. (1),(2),(3),(4)	14. (1),(2),(4)	15. (1),(2)
16. (1),(2),(3)	17. (3),(4)	18. (1),(3)
19. (1),(2),(4)	20. (1),(4)	21. (2),(4)
22. (1),(3),(4)	23. (1),(3)	24. (1),(2),(3)
25. (1),(2),(3)	26. (1),(3),(4)	27. (1),(3),(4)
28. (1),(2),(3),(4)	29. (1),(2),(4)	30. (1),(3),(4)
31. (1),(3)	32. (1),(3)	33. (1),(3)
34. (2),(3)		

Linked Comprehension Type

1. (4)	2. (3)	3. (2)	4. (2)	5. (1)
6. (3)	7. (4)	8. (1)	9. (3)	10. (1)
11. (3)	12. (2)	13. (4)	14. (1)	15. (2)
16. (2)	17. (1)	18. (1)	19. (3)	20. (2)
21. (3)	22. (1)	23. (3)	24. (1)	25. (4)
26. (1)	27. (2)	28. (1)	29. (2)	30. (3)
31. (3)	32. (1)	33. (4)	34. (3)	

Matrix Match Type

1. i. → b.; ii. → c.; iii. → a.
2. i. → c., d.; ii. → e.; iii. → b.; iv. → c., a.
3. i. → d.; ii. → c.; iii. → a.; iv. → a.
4. i. → d.; ii. → c.; iii. → a.; iv. → b.
5. i. → d.; ii. → c.; iii. → a.; iv. → b.
6. (2)

Numerical Value Type

1. (1)	2. (5)	3. (5)	4. (2)	5. (1)
6. (12.40)	7. (32)	8. (2.00)	9. (37.50)	10. (46.00)
11. (25.00)	12. (6.00)	13. (64.00)	14. (30.00)	15. (6.00)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3)	2. (2)	3. (3)	4. (1)

JEE Advanced

Multiple Correct Answers Type

1. (1),(2),(4) 2. (1),(4)

Linked Comprehension Type

1. (3) 2. (1)

Numerical Value Type

1. (6)

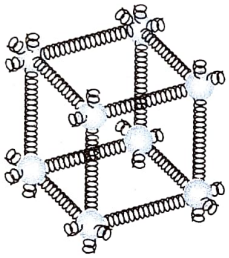
7 Elasticity

INTRODUCTION

The elastic behaviour of material plays an important role in engineering design and also constructing bridges, automobiles, roadways, etc. In this section, we will study the elastic behaviour and the mechanical properties of the solids.

ELASTICITY

We might have observed that a spring returns to its original shape when the force compressing or stretching it is removed. In fact, all materials change their shape in some way when they are squeezed or stretched, and many of them, such as rubber, return to their original shape when the action on the material is stopped. Such materials are said to be "elastic."



From an atomic viewpoint, elastic behaviour has its origin in the forces that atoms exert on each other. Atoms in a solid are held in place by forces between atoms termed as interatomic forces that can be modeled as springs. It is because of these atomic-level "springs" that a material tends to return to its initial shape once the forces that cause the deformation are removed. The interatomic forces that hold the atoms of a solid together are particularly strong, so considerable force has to be applied to stretch or compress a solid object.

In nature all rigid objects are somewhat elastic, even though they do not appear to be, compressing, pulling or twisting can deform a rigid object. If a rigid object is deformed by a small amount, it will return to its original size and shape when the deforming force is removed. If a rigid object is deformed past a point called its elastic limit, it will not return to its original size and shape but will remain permanently deformed and will break if the applied force continues.

ELASTIC PROPERTIES OF SOLIDS

STRESS AND STRAIN

All objects can be deformed to some extent. That is, it is possible to change the shape or the size (or both) of an object by applying external forces. When a force is applied on a body, there will be relative displacement of the particles and due to property of elasticity, an internal restoring force is developed which tends to restore the body to its original state. The internal restoring force acting per unit area of cross section of the deformed body is called stress. If external force F is applied on the area A of a body then,

$$\text{Stress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$$

Unit of stress: N/m^2 (SI); Dimensional formula: $[ML^{-1}T^{-2}]$

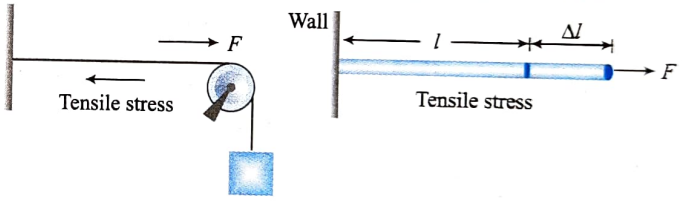
Stress developed in a body depends on how the external forces are applied over it. The result of a stress is **strain**, which is a measure of the degree of deformation. The ratio of change in shape to the original shape is called strain.

On this basis there are two types of stresses: Normal and Shear (or tangential) stress. When the force is applied normal to the surface of the object, normal stress develops. It is again of two types: Longitudinal and Bulk (or volume) stress.

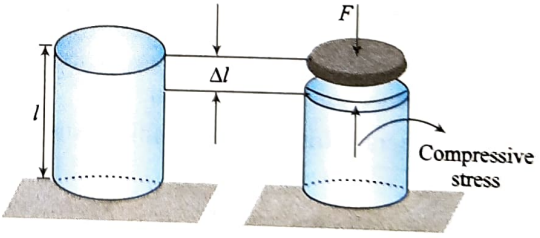
Longitudinal Stress

It occurs only in solids and comes into picture when one of the three dimensions viz. length, breadth, height is much greater than other two. Area taken for calculating the stress is the area of cross-section. Deforming force is applied parallel to the length and causes increase or decrease in length.

- Longitudinal stress produced due to increase in length of a body under a deforming force is called tensile stress.

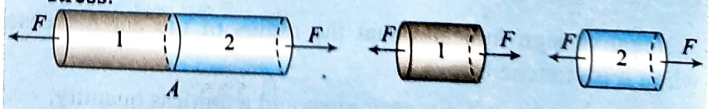


- Longitudinal stress produced due to decrease in length of a body under a deforming force is called compressive stress.



Note:

- If the two parts of the body on the two sides of cross-sectional area pull each other, then the normal stress is known as tensile stress.



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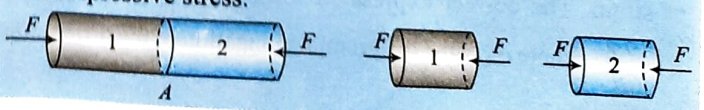
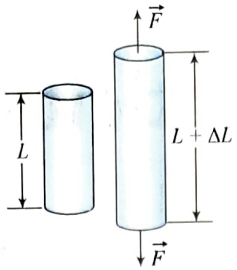


Longitudinal Strain

The result of longitudinal stress is longitudinal strain or linear strain.

If the deforming force produces a change in length alone, the strain produced in the body is called linear strain or tensile strain.



$$\text{Linear strain } (\varepsilon) = \frac{\text{Change in length}(\Delta L)}{\text{Original length}(L)}$$

Linear strain along the direction of deforming force is called longitudinal strain and, in a direction, perpendicular to force is called lateral strain.

Under certain conditions the strain produced in a rod is assumed to be uniform. These conditions are as follows:

(i) The specimen must be of uniform cross section.

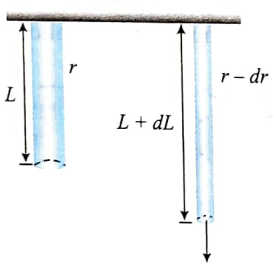
(ii) The material must be homogeneous.

(iii) The load must be axial so as to produce uniform stress.

Since strain represents a change in length divided by the original length, strain is a dimensionless quantity. Sometimes, its units are used as metre per metre (m/m). However, commonly it is used in percentage.

Poisson's Ratio

When a long bar is stretched by a force along its length then its length increases, its radius decreases as shown in the figure.



Lateral strain: The ratio of the change in its radius or diameter to its original radius or diameter is called lateral strain.

Longitudinal strain: The ratio of change in its length to its original length is called longitudinal strain.

The ratio of lateral strain to longitudinal strain is called Poisson's ratio (ν).

$$\text{i.e. } \nu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$\Rightarrow \nu = \frac{-dr/r}{dL/L}$$

Negative sign indicates that the radius of the bar decreases when it is stretched.

Poisson's ratio is a dimensionless and a unitless quantity.

Note: The above expression measures only the average value of strain. The correct expression for strain at any position is $\epsilon = \frac{d\chi}{dL}$, where $d\chi$ is the differential elongation of the differential length dL.

ILLUSTRATION 7.1

A steel wire 2 m long is suspended from the ceiling. When a mass is hung at its lower end, the increase in length recorded is 1 cm. Determine the strain in the wire.

Sol. Original length = 2 m

Increase in length = 1 cm = 1×10^{-2} m

$$\text{Longitudinal strain} = \frac{\text{Increase in length}}{\text{Original length}}$$

$$= \frac{1 \times 10^{-2}\ \text{m}}{2\ \text{m}} = 5 \times 10^{-3}$$

ILLUSTRATION 7.2

Two stretched cables both experience the same stress. The first cable has a radius of 3.5×10^{-3} m and is subject to a stretching force of 270 N. The radius of the second cable is 5.1×10^{-3} m. Determine the stretching force acting on the second cable.

Sol. The stress in either cable is the ratio F/A of the magnitude F of the stretching force to the cross-sectional area A of the cable. The cables have circular cross-sections, so the area of each cable is given by $A = \pi r^2$. Therefore, the stress in either cable is given by,

$$\text{Stress} = \frac{F}{\pi r^2} \qquad \ldots(\text{i})$$

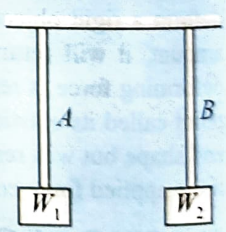
Now, set the stresses in the cables equal and obtain the stretching force in the second cable using Eq. (i).

$$\frac{F_2}{\pi r_2^2} = \frac{F_1}{\pi r_1^2}$$

$$\text{or } F_2 = \frac{F_1 r_2^2}{r_1^2} = \frac{(270\text{N})(5.1\times10^{-3}\ \text{m})^2}{(3.5\times10^{-3}\ \text{m})^2} = 570\,\text{N}$$

ILLUSTRATION 7.3

Two rods A and B, each of equal length but different materials are suspended from a common support as shown in the figure. The rods A and B can support a maximum load of $W_1 = 600$ N and $W_2 = 6000$ N, respectively. If their cross-sectional areas are $A_1 = 10$ mm^2 and $A_2 = 1000$ mm^2, respectively, then identify the stronger material.



Sol. At first sight, it looks that rod B is stronger as it supports a larger load. But we cannot compare strengths without having a common basis of comparison. Since the rods have different cross-sectional area, therefore, the strengths can be compared by determining the load capacity per unit area.

For rod A

$$\sigma_1 = \frac{W_1}{A_1} = \frac{600\,\text{N}}{10\,\text{mm}^2} = \frac{600}{10\times10^{-6}} = 60\times10^6\ \text{Nm}^{-2}$$

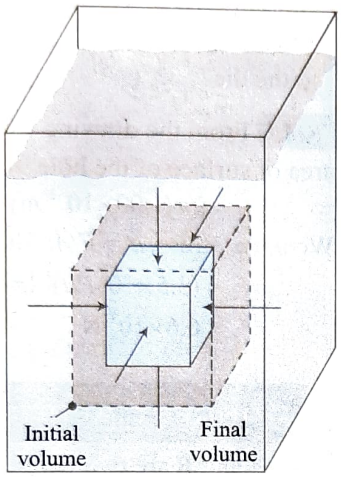
For rod B

$$\sigma_2 = \frac{W_2}{A_2} = \frac{6000\,\text{N}}{1000\,\text{mm}^2} = \frac{6000}{1000\times10^{-6}} = 6\times10^6\ \text{Nm}^{-2}$$

Since $\sigma_1 > \sigma_2$, therefore, the material of rod A is stronger than that of rod B.

Bulk or Volume Stress

When a compressive force is applied along one dimension of a solid, the length of that dimension is decreased. It is also possible to apply compressive forces so that the size of every dimension (length, width, and depth) decreases, leading to a decrease in volume, as figure illustrates. This kind of overall compression occurs, for example, when an object is submerged in a liquid, and the liquid presses in inward direction everywhere on the object.



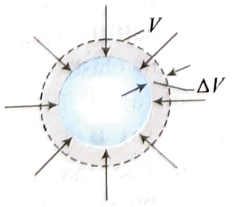
The forces acting in such situations are applied perpendicular to every surface, and it is more convenient to speak of the perpendicular force per unit area, rather than the amount of any one force in particular. The magnitude of the perpendicular force per unit area is called the **bulk or volume stress**.

Important Points:

- It occurs in solids, liquids or gases.
- In case of fluids only bulk stress can be found.
- It produces change in volume and density, the shape remains the same.
- Deforming force is applied normal to surface at all points.
- Area for calculation of stress is the complete surface area perpendicular to the applied forces.
- It is equal to change in pressure because change in pressure is responsible for change in volume.

Volumetric Strain

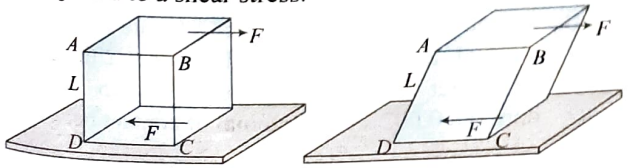
If the force causing deformation produces a change in volume alone the strain produced in the body is called volumetric strain.



$$\text{Volumetric strain} = \frac{\text{Change in volume}(\Delta V)}{\text{Original volume}(V)}$$

Shear or Tangential Stress

One more type of deformation occurs when an object is subjected to a force parallel to one of its faces while the opposite face is held fixed by another force. The stress in this case is called a **shear stress**. If the original object is a rectangular block, a shear stress results in a shape whose cross section is a parallelogram. For example: A book pushed sideways (as shown in figure) is being subjected to a shear stress.



To a first approximation (for small distortions), no change in volume occurs with this deformation. It comes into picture when successive layers of solid move on each other i.e., when there is a relative displacement between various layers of solid.

Note that:

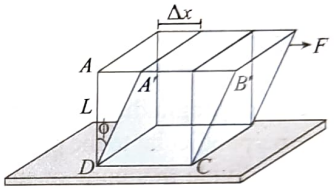
(i) Here the force causing the deformation is applied tangential to one of the faces.

(ii) Area for calculation is the area of the face on which force is applied.

(iii) It produces change in shape, but volume remain the same.

Shearing Strain

If the force causing deformation produces a change in the shape of the body without changing its volume, strain produced is called shearing strain. This shearing force causes the consecutive horizontal layers of the cube to be slightly displaced or sheared relative to one another, such as AD or BC in the cube is rotated through an angle ϕ by this shear.



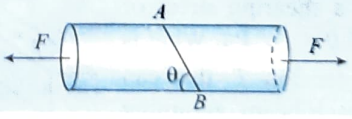
It is defined as angle ϕ in radians through which a plane (line) perpendicular to the fixed surface of the cubical body gets turned under the effect of tangential force.

$$\phi = \frac{\Delta x}{L}$$

Note: A shearing strain is never accompanied by a change in volume, i.e., in a shear strain, the body gets distorted, resulting only in an alternation in its shape but not in size.

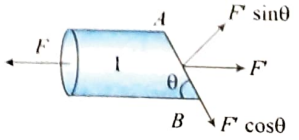
ILLUSTRATION 7.4

Two equal and opposite forces F and $-F$ act on a rod of uniform cross-sectional area A, as shown in the figure. Find the **(i)** shearing **(ii)** longitudinal stress on the section AB.



Sol.

(i) As the net force acting on the rod is zero, it is in equilibrium. Let the tension in the segment AB be F'. Applying Newton's second law for the segment 1.



$$F_{net} = F' - F = ma$$

where $a = 0$

This gives $F' = F$

Resolving the force F' parallel and perpendicular to the given area AB ($= A'$, say), we have

$$F_{tan} = F'\cos\theta \text{ and } F_{long} = F'\sin\theta$$

Then $P_{shearing} = \dfrac{F_{tan}}{A'} = \dfrac{F'\cos\theta}{A'}$

where $A' = \dfrac{A}{\sin\theta}$ and $F' = F$

This gives $P_{shearing} = \dfrac{F\sin\theta\cos\theta}{A}$

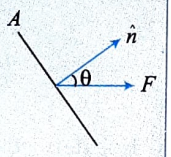
(ii) Similarly, $P_{long} = \dfrac{F_{long}}{A'} = \dfrac{F'\sin\theta}{A'}$

where $A' = \dfrac{A}{\sin\theta}$ and $F' = F$

This gives $P_{long} = \dfrac{F}{A}\sin^2\theta$

Important Points:

- To find the linear (or longitudinal) stress, resolve the force perpendicular to the plane of a given area A, divide this component $F_\perp$ (or F_{long}) by the area A to obtain



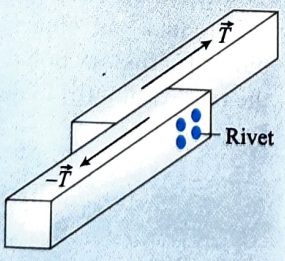
$$P_{long} = \frac{F_\perp}{A} = \frac{F\cos\theta}{A}$$

- To find shearing stress, resolve F parallel to the plane of the given area and then divide $F_\parallel$ by the area A to obtain

$$P_{shearing} = \frac{F_\parallel}{A} = \frac{F\sin\theta}{A}$$

ILLUSTRATION 7.5

Two metal beams are joined together by four rivets, as the drawing indicates. Each rivet has a radius of 5.0×10^{-3} m and is to be exposed to a shearing stress of no more than 5.0×10^8 Pa. What is the maximum tension $\vec{T}$ that can be applied to each beam, assuming that each rivet carries one-fourth of the total load?



Sol. The shearing stress is equal to the force per unit area applied to the rivet. Thus, when a shearing stress of 5.0×10^8 Pa is applied to each rivet, the force experienced by each rivet is

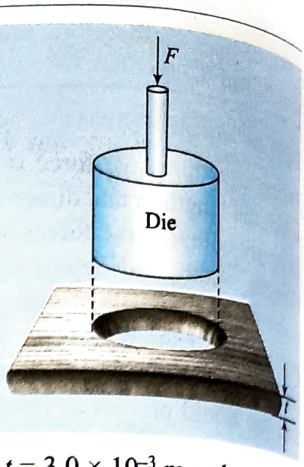
$$F = (\text{Stress})A = (\text{Stress})(\pi r^2)$$

$$= (5.0\times10^8\,\text{Pa})\left[\pi(5.0\times10^{-3})^2\right] = 3.9\times10^4\,\text{N}$$

Therefore, the maximum tension T that can be applied to each beam, assuming that each rivet carries one-fourth of the total load, is $4F = 1.6 \times 10^5$ N.

ILLUSTRATION 7.6

A die is designed to punch holes with a radius of $R = 1.00 \times 10^{-2}$ m in a metal sheet of thickness $t = 3.0 \times 10^{-3}$ m thick, as the figure illustrates. To punch through the sheet, the die must exert a shearing stress of $\sigma = 3.5 \times 10^8$ Pa. What force $\vec{F}$ must be applied to the die?



Sol. From the drawing we have $t = 3.0 \times 10^{-3}$ m and area of surface of the hole $A = 2\pi rt$

$\Rightarrow$ $A = 2\pi(1.00\times10^{-2}\text{ m})(3.0\times10^{-3}\text{ m})$

We have, stress, $\sigma = F/A$. Therefore, $F = (\sigma)A$

$$F = (3.5\times10^8\,Pa)[2\pi(1.00\times10^{-2}\text{ m})(3.0\times10^{-3})\text{m}]$$
$$= 6.6\times10^4\text{ N}$$

CONCEPT APPLICATION EXERCISE 7.1

1. A and B are two wires. The radius of A is twice that of B. they are stretched by the same load. Then the stress on B is
   (a) equal to that on A (b) four times that on A
   (c) two times that on A (d) half that on A
2. A cube of aluminium of sides 0.1 m is subjected to a shearing force of 100 N. The top face of the cube is displaced through 0.02 cm with respect to the bottom face. The shearing strain would be
   (a) 0.02 (b) 0.1
   (c) 0.005 (d) 0.002
3. A steel wire can support a maximum load of W before reaching its elastic limit. How much load can another wire, made out of identical steel, but with a radius one half the radius of the first wire, support before reaching its elastic limit?
   (a) W (b) $W/2$
   (c) $W/4$ (d) $4W$
4. The maximum load a wire can withstand without breaking, when its length is reduced to half of its original length, will
   (a) be double (b) be half
   (c) be four times (d) remain same.
5. The dimensions of four wires of the same material are given below. In which wire the increase in length will be maximum when the same tension is applied?
   (a) Length 100 cm, diameter 1 mm
   (b) Length 200 cm, diameter 2 mm
   (c) Length 300 cm, diameter 3 mm
   (d) Length 50 cm, diameter 0.5 mm
6. (i) Estimate the force with which a karate master strikes a board, assuming the hand's speed at the moment of impact is 10.0 m s^{-1}, decreasing to 1.00 m s^{-1} during a 0.002 s time internal of contact between the

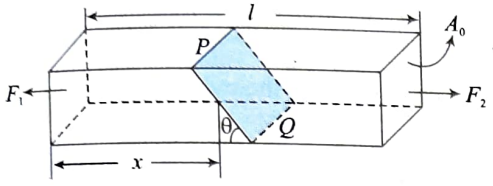
hand and the board. The mass of his hand and arm is 1.00 kg.

(ii) Estimate the shear, assuming this force it exerted on a 1.00 cm thick pine board that is 10.0 cm wide.

(iii) If the maximum shear stress that a pine board can support before breaking is 3.60×10^6 N/m^2, will the board break?

7. The diameter of a brass rod is 4 mm having Young's modules 9×10^9 N/m^2. Find the force required to stretch the rod by 0.1% of its length.

8. A rod is pulled with forces F_1 and F_2 at its ends. Find the:



(i) tension at a distance x from left end of the rod,

(ii) longitudinal and shearing stresses at the given cross-section PQ if the cross-section of the rod is A_0,

(iii) conditions for maximum longitudinal and shearing stresses at the given cross-section PQ.

ANSWERS

1. (b) **2.** (d) **3.** (c) **4.** (d) **5.** (d)

6. (i) 4500 N (ii) 4.5 MN/m^2 (iii) Yes

7. 1.13×10^2 N

8. (i) $(F_2-F_1)x/l + F_1$ (ii) $T\sin2\theta/2A_0$, $T\sin^2\theta/A_0$

MODULUS OF ELASTICITY AND HOOKE'S LAW

It is found that, for sufficiently small stresses, stress is proportional to strain, a relationship first discovered by Robert Hooke and now referred to as Hooke's law. According to this law, within the elastic limit, the stress is proportional to the strain.

The constant of proportionality depends on the material being deformed and on the nature of the deformation. We call this proportionality constant the **elastic modulus.** The elastic modulus is therefore defined as the ratio of the stress to the resulting strain

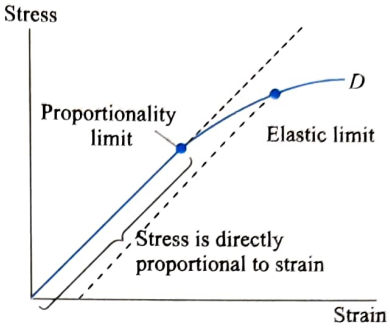
i.e. Stress $\propto$ Strain or $\dfrac{\text{stress}}{\text{strain}} = \text{constant} = E$

The constant E is called the modulus of elasticity.

- Its value depends upon the nature of material of the body and the manner in which the body is deformed.
- Its value depends upon the temperature of the body.
- Its value is independent of the dimensions (length, volume etc.) of the body.

As long as stress remains proportional to the strain, a plot of stress versus strain will be a straight line. The point on the graph where the material begins to deviate from straight-line behaviour is called the "proportionality limit". Beyond the proportionality limit stress and strain are no longer directly proportional. For relatively small stresses, the bar returns to its initial length when the force is removed. If the stress exceeds the "elastic limit" of the material, the object will not return to its original size and shape once the stress is removed. The **elastic limit** is the point beyond which the object no longer returns to its original size and shape when the stress is removed; the object becomes permanently deformed.

It is possible to exceed the elastic limit of a substance by applying a sufficiently large stress as seen in figure.



Initially, a stress-versus strain curve is a straight line. As the stress increases, however, the curve is no longer in the shape of a straight line. When the stress exceeds the elastic limit, the object is permanently distorted and does not return to its original shape after the stress is removed. As the stress is increased even further, the material ultimately breaks

The elastic modulus in general relates what is done to a solid object (a force is applied) to how that object responds (deformation in object). We consider three types of deformation and define an elastic modulus for each:

1. Young's modulus measures the resistance of a solid to a change in its length.
2. Shear modulus measures the resistance to motion of the planes within a solid parallel to each other.
3. Bulk modulus measures the resistance of solids or liquids to changes in their volume.

YOUNG'S MODULUS (Y): ELASTICITY IN LENGTH

Let us take a long bar of cross-sectional area A and initial length L_i that is clamped at one end. When an external force is applied perpendicular to the cross section, internal molecular forces in the bar resist the distortion ("stretching"), but the bar reaches a situation of equilibrium in which its final length L_f is greater than L_i and in which the external force is exactly balanced by the internal forces. In such a situation, the bar is said to be stressed. We define the tensile stress as the ratio of the magnitude of the external force F to the cross-sectional area A, where the cross section is perpendicular to the force vector. In this case the tensile strain is defined as the ratio of the change in length ΔL to the original length L_i. We define Young's modulus by a combination of these two ratios:

$$Y = \frac{\text{Normal stress}}{\text{longitudinal strain}} = \frac{F/A}{\Delta L/L_i} = \frac{FL_i}{A\Delta L} \quad \text{...(i)}$$

Equation (i) may be rearranged as

$$\Delta L = \left(\frac{FL_i}{YA}\right) \quad \text{...(ii)}$$

Important Points:

- If force is applied on a wire of radius r and length L by hanging a weight of mass M, then
$$Y = \frac{MgL}{\pi r^2 l}$$
- Extension is proportional to the deforming force and length of the rod, i.e., greater the force larger the deformation; longer the rod larger the extension.
- Extension is inversely proportional to the cross-sectional area A and the modulus of elasticity, i.e., a thicker rod produces less deformation and a stiffer material produces less deformation.

ILLUSTRATION 7.7

A tow truck is pulling a car out of a ditch by means of a steel cable that is 9.1 m long and has a radius of 0.50 cm. When the car just begins to move, the tension in the cable is 890 N. How much has the cable stretched? Young's modulus of steel, $Y_{steel} = 2.0\times10^{11}\ \text{N/m}^2$

Sol. When the tow truck pulls the car out of the ditch, the cable stretches, and a tension exists in it. This tension is the force that acts on the car. The amount ΔL that the cable stretches depends on the tension F, the length L_0 and cross-sectional area A of the cable, as well as Young's modulus Y for steel.

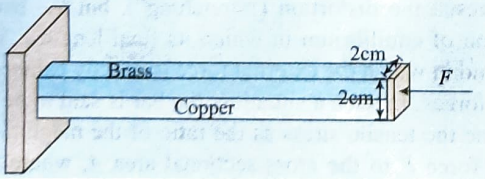
$$Y = \frac{F/A}{\Delta L/L_0}$$

Hence change in the length of cable,

$$\Delta L = \frac{FL_0}{AY} = \frac{(890\ \text{N})(9.1\ \text{m})}{\pi(0.50\times10^{-2}\ \text{m})^2(2.0\times10^{11}\ \text{N/m}^2)} = 5.2\times10^{-4}\ \text{m}$$

ILLUSTRATION 7.8

A bimetallic rod made of brass and copper has a length of 0.5 m and is of rectangular cross-section as shown in figure. If the rod is subjected to a compressive force of $F = 5\times10^4$ N normal to its cross-section, then find the compressions in each of its two metallic parts. Given that Young's modulus of brass $Y_{brass} = 9\times10^{10}\ \text{N m}^{-2}$ and Young's modulus of copper $Y_{copper} = 11\times10^{10}\ \text{N m}^{-2}$. Which way will the rod bend?



Sol. Let l_{brass} and l_{copper} are the respective compressions in the two wires, then

$$l_{brass} = \frac{FL}{aY_{brass}} = \frac{5\times10^4\times0.5}{2\times10^{-4}\times9\times10^{10}} = 1.389\times10^{-3}\ \text{m}$$

and $$l_{copper} = \frac{FL}{aY_{copper}} = \frac{5\times10^4\times0.5}{2\times10^{-4}\times11\times10^{10}} = 1.136\times10^{-3}\ \text{m}$$

Since length of the brass rod becomes smaller than that of the copper rod, it will bend towards brass.

ILLUSTRATION 7.9

A solid steel cylinder is standing vertically on the floor. The length of the cylinder is 3.6 m and its radius is 65 cm. When an object is placed on top of the cylinder, the cylinder compresses by an amount of 5.7×10^{-7} m. What is the weight of the object? (Given $Y_{steel} = 2.0\times10^{11}\ \text{N/m}^2$)

Sol. When the object is placed on top of the cylinder, the cylinder is compressed due to the weight of the object. The amount ΔL of compression depends on the weight F, the length L_0 and cross-sectional area A of the cylinder, as well as Young's modulus Y for steel.

$\therefore$ The weight F of the object is given by $F = Y\left(\frac{\Delta L}{L_0}\right)A$

The circular cross section of the cylinder is a circle of area,

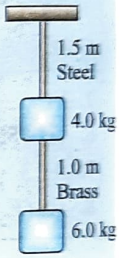
$$A = \pi r^2$$

$$\Rightarrow \quad F = (2.0\times0^{11}\ \text{N/m}^2)\left(\frac{5.7\times10^{-7}\ \text{m}}{3.6\ \text{m}}\right)\pi(65\times10^{-2}\ \text{m})^2$$

$$= 4.2\times10^4\ \text{N}$$

ILLUSTRATION 7.10

Two wires of diameter 0.25 cm, one made of steel and the other made of brass are loaded as shown in figure. The unloaded length of steel wire is 1.5 m and that of brass wire is 1.0 m. Compute the elongations of the steel and the brass wires.
(Young's modulus for brass = $9\times10^{10}\ \text{N m}^{-2}$ and Young's modulus for steel = $2\times10^{11}\ \text{N m}^{-2}$)



Sol. **For steel wire:**

Total force on steel wire: $F_1 = (4+6)\times10 = 100\ \text{N}$

Radius of steel wire: $r_1 = \left(\frac{0.25}{2}\right)\text{cm} = 0.125\times10^{-2}\ \text{m}$

Young's modulus of steel: $Y_1 = 2.0\times10^{11}\ \text{Pa}$

As $$Y_1 = \frac{F_1\times l_1}{A_1\times\Delta l_1} = \frac{F_1\times l_1}{\pi r_1^2\times\Delta l_1}$$

Elongation of the steel wire, $$\Delta l_1 = \frac{F_1\times l_1}{\pi r_1^2\times Y_1}$$

$$= \frac{100\times1.5\times7}{22\times(0.125\times10^{-2})^2\times2\times10^{11}} = 1.49\times10^{-4}\ \text{m}$$

For brass wires:

Total force on brass wire: $F_2 = 6\times10 = 60\ \text{N}$

Radius of brass wire $r_2 = \left(\frac{0.25}{2}\right)\text{cm} = 0.125\times10^{-2}\ \text{m}$

$Y_2 = 9\times10^{11}\ \text{Pa}$, $l_2 = 1.0\ \text{m}$, $\Delta l_2 = ?$

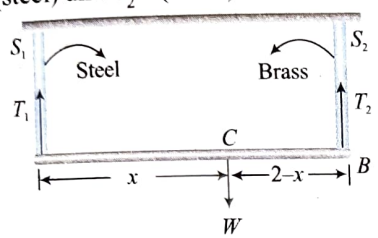
Elongation of the brass wire, $$\Delta l_2 = \frac{F_2\times l_2}{\pi r_2^2\times Y_2}$$

$$= \frac{60\times1.0\times7}{22\times(0.125\times10^{-2})^2\times(0.9\times10^{11})} = 1.3\times10^{-4}\ \text{m}$$

ILLUSTRATION 7.11

A light rod of length 2 m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section 10^{-3} m^2 and the other is of brass of cross-section 2×10^{-3} m^2. Find out the position along the rod at which a weight may be hung to produce:
(a) equal stresses in both the wires,
(b) equal strains in both wires.
Young's modulus for brass = 10^{11} N m^{-2} and
Young's modulus for steel = 2×10^{11} N m^{-2}

Sol. Figure shows a light rod AB suspended by means of the two wires S_1A (steel) and S_2B (brass).



Let a_1, and a_2 be the areas of cross-section of the two wires. Suppose that a weight W is suspended from a point C at a distance x from the end A of the rod. As a result, let T_1 and T_2 be the tensions in the wires S_1A and S_2B respectively.

(a) When stresses in the two wires are equal:

Stress in the steel wire $= \dfrac{T_1}{a_1}$

and stress in the brass wire $= \dfrac{T_2}{a_2}$

Since stresses in the two wires are equal,

$$\frac{T_1}{a_1} = \frac{T_2}{a_2} \text{ or } \frac{T_1}{T_2} = \frac{a_1}{a_2} = \frac{10^{-3}}{2\times10^{-3}}$$

or $\dfrac{T_1}{T_2} = 0.5$ ...(i)

As the suspended system of the rod and the two wires is in equilibrium,

$$T_1 \times AC = T_2 \times BC \text{ or } T_1 \times x = T_2 (2-x)$$

or $\dfrac{T_1}{T_2} = \dfrac{2-x}{x}$ ...(ii)

From Eqs. (i) and (ii), we have

$$\frac{2-x}{x} = 0.5 \text{ or } 1.5\,x = 2$$

or $x = 1.333$ m $= 133.3$ cm (from steel wire)

(b) When strains in the two wires are equal:

Now, $Y = \dfrac{\text{stress}}{\text{strain}}$ or $\text{strain} = \dfrac{\text{stress}}{Y}$

Therefore, strain in the steel wire $= \dfrac{T_1/a_1}{Y_1} = \dfrac{T_1}{a_1 Y_1}$

and strain in the brass wire $= \dfrac{T_2/a_2}{Y_2} = \dfrac{T_2}{a_2 Y_2}$

Since strains in the two wires are equal, $\dfrac{T_1}{a_1 Y_1} = \dfrac{T_2}{a_2 Y_2}$

or $\dfrac{T_1}{10^{-3}\times2\times10^{11}} = \dfrac{T_2}{2\times10^{-3}\times10^{11}}$

or $T_1 = T_2$

Again, for the equilibrium of the system,

$T_1 \times AC = T_2 \times BC$

or $T_1 \times x = T_2 \times (2-x)$

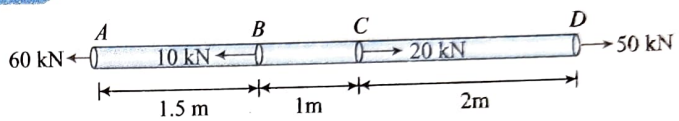
or $x = 2 - x$ $(\because T_1 = T_2)$

or $x = 1$ m $= 100$ cm (from steel wire)

ILLUSTRATION 7.12

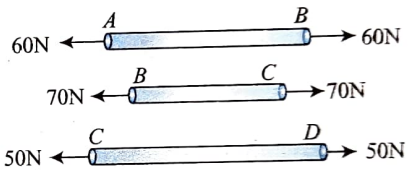
A steel rod of cross-sectional area 1 m^2 is acted upon by forces shown in figure. Determine the total elongation of the bar. Take $Y = 2.0\times10^{11}$ N/m^2.

Sol. The action of forces on each part of rod is shown in figure.



We know that the extension due to external force F is given by,

$$\Delta L = \frac{FL}{AY}$$



$$\Delta L_{AB} = \frac{(60\times10^3)\times1.5}{1\times2\times10^{11}} = 4.5\times10^{-7} \text{ m}$$

$$\Delta L_{BC} = \frac{(70\times10^3)\times1}{1\times2\times10^{11}} = 3.5\times10^{-7} \text{ m}$$

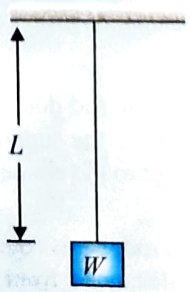
and $\Delta L_{CD} = \dfrac{(50\times10^3)\times2}{1\times2\times10^{11}} = 5.0\times10^{-7}$ m

The total extension,

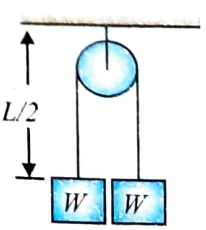
$$\Delta L = \Delta L_{AB} + \Delta L_{BC} + \Delta L_{CD}$$
$$= 4.5\times10^{-7} + 3.5\times10^{-7} + 5.0\times10^{-7} = 13\times10^{-7} \text{ m}$$

ILLUSTRATION 7.13

When a weight W is hung from one end of a wire of length L (other end being fixed), the length of the wire increases by l. If the same wire is passed over a pulley and two weights W each are hung at the two ends, what will be the total elongation in the wire?



(a)



(b)

Sol. In case (a), let Y be Young's modulus of the material of the wire. If a is its area of cross section, then we can write

$$Y = \frac{F/a}{l/L}\frac{F/a}{l/L} = \frac{WL}{al} \qquad (\because F = W)$$

Increase in length of wire, $l = \dfrac{WL}{aY}$

In case (b), we can treat either segment of length $L/2$ in Fig. (b) as if one of its ends is fixed while the other end is attached to the load W because the point M remains stationary, i.e., fixed all the time due to symmetrical load. Thus, when the wire is passed over the pulley, let l' be the increase in the length of each segment. Since each segment is of length $L/2$, we have

$$Y = \frac{W(L/2)}{al'}$$

Increase in length of wire of one side of pulley, $l' = \dfrac{1}{2}\dfrac{WL}{aY} = \dfrac{l}{2}$

Therefore, increase in length of both the segments of the wire $= l' + l' = l/2 + l/2 = l$.

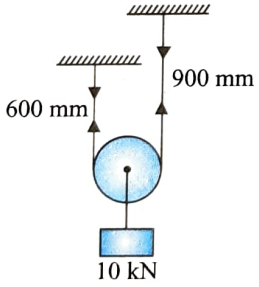
So the increase in length remains the same.

ILLUSTRATION 7.14

A load of 10 kN is supported from a pulley which in turn is supported by a rope of sectional area 1×10^3 mm^2 and modulus of elasticity 10^3 Nmm^{-2}, as shown in figure. Neglecting the friction at the pulley, determine the deflection of the load.

Sol. Let T be the tension in the rope. Then

$2T = 10$ kN $\Rightarrow T = 5$ kN



Hence longitudinal stress in the rope is

$$\sigma = \frac{T}{A} = \frac{5\text{ kN}}{10^3\text{ mm}^2} = 5\text{ N/mm}^2$$

Extension in the rope $= \dfrac{\text{Stress}}{Y} \times L$

$$= -\frac{5\text{ N/mm}^2}{10^3\text{ N/mm}^2} \times 1500\text{ mm} = 7.5\text{ mm}$$

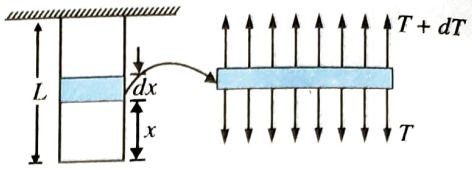
Therefore, deflection of the load $\delta = \dfrac{7.5}{2} = 3.75$ mm.

ILLUSTRATION 7.15

A rod of uniform cross-sectional area A and length L has a weight W. It is suspended vertically from a fixed support. If the material of the rod is homogeneous and its modulus of elasticity is Y, then determine the total elongation produced in the rod due to its own weight.

Sol. In this case, different parts of the rod does not elongate to the same extent. The element closer to the support elongates more as the stress is higher with respect to the elements closer to the free end of the rod.

To determine total elongation of the rod, let us consider a small element of differential length dx at a distance x from the free end of the rod, as shown in the figure.



The stress at the position of this element is produced by the weight of the rod of length x lying below it, i.e., $(W/L)x$.

Therefore, stress at this section

$$\sigma = \frac{(W/L)x}{A} = \frac{Wx}{AL}$$

The elongation $d\delta$ produced in the differential element dx is

$$d\delta = \frac{\sigma}{Y}dx = \frac{W}{YAL}x\,dx$$

Thus, total elongation produced in the rod can be calculated as

$$\delta = \int d\delta$$

$$\delta = \frac{W}{YAL}\int_0^L x\,dx = \frac{W}{YAL}\left[\frac{x^2}{2}\right]_0^L$$

or $$\delta = \frac{1}{2}\frac{WL}{YA} \qquad \text{...(i)}$$

Note: Equation (i) may be visualized as

$$\delta = \frac{(W/2)L}{YA}$$

which suggests that a rod of uniform mass is equivalent to a massless rod with a concentrated load equal to half weight $(W/2)$ of the rod acting at its lower end, as shown in figure.

If σ is the density of the material of the rod, then Eq. (i) may be written as

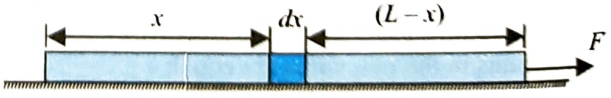
$$\delta = \frac{1}{2}\frac{\rho g L^2}{Y}$$

It indicates that the deflection of the rod is proportional to the square of the length of the rod.

ILLUSTRATION 7.16

A uniform rod of length L and mass M is pulled horizontally on a smooth surface with a force F. Determine the elongation of rod if Young's modulus of the material is Y.

Sol. Under the action of the external force, the rod accelerates with $a = F/m$. The magnitude of stress decreases as one moves away from the point of application of the force. Therefore, the strain also decreases as one moves towards the free end.



To determine the total elongation of the rod, let us consider a small element of length dx at a distance x from the free end of the rod. The magnitude of force at this section is $F' = Fx/L$.

Therefore, the stress at this section is

$$\sigma = \frac{F'}{A} = \frac{F}{A}\frac{x}{L}$$

and elongation $d\delta$ produced in this differential element is

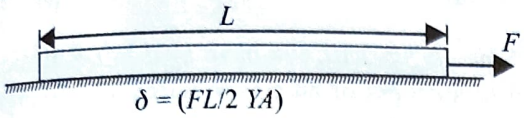
$$d\delta = \frac{F}{YAL}x\,dx$$

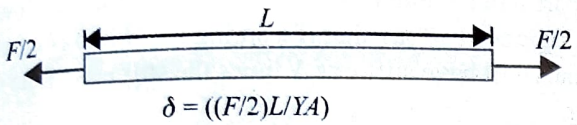
Thus, total elongation is

$$\delta = \frac{F}{YAL}\int_0^L x\,dx = \frac{F}{YAL}\left[\frac{x^2}{2}\right]_0^L$$

or $\delta = \frac{1}{2}\frac{FL}{YA}$

Note: The elongation produced in the rod is equivalent to the case when a rod of length L is being pulled with two equal and opposite forces, each of magnitude $F/2$, as shown in figure.

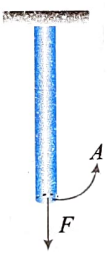




If the rod is pulled with two equal and opposite forces each of magnitude F, then the elongation in the rod is FL/YA. Here, since the rod is being pulled only by a single force, therefore, deformation in the rod reduces to one-half.

BREAKING OF WIRE

When the wire is loaded beyond the elastic limit, then the strain on it increases much more rapidly. The maximum stress after which the wire begins to flow and breaks, is called its breaking stress or its tensile strength and the force by application of which the wire breaks is called the breaking force. Breaking force depends upon the area of cross-section of the wire i.e., Breaking force $\propto A$



$\therefore$ Breaking force, $F = P \times A$

Here P is a constant of proportionality and known as breaking stress.

Breaking stress is a constant for a given material and it does not depend upon the dimension (length or thickness) of wire.

Important Points:

- If a wire of length L is cut into two or more parts, then again, its each part can hold the same weight. Since breaking force is independent of the length of wire.
- If a wire can bear a maximum force F, then a wire of same material but of double thickness can bear a maximum force of $4F$.
- The working stress is always kept lower than that of a breaking stress.

  So that safety factor $= \frac{\text{breaking stress}}{\text{working stress}}$, may have large value.
- **Breaking of wire under its own weight.**

  Breaking force = Breaking stress × Area of cross-section

  Weight of wire = $Mg = ALdg = PA$ [P =Breaking stress]

  [As mass = volume × density = ALd]

  $\therefore \quad Ldg = P \quad \therefore L = \frac{P}{dg}$

  This is the length of wire if it is to break by its own weight.

ILLUSTRATION 7.17

The femur is a bone in the leg whose minimum cross-sectional area is about 4.0×10^{-4} m^2. A compressional force more than 6.8×10^4 N will fracture this bone.

(a) Find the maximum stress that this bone can withstand.

(b) What is the strain that exists under a maximum-stress condition? ($Y = 9.4 \times 10^9$ N/m^2 for bone compression)

Sol.

(a) The maximum stress is equal to the maximum compressional force divided by the cross-sectional area of the femur.

$$\text{Maximum stress} = \frac{F}{A} = \frac{6.8\times10^4 \text{ N}}{4.0\times10^{-4} \text{ m}^2} = 1.7\times10^8 \text{ N/m}^2$$

(b) The strain $\Delta L/L$ is given by

$$\frac{\Delta L}{L_0} = \frac{1}{Y}\left(\frac{F}{A}\right) = \left(\frac{1}{9.4\times10^9 \text{ N/m}^2}\right)(1.7\times10^8 \text{ N/m}^2) = 1.8\times10^{-2}$$

FORCE CONSTANT OR STIFFNESS OF WIRE

The force required to produce a unit elongation in a wire is called the force constant or stiffness of the material of the wire.

It is denoted by k, $k = \frac{F}{l}$ ...(i)

but from the definition of Young's modulus,

$$\frac{F}{l} = \frac{YA}{L} \quad \text{...(ii)}$$

From (i) and (ii), $k = \frac{YA}{L}$

It is clear that the value of force constant depends upon the dimension (length and area of cross section) and material of a substance.

Note: Longer the rod, lesser the stiffness and thicker the rod, greater the stiffness.

ILLUSTRATION 7.18

Length of a rod under the tension T_1, becomes L_1 and under the tension T_2, its length becomes L_2. Find the actual length of the rod.

Sol. Let the actual length of the wire be L. Length of a rod under the tension T_1, becomes L_1 then $L_1 = L + l_1$,

where l_1 is the increase in length of the rod under tension T_1. We have,

$$T_1 = kl_1 \Rightarrow l_1 = \frac{T_1}{k}$$

$$\Rightarrow \quad L_1 = L + \frac{T_1}{k} \quad \text{...(i)}$$

Similarly, $L_2 = L + l_2$

$$\Rightarrow \quad L_2 = L + \frac{T_2}{k} \quad \text{...(ii)}$$

From (i) and (ii), we get $L = \frac{L_1T_2 - L_2T_1}{T_2 - T_1}$

ILLUSTRATION 7.19

An archer, about to shoot an arrow, is applying a force of +240 N to a drawn bowstring. The bow behaves like an ideal spring whose spring constant is 480 N/m. What is the displacement of the bowstring?

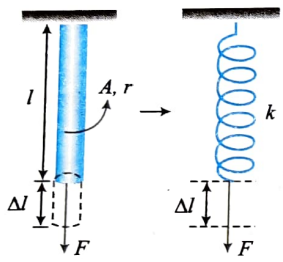
Sol. The applied force required to stretch the bow string is given by $F = kx$, where k is the spring constant and x is the displacement of the string.

Solving for the displacement, the displacement of the bow string,

$$x = \frac{F_x^{\text{Applied}}}{k} = \frac{+240\text{ N}}{480\text{ N/m}} = +0.50\text{ m}$$

ANALOGY OF ROD AS A SPRING

A thin wire or a rod can be imagined as a series combination of arrays of molecular springs. When we pull a wire, we really pull the springs. Let us take an elastic rod of length l, area of cross-section A and modulus of elasticity Y and apply a force F as shown. If the rod elongates under the action of force Δl then we can write,



Young's modulus, $Y = \dfrac{\text{Stress}}{\text{Strain}} \Rightarrow Y = \dfrac{Fl}{A\Delta l}$ or $F = \dfrac{AY}{l}\Delta l$

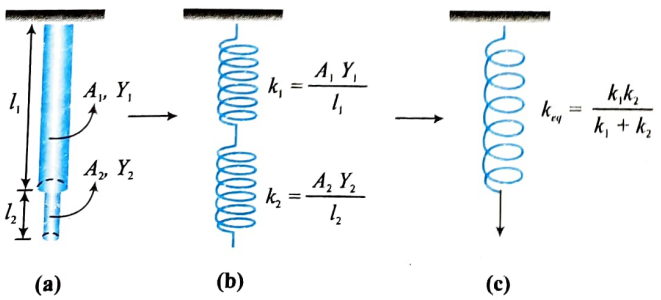
$\dfrac{AY}{l} = k$ (constant) depends on type of material and geometry of rod. Now,

$$F = k\Delta l \Rightarrow F \propto \Delta l \qquad \ldots(i)$$

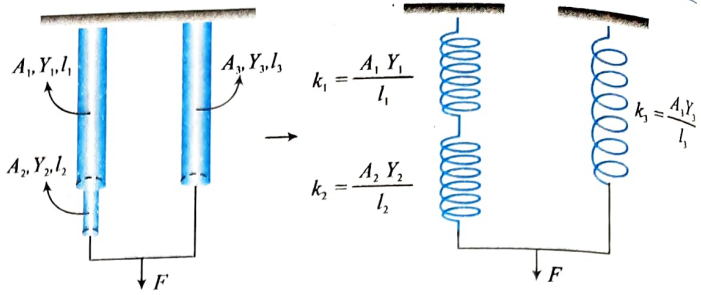
If we compare the relation in Eq. (i) with spring force ($F_{\text{spring}} = kx$) we can say k in Eq. (i) is equivalent to force constant of the spring. Hence, we can write,

$k = \dfrac{AY}{l}$ = equivalent spring constant.

For the system of rods shown in Fig. (a), the replaced spring system is shown in Fig. (b) (two springs in series). Figure (c) represents the equivalent spring system. Figure (d) represents another combination of rods and their replaced spring system.



(a) (b) (c)

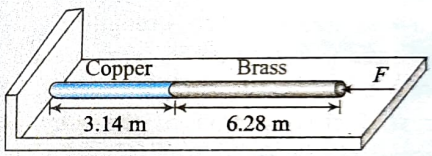


(d) (e)

Note: Since $k_{eq} = YA/l$, we have $k_{eq}\, l = YA$. It means that the product of stiffness of an elastic wire and its length is always constant. If we cut the wire into two equal parts, the stiffness of each part will be double that of the original wire. Following the same logic we can say that if a spring is cut into N equal parts, each part will have stiffness N times the stiffness of the original spring.

ILLUSTRATION 7.20

A copper cylinder and a brass cylinder are stacked end to end, as shown in figure. Each cylinder has a radius of 0.25 cm. A compressive force of F = 9000 N is applied to the right end of the brass cylinder. Find the amount by which the length of the stack decreases. (Given: $Y_{\text{copper}} = 1.0\times10^{11}$ N/m^2 and $Y_{\text{brass}} = 9.0\times10^{10}$ N/m^2)



Sol. Both the cylinders experience the same amount of force F. The magnitude of this force is related to the change in length of each cylinder according to equation:

$$Y = \left(\frac{F/A}{\Delta L/L}\right)$$

Each cylinder decreases in length; the total decrease being the sum of the decreases for each cylinder.

The length of the copper cylinder decreases by,

$$\Delta L_{\text{Copper}} = \frac{FL_C}{Y_{\text{Copper}}A} = \frac{FL_C}{Y_{\text{Copper}}(\pi r^2)}$$

Similarly, the length of the brass decreases by,

$$\Delta L_{\text{Brass}} = \frac{FL_B}{Y_{\text{Brass}}A} = \frac{FL_B}{Y_{\text{Brass}}(\pi r^2)}$$

Total change in length, $\Delta L = \Delta L_{\text{Copper}} + \Delta L_{\text{Brass}}$

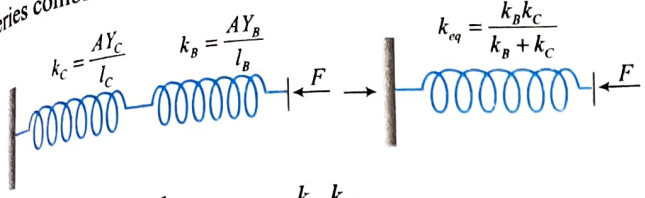
$$\Delta L = \frac{FL_C}{Y_{\text{Copper}}A} + \frac{FL_B}{Y_{\text{Brass}}A} = \frac{F}{A}\left(\frac{L_C}{Y_C} + \frac{L_B}{Y_B}\right)$$

$$= \frac{9000}{\pi(0.25\times10^{-2})^2}\left[\frac{3.14}{1.0\times10^{11}} + \frac{6.28}{9.0\times10^{10}}\right]$$

$$= 4.64\times10^{-2}\text{ m}$$

Therefore, the amount by which the length of the stack decreases is 4.64×10^{-2} m.

Alternative method: The composite bar can be replaced with series combination of two springs.



$$\frac{1}{k_{eq}} = \frac{1}{k_C} + \frac{1}{k_B} \Rightarrow k_{eq} = \frac{k_B.k_C}{k_B + k_C}$$

Deformation of equivalent spring

$$\Delta L = \frac{F}{k_{eq}} = F\left(\frac{k_B + k_C}{k_B.k_C}\right) = F\left(\frac{1}{k_B} + \frac{1}{k_C}\right)$$

But, $k_B = \frac{Y_C A}{L_C}$ and $k_B = \frac{Y_B A}{L_B}$

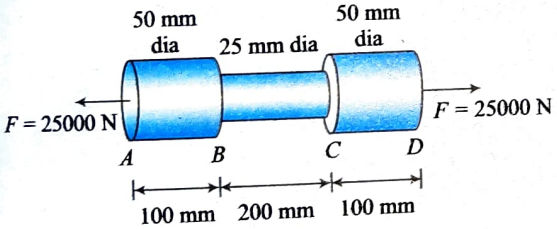
Hence, $\Delta L = \frac{F}{A}\left(\frac{L_C}{Y_C} + \frac{L_B}{Y_B}\right)$

$$= \frac{9000}{\pi(0.25\times10^{-2})^2}\left[\frac{3.14}{1.0\times10^{11}} + \frac{6.28}{9.0\times10^{10}}\right]$$

$$= 4.64\times10^{-2} \text{ m}$$

ILLUSTRATION 7.21

A steel bar $ABCD$ 40 cm long is made up of three parts AB, BC and CD, as shown in figure. The rod is subjected to a pull of 25 kN. Young's modulus for steel = 2×10^{11} Nm^{-2}.



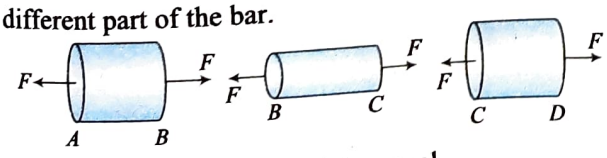
Determine:

(i) the stress in the three parts

(ii) the total extension of the rod

Sol.

(i) If F is the stretching force applied on the given steel bar, then the same axial force 25 kN is transmitted longitudinally to each of the three bars. However, stresses induced will be different and hence elongations of each bar will also be different. Figure shows the free-body diagram of the different part of the bar.



Stress in part AB and CD will be equal

$$\sigma_{AB} = \sigma_{CD} = \frac{F}{A_{AB}} = \frac{25000 \text{ N}}{\frac{\pi}{4}(50)^2 \text{ mm}^2} = \frac{40}{\pi} = 12.73 \text{ N/mm}^2$$

Stress in part BC, $\sigma_{BC} = \frac{25000}{\frac{\pi}{4}(25)^2} = 50.93 \text{ N/mm}^2$

(ii) Total extension of the rod

Δl = extension in the parts $AB + BC + CA$

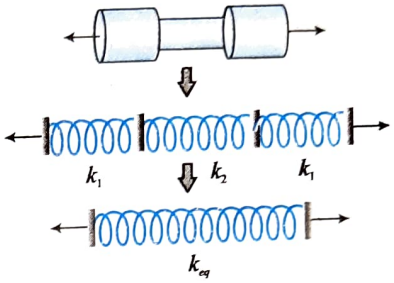
$$\Rightarrow \Delta l = \Delta l_{AB} + \Delta l_{BC} + \Delta l_{CD} = 2\Delta l_{AB} + \Delta l_{BC}$$

$$= 2\left(\frac{\sigma_{AB}}{Y} \times l_{AB}\right) + \left(\frac{\sigma_{BC}}{Y} \times l_{BC}\right)$$

Given: $Y = 2\times10^{11} \text{ N/m}^2 = 2\times10^5 \text{ N/mm}^2$

$$\Rightarrow \Delta l = 2\left(\frac{12.73}{2\times10^5} \times 100\right) + \frac{50.93}{2\times10^5} \times 200 = 0.0637 \text{ mm}$$

Alternative method: The steel bar $ABCD$ can be replaced with series combination of three springs.



In series combination of springs, the force in each spring should be equal. Hence, each rod will experience same force F. Hence, stress in rods AB and CD is

$$\sigma_{AB} = \sigma_{CD} = \frac{F}{A_1} = \frac{25\times10^3}{\left(\frac{\pi}{4}\times50^2\right)} = \frac{40}{\pi} \text{ N/mm}^2$$

Stress in rod BC is $\sigma_{BC} = \frac{25\times10^3}{\left(\frac{\pi}{4}\times25^2\right)} = \frac{160}{\pi} \text{ N/mm}^2$

Equivalent spring contact can be written as

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_1}$$

Here $k_1 = \frac{YA_1}{l_1}$ and $k_2 = \frac{YA_2}{l_2}$

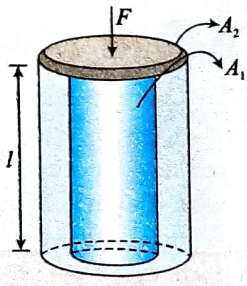
which gives $k_{eq} = \frac{Y(A_1A_2)}{(2A_2l_1 + A_1l_2)}$

Extension of composite rod $x = \frac{F}{k_{eq}}$

This gives $x = \frac{F(2A_2l_1 + A_1l_2)}{YA_1A_2} = \frac{1}{5\pi} \text{ mm} = 0.0637 \text{ mm}$

ILLUSTRATION 7.22

Consider a column (or strut) made up of two different materials. Figure shows the inner section of area A_1 having Young's modulus Y_1 and the outer section of area A_2 having Young's modulus Y_2. A compressive force F acts on this column. Find the load supported by inner and outer section of the column.



Sol. Let σ_1 and σ_2 be the stresses induced in the two materials, then sum of restoring forces in the two materials should balance the externally applied load F.

$$\therefore \quad \sigma_1 A_1 + \sigma_2 A_2 = F \qquad ...(i)$$

Also, the compression in the two portions should be the same. But since their original lengths are same, so strains will be same.

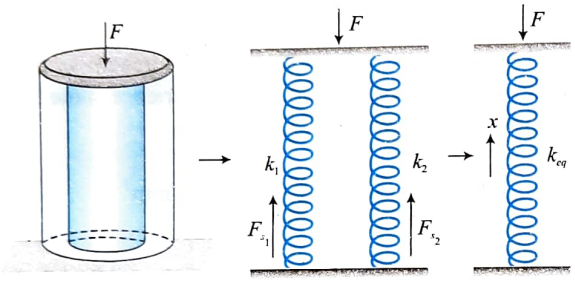
$$\therefore \quad \varepsilon_1 = \varepsilon_2 \Rightarrow \frac{\sigma_1}{Y_1} = \frac{\sigma_2}{Y_2} \qquad ...(ii)$$

Solving Eqs. (i) and (ii), $\sigma_1 = \dfrac{FY_1}{A_1Y_1 + A_2Y_2}$

Load supported by inter section, $F_1 = \sigma_1 A_1 = \dfrac{FY_1A_1}{A_1Y_1 + A_2Y_2}$

and load supported by outer section, $F_2 = \sigma_2 A_2 = \dfrac{FY_2A_2}{A_1Y_1 + A_2Y_2}$

Alternative method: We can replace the composite section as combination of two springs in parallel.



As springs are connected in parallel, the compressions in each spring will be same.

$$x_1 = x_2 \Rightarrow \frac{F_{s_1}}{k_1} = \frac{F_{s_2}}{k_2}$$

Here $k_1 = \dfrac{A_1Y_1}{l_1}$ and $k_2 = \dfrac{A_2Y_2}{l_2}$

The series combination of spring,

$$k_{eq} = k_1 + k_2 = \frac{1}{l}(A_1Y_1 + A_2Y_2)$$

Compression in each spring,

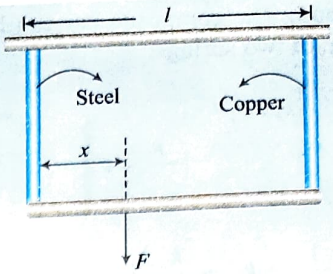
$$x_1 = x_2 = x = \frac{F}{k_{eq}} = \frac{Fl}{(A_1Y_1 + A_2Y_2)}$$

Load supported by inner section, $F_{s_1} = k_1x_1 = \dfrac{FY_1A_1}{(A_1Y_1 + A_2Y_2)}$

Load supported by outer section, $F_{s_2} = k_2x_2 = \dfrac{FY_2A_2}{(A_1Y_1 + A_2Y_2)}$

ILLUSTRATION 7.23

Two vertical rods of equal lengths, one of steel and the other of copper, are suspended from the ceiling, at a distance l apart and are connected rigidly to a rigid horizontal bar at their lower ends. The area of cross section of steel and copper wires are A_S and A_C, and Y_S and Y_C, their respective Young's moduli of elasticity.



(i) Find the tension in steel and copper wires.

(ii) Where should a vertical force F be applied to the horizontal bar in order to keep the bar horizontal?

Sol.

(i) Let the force F be applied at a distance x from the steel bar, measured along the horizontal bar. Let T_S and T_C be the loads on steel and copper rods, respectively. Then,

$$T_S + T_C = F \qquad ...(i)$$

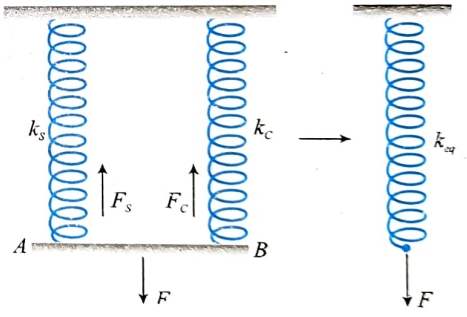
Since the rigid horizontal bar remains horizontal, the extensions produced in the two rods and strains remain the same. That is

$$\frac{T_S}{A_SY_S} = \frac{T_C}{A_CY_C} \qquad ...(ii)$$

Solving Eqs. (i) and (ii), we get

$$T_S = \frac{FA_SY_S}{A_SY_S + A_CY_C} \text{ and } T_C = \frac{FA_CY_C}{A_SY_S + A_CY_C}$$

Alternative method: We can treat two rods as two springs, connected in parallel combination



Equivalent spring constant of steel and copper rod

$$k_S = \frac{A_SY_S}{l} \text{ and } k_C = \frac{A_CY_C}{l}$$

As the bar always remains horizontal, it means the elongation of the rods should be equal.

$$\Rightarrow \quad \frac{F_S}{k_S} = \frac{F_C}{k_C} = \frac{F}{(k_S + k_C)}$$

It means $F_S = \dfrac{F \cdot k_S}{k_S + k_C} = \dfrac{FA_SY_S}{A_SY_S + A_CY_C}$

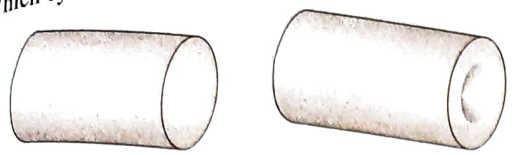
and $F_C = \dfrac{Fk_C}{k_S + k_C} = \dfrac{FA_CY_C}{A_SY_S + A_CY_C}$

(ii) Taking moments about the steel bar,

$$T_Cl = F \cdot x \Rightarrow x = \frac{T_C}{F}l = \frac{A_CY_Cl}{A_SY_S + A_CY_C}$$

CONCEPT APPLICATION EXERCISE 7.2

1. Drawings A and B show two cylinders that are identical in all respects, except that one is hollow. Identical forces are applied to each cylinder in order to stretch them. Which cylinder, if either, stretches more?



(A) (B)

(a) A and B both stretch by the same amount.
(b) A stretch more than B.
(c) B stretches more than A.
(d) Insufficient information is given for an answer.

2. The Young's modulus of a wire of length L and radius r is Y. If the length is reduced to $L/2$ and radius is $r/2$ then the Young's modulus will be
(a) $Y/2$ (b) Y
(c) $2Y$ (d) $4Y$

3. A wire of length L and radius r is rigidly fixed at one end. On stretching the other end of the wire with a force F, the increase in its length is l. If another wire of same material but of length $2L$ and radius $2r$ is stretched with a force of $2F$, the increase in its length will be
(a) l (b) $2l$
(c) $\frac{l}{2}$ (d) $\frac{l}{4}$

4. Two wires A and B are of same materials. Their lengths are in the ratio 1:2 and diameters are in the ratio 2:1 when stretched by force F_A and F_B respectively they get equal increase in their lengths. Then the ratio F_A/F_B should be
(a) 1 : 2 (b) 1 : 1
(c) 2 : 1 (d) 8 : 1

5. A cable is replaced by another one of the same length and material but twice the diameter.
(i) How will this affect the elongation under a given load?
(ii) How does this affect the maximum load it can support without exceeding the elastic limit?

6. Two exactly similar wires of steel and copper are stretched by equal forces. If the total elongation is 1 cm, find by how much is each wire elongated? Given Y for steel $= 20 \times 10^{10}$ N/m^2 and Y for copper $= 12 \times 10^{10}$ N/m^2.

7. A weightlifter raises a weight of 600 N over her head. Assume each of her legs supports the same weight and the legs are parallel and vertical, each femur has an effective cross-sectional area of 7.5×10^{-4} m^2 and a length of 0.52 m. Determine the amount by which each femur compresses.
(Given Young's modulus for bone $= 7 \times 10^9$ N/m^2.)

8. A 100 kg mass, fastened to the end of a steel wire of unstretched length 1.0 m, is whirled in a vertical circle with a speed of 5 m/s at the bottom of the circle. The cross-sectional area of the wire is 0.070 cm^2. Calculate the elongation of the wire when the mass is at the lowest point of its path. ($Y_{steel} = 200$ GPa).

9. The breaking stress for aluminium is 7.5×10^7 N/m^2. Find the greatest length of aluminium wire that can hang vertically without breaking. Density of aluminium is 2.7×10^3 kg /m^3.

ANSWERS

1. (c)	**2.** (b)	**3.** (a)	**4.** (d)
7. 0.03 mm	**8.** 2.5 mm	**9.** 28 m	

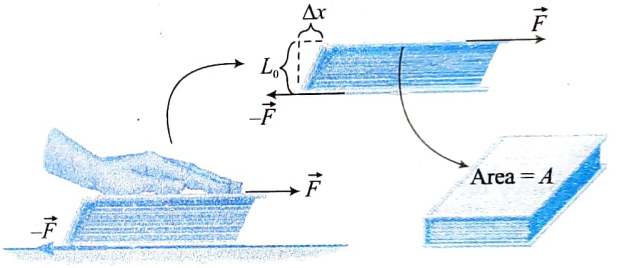
SHEAR MODULUS: ELASTICITY OF SHAPE

The modulus of rigidity, also known as shear modulus defined within limits of proportionality as the ratio of tangential stress (Shearing stress) to the shearing strain and denoted by η.

$$\eta = \frac{\text{Shearing stress}}{\text{Shearing strain}}$$

In this case the shape of a body changes but its volume remains unchanged.

Consider thick book whose lower face is fixed and acted upon by a tangential force F at its upper surface having area A.



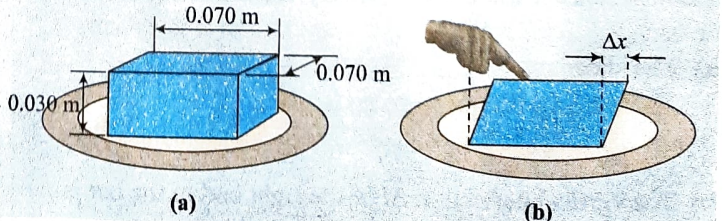
(a) (b)

The shearing stress, $\sigma = \frac{F}{A}$ and shearing strain $\theta = \Delta x/L_0$

So, $\eta = \frac{\text{Shearing stress}}{\text{Shearing strain}} = \frac{F/A}{\Delta x/L_0} = \frac{FL_0}{A\Delta x}$

ILLUSTRATION 7.24

A block of soft butter is resting on a plate. Figure (a) gives the dimensions of the butter block. You are bored, impatiently waiting for dinner, and push it tangentially across the top surface with a force of $F = 0.45$ N, as in Fig. (b). The top surface moves a distance $\Delta x = 6.0 \times 10^{-3}$ m relative to the bottom surface. Use this idle gesture to measure the shear modulus of butter block.



(a) (b)

Sol. The finger applies a force that is parallel to the top surface of the butter block. The shape of the block changes, because the top surface moves a distance Δx relative to the bottom surface.

The magnitude of the force required to produce this change in shape is given by

$$F = \eta A \frac{\Delta x}{L_0} \text{ or } \eta = \frac{FL_0}{A\Delta x}$$

where $A = (0.070\text{ m}) \times (0.070\text{ m})$ is the area of the top surface, and $L_0 = 0.030$ m is the thickness of the block.

$$\eta = \frac{FL_0}{A\Delta x} = \frac{(0.45\text{ N})(0.030\text{ m})}{(0.070\text{ m})(0.070\text{ m})(6.0\times10^{-3}\text{ m})} = 460\text{ N/m}^2$$

Butter can be deformed easily, so its shear modulus is significantly less than that of a more rigid material like steel.

ILLUSTRATION 7.25

Between each pair of vertebrae in the spinal column is a cylindrical disc of cartilage. Typically, this disc has a radius of about 3.0×10^{-2} m and a thickness of about 7.0×10^{-3} m. The shear modulus of cartilage is 1.2×10^7 N/m^2. Suppose that a shearing force of magnitude 11 N is applied parallel to the top surface of the disc while the bottom surface remains fixed in place. How far does the top surface move relative to the bottom surface?

Sol. The distance Δx that the top surface of the disc moves relative to the bottom surface is given by

$$\Delta x = \frac{FL_0}{\eta A}$$

where F is the magnitude of the shearing force, L_0 is the thickness of the cartilage, A is the cross-sectional area, and η is the shear modulus. Since the cross-section is circular, $A = \pi r^2$, where r is the radius.

$$\Delta x = \frac{(11\text{ N})(7.0\times10^{-3}\text{ m})}{(1.2\times10^7\text{ N/m}^2)\pi(3.0\times10^{-2}\text{ m})^2} = 2.3\times10^{-6}\text{ m}$$

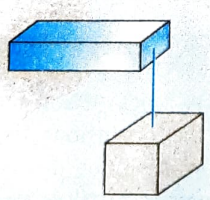
ILLUSTRATION 7.26

The figure shows a 160 kg crate hanging from the end of a steel bar. The length of the bar is 0.10 m, and its cross-sectional area is 3.2×10^{-4} m^2. Neglect the weight of the bar itself and determine:

(a) the shear stress on the bar and

(b) the vertical deflection of the right end of the bar.

(Shear modulus η of steel 8.1×10^{10} N/m^2)



Sol. The shear stress is equal to the magnitude of the shearing force exerted on the bar divided by the cross-sectional area of the bar.

(a) The shear stress,

$$\frac{F}{A} = \frac{mg}{A} = \frac{(160\text{ kg})(9.80\text{ m/s}^2)}{3.2\times10^{-4}\text{ m}^2} = 4.9\times10^6\text{ N/m}^2$$

(b) The vertical deflection Δl of the right end of the bar is

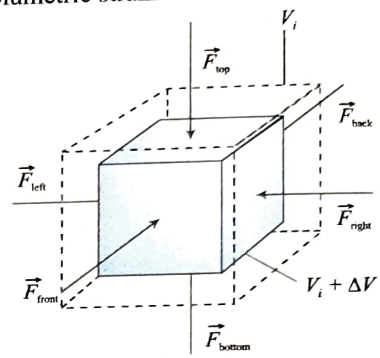
$$\Delta l = \left(\frac{F}{A}\right)\frac{L_0}{\eta} = (4.9\times10^6\text{ N/m}^2)\frac{0.10\text{ m}}{8.1\times10^{10}\text{ N/m}^2}$$

$$= 6.0\times10^{-6}\text{ m}$$

BULK MODULUS: VOLUME ELASTICITY

If a solid or fluid (liquid or gas) is subjected to a uniform pressure all over its surface, such that the shape remains the same, then there is a change in volume. Bulk modulus characterizes the response of an object, to changes in the forces of uniform magnitude applied perpendicularly over the entire surface of the object as shown in figure. The ratio of normal stress to the volumetric strain within the elastic limits is called as Bulk modulus. This is denoted by K.

$$K = \frac{\text{Normal stress}}{\text{volumetric strain}}$$



If the pressure on an object changes by an amount ΔP, the normal stress developed on the object will be equal to change in pressure. Then

$$K = \frac{\Delta P}{-(\Delta V/V)} = -V\left(\frac{\Delta P}{\Delta V}\right)$$

where V = original volume; ΔV = change in volume

A negative sign is inserted in this defining equation so that B is a positive number. This exercise is necessary because an increase in pressure (positive ΔP) causes a decrease in volume (negative ΔV) and vice versa.

The reciprocal of bulk modulus is called compressibility.

$$C = \text{compressibility} = \frac{1}{K} = -\frac{\Delta V}{PV}$$

SI unit of compressibility is N^{-1}m^2 and CGS unit is dyne^{-1} cm^2.

Important Point:

We define a bulk modulus for both solids and liquids. Liquids have no shear modulus and no Young's modulus, because a liquid does not sustain a shearing stress or a tensile stress. If a shearing force or a tensile force is applied to a liquid, the liquid simply flows in response.

DENSITY OF COMPRESSED LIQUID

If a liquid of density ρ, volume V and bulk modulus K is compressed, then its density increases.

As density $\rho = \frac{m}{V}$ so $\frac{\Delta\rho}{\rho} = \frac{-\Delta V}{V}$ ...(i)

But by definition, $K = \frac{-V\Delta P}{\Delta V} \Rightarrow -\frac{\Delta V}{V} = \frac{\Delta P}{K}$ ...(ii)

From (i) and (ii)

$$\frac{\Delta\rho}{\rho}=\frac{\rho'-\rho}{\rho}=\frac{\Delta P}{K} \qquad [\text{As } \Delta\rho=\rho'-\rho]$$

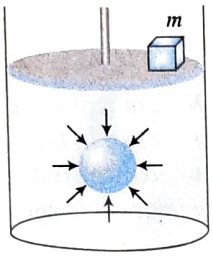
or $$\rho'=\rho\left[1+\frac{\Delta P}{K}\right]=\rho[1+C\Delta P]$$

$$\left[\text{As } \frac{1}{K}=C=\text{compresibility}\right]$$

FRACTIONAL CHANGE IN THE RADIUS OF SPHERE

A solid sphere of radius R made of a material of bulk modulus K is surrounded by a liquid in a cylindrical container.

A massless piston of area A floats on the surface of the liquid.



Volume of the spherical body

$$V=\frac{4}{3}\pi R^3 \text{ or } V\propto R^3$$

or $$\frac{\Delta V}{V}=3\frac{\Delta R}{R}$$

$$\therefore \quad \frac{\Delta R}{R}=\frac{1}{3}\frac{\Delta V}{V} \qquad ...(i)$$

Bulk modulus $$K=-V\frac{\Delta P}{\Delta V}$$

$$\therefore \quad \left|\frac{\Delta V}{V}\right|=\frac{\Delta P}{K}=\frac{mg}{AK} \qquad \left[\text{As } \Delta P=\frac{mg}{A}\right] \qquad ...(ii)$$

Substituting the value of $\frac{\Delta V}{V}$ from Eq. (ii) in Eq. (i), we get

$$\frac{\Delta R}{R}=\frac{1}{3}\frac{mg}{AK}$$

ILLUSTRATION 7.27

The pressure increases by 1.0×10^4 N/m^2 for every meter of depth beneath the surface of the ocean. At what depth does the volume of a Pyrex glass cube, 1.0×10^{-2} m on an edge at the ocean's surface, decrease by 1.0×10^{-10} m^3? [$K=2.6\times10^{10}$ N/m^2.]

Sol. We know change in pressure,

$$\Delta P=-K\frac{\Delta V}{V_0}$$

$$=-(2.6\times10^{10}\text{ N/m}^2)\frac{-1.0\times10^{-10}\text{ m}^3}{1.0\times10^{-6}\text{ m}^3}=2.6\times10^6\text{ N/m}^2$$

where we have expressed the volume V_0 of the cube at the ocean's surface as $V_0=(1.0\times10^{-2}\text{ m})^3=1.0\times10^{-6}$ m^3.

Since the pressure increases by 1.0×10^4 N/m^2 per meter of depth, the depth is

$$\frac{2.6\times10^6\text{ N/m}^2}{1.0\times10^4\frac{\text{N/m}^2}{\text{m}}}=260\text{ m}$$

ILLUSTRATION 7.28

What is the density of water at a depth where the pressure is 501 atm, given its density at the surface is 1.0×10^3 kg/m^3? Given: Bulk modulus of water $K=2.0\times10^9$ Pa, 1atm $=1.01\times10^5$ Pa

Sol. Change in pressure, ΔP = 501 atm – 1 atm
= 500 atm = $500\times1.01\times10^5$ Pa

Let ρ' be the density of water at the given depth. If V and V are volume of certain mass M of ocean water at surface and at a given depth,

As $$B=\frac{\Delta P\times V}{\Delta V},\ \frac{\Delta V}{V}=\frac{\Delta P}{B}$$

or $$\frac{\Delta V}{V}=\frac{(500\times1.01\times10^5)}{2.0\times10^9}=2.525\times10^{-2}$$

Further, $$\frac{\Delta V}{V}=\frac{(M/\rho)-(M/\rho')}{(M/\rho)}=1-\frac{\rho}{\rho'}$$

(as volume = mass/density)

or $$\frac{\rho}{\rho'}=1-\frac{\Delta V}{V} \text{ or } \rho'=\frac{\rho}{1-\frac{\Delta V}{V}}$$

or $$\rho'=\frac{1.0\times10^3}{1-2.525\times10^{-2}}=1.025\times10^3\text{ kg/m}^3$$

CONCEPT APPLICATION EXERCISE 7.3

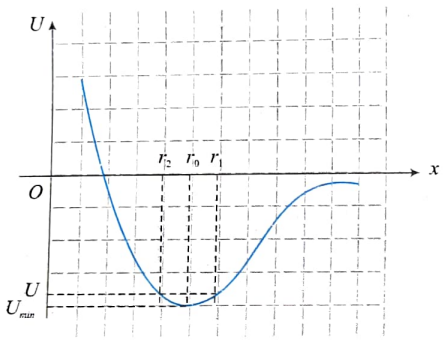
1. Fluids can develop
   (a) longitudinal strain only
   (b) longitudinal and shearing strain
   (c) longitudinal, shearing and volumetric strain
   (d) volumetric strain only
2. Modulus of rigidity of ideal liquids is
   (a) infinity (b) zero
   (c) unity (d) non-zero constant value.
3. The marina trench is located in the Pacific Ocean and at one place it is nearly 11 km beneath the surface of water. The water pressure at the bottom of the trench is about 1.1×10^8 Pa. A steel ball of initial volume 0.32 m^3 is dropped into the ocean and falls to the bottom of the trench. What is the change in the volume of the ball when it reaches to the bottom?
   (Bulk modulus for steel = 160 GPa)
4. Two strips of metal are riveted together at their ends by four rivets, each of diameter 6.0 mm. What is the maximum tension that can be exerted by the riveted strip if the shearing stress on the rivet is not to exceed 6.9×10^7 Pa? Assume that each rivet is to carry one quarter of the load.
5. An India rubber cube of side 7 cm has one side fixed while a tangential force equal to weight of 200 kg is applied to the opposite face. Find the shear strain produced and the distance through which the strained side moves. Modulus of rigidity for rubber is 2×10^6 N/m^2.

ANSWERS

1. (d) **2.** (b) **3.** 2.24×10^{-4} m^3 **4.** 7800 N
5. 0.2, 1.4 cm

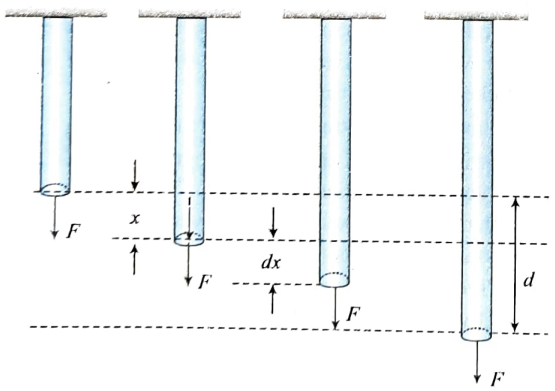
ELASTIC ENERGY

Whenever we squeeze or twist an object, the interatomic distance either increases to r_1, or decreases to r_2, say, from equilibrium distance r_0. Referring to U-r graph between two atoms, we can understand that the electrostatic potential energy between two atoms is minimum at $r = r_0$ (stable equilibrium position). For any other values of r, U will increase. It means that in a deformed body, the potential energy U between any two nearest atoms or molecules is greater than the minimum value. The difference in these two energies $|U - U_{min}|$ is the increase in energy between the atoms or molecules of a deformed body. Macroscopically, this excess of the energy of interaction between all atoms/molecules of a deformed object is called elastic energy. Hence, elastic energy is a gross (macroscopic) manifestation of microscopic electrostatic energy between the atoms. When we remove the external load (stress), the body becomes undeformed and the elastic energy in principle, will be retrieved back. However, the elastic energy will be, in practice, converted into vibrational energy followed by heat, light, sound etc.



WORK DONE IN STRETCHING A WIRE

We know that when a metal rod is elongated under a gradually increasing load, work is done by the external forces on the rod which is transferred either partially or completely into potential energy of strain. If the strain remains within the elastic limit, the work done is completely transformed into potential energy and can be recovered during a gradual unloading of the strained rod.



To produce x elongation in a rod of length L and cross-sectional area A, force required is given by

$$F = \frac{YA}{L}x$$

The work done produces an additional deformation dx is given by

$$dW = \frac{YA}{L}x\,dx$$

If total deformation of rod is δ. The total work done is

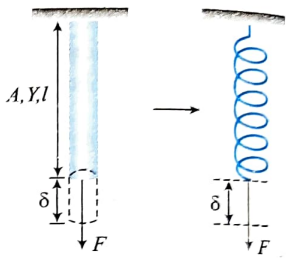
$$W = \frac{YA}{L}\int_0^{\delta} x\,dx \Rightarrow W = \frac{1}{2}\frac{YA}{L}\delta^2$$

According to work–energy theorem, $W = \Delta U$. Thus, the potential energy stored in the rod is

$$U = \frac{1}{2}\frac{YA}{L}\delta^2$$

ELASTIC POTENTIAL ENERGY STORED IN A STRETCHED WIRE OR IN A ROD

We can replace a rod with a spring as shown in the figure. The energy stored in the rod is the energy stored in equivalent spring is



$$U = \frac{1}{2}k\delta^2 = \frac{1}{2}\left(\frac{AY}{l}\right)\delta^2$$

where $\delta = \dfrac{Fl}{AY}$

$\therefore$ The potential energy stored in the rod,

$$U = \frac{1}{2}\frac{AY}{l}\frac{F^2l^2}{A^2Y^2} = \frac{1}{2}\frac{F^2l}{AY} \qquad \text{...(i)}$$

or $U = \dfrac{1}{2}F\delta$

Equation (i) can be rearranged as

$$U = \frac{1}{2}\frac{F^2}{A^2}\cdot\frac{lA}{Y}$$

(As, lA = volume of the rod, F/A = stress)

$$U = \frac{1}{2}\frac{(\text{Stress})^2}{Y}\cdot \text{Volume}$$

Again, $U = \dfrac{1}{2}\dfrac{F}{A}\times\dfrac{F}{AY}\times Al$ $\qquad \left(\text{Strain} = \dfrac{F}{AY}\right)$

$$U = \frac{1}{2}\times \text{Stress}\times\text{Strain}\times\text{Volume}$$

Again, $U = \dfrac{1}{2}\dfrac{F^2}{A^2Y^2}AlY$

$$U = \frac{1}{2}Y(\text{Strain})^2\times\text{Volume}$$

$\therefore$ The energy density, $u = \dfrac{\text{Strain energy}}{\text{Volume}} = \dfrac{1}{2}\dfrac{(\text{Stress})^2}{Y}$

$$u = \frac{1}{2}Y(\text{Strain})^2 = \frac{1}{2}\text{Stress}\times\text{Strain}$$

So, $u = \dfrac{U}{AL} = \dfrac{\rho^2}{2Y} = \dfrac{1}{2}Y\epsilon^2 = \dfrac{1}{2}\epsilon\rho$

where $\epsilon = \delta / L$ and $\rho = F/A$.

From above equations, we can conclude that:

1. The area under the stress–strain diagram gives the potential energy (or strain energy) stored per unit volume of the specimen.
2. If stress applied is not constant we can write energy stored in any elastic rod

$$U=\frac{1}{2Y}\int \rho^2\, dV$$

where ρ = stress and dV = change in volume of rod.

Note:

The potential energy stored in the rod in terms of stiffness constant k is given by

$$U=\frac{1}{2}k\delta^2, \text{ where } k = YA/L$$

The potential energy stored in the rod in terms of the maximum stretching force is

$$U=\frac{F^2L}{2YA}=\frac{1}{2}F\delta$$

Total energy stored in wire (U)	Energy stored in per unit volume of wire (U_V)
$\frac{1}{2}Fl$	$\frac{1}{2}\frac{Fl}{\text{volume}}$
$\frac{1}{2}\times$stress$\times$strain$\times$volume	$\frac{1}{2}\times$stress$\times$strain
$\frac{1}{2}\times Y\times(\text{strain})^2\times$volume	$\frac{1}{2}\times Y\times(\text{strain})^2$
$\frac{1}{2Y}\times(\text{stress})^2\times$volume	$\frac{1}{2Y}\times(\text{stress})^2$

ILLUSTRATION 7.29

A steel wire of length 2 m and 1.2×10^{-7} m^2 in cross sectional area is stretched by a force of 36 N. Calculate stress, strain, increase in length and work done in stretching the wire ($Y=1.8\times10^{11}$ N/m^2).

Sol. The stress in steel wire,

$$\sigma=\frac{F}{A}=\frac{36}{1.2\times10^{-7}}=3\times10^8 \text{ N/m}^2$$

As Young's modulus of elasticity, $Y=\dfrac{\text{Stress}}{\text{Strain}}$

$$\text{Strain}=\frac{\text{Stress}}{Y}=\frac{3\times10^8}{1.8\times10^{11}}=\frac{5}{3}\times10^{-3}$$

$$\text{Increase of length}=\Delta l=(l)(\text{strain})=1.67\times10^{-3}\times2$$

$$=3.34\times10^{-3} \text{ m}$$

$$\text{Workdone}=\frac{1}{2}(\text{stress})(\text{strain})\text{ volume}$$

$$=\frac{1}{2}\times3\times10^8\times\frac{5}{3}\times10^{-3}(2\times1.2\times10^{-7})$$

$$=6\times10^{-2} \text{ J}$$

ILLUSTRATION 7.30

A catapult consists of two parallel rubber strings, each of lengths 10 cm and cross-sectional area 10 mm^2. When stretched by 5 cm, it can throw a stone of mass 100 g to a vertical height of 25 m. Determine Young's modulus of elasticity of rubber.

Sol. A stretched catapult has elastic potential energy stored in it (Elastic potential energy stored in both the rubber strings)

$$U=\left(\frac{1}{2}\frac{YAl^2}{L}\right)\times2$$

This energy, when imparted to the stone, it flies off a height h. Energy possessed by the stone = mgh. Now,

$$U=mgh \Rightarrow \frac{YAl^2}{L}=mgh$$

$$Y=\frac{mghL}{Al^2}=\frac{100\times10^{-3}\times10\times25\times10^{-1}}{10\times10^{-6}\times(5\times10^{-2})^2}$$

$$\therefore \quad Y=1\times10^8 \text{ N/m}^2$$

ILLUSTRATION 7.31

When the load on a wire is increased slowly from 3 kg-wt to 5 kg-wt, the elongation increases from 0.61 mm to 1.02 mm. How much work is done during the extension of the wire?

Sol. Work done in stretching the wire from its initial length through 0.61 mm (= 0.61×10^{-3} m) when the stretching force is 3 kg wt (= 3×9.8 N), i.e.,

$$W_1=\frac{1}{2}\times\text{ stretching force}\times\text{extension}$$

$$=\frac{1}{2}\times(3\times9.8)\times(0.61\times10^{-3})=0.9\times10^{-2} \text{ J}$$

Work done in stretching the wire from its initial length through 1.02 mm (= 1.02×10^{-3} m) when the stretching force is 5 kg-wt (= 5×9.8 N), i.e.,

$$W_2=\frac{1}{2}\times(5\times9.8)\times(1.02\times10^{-3})=2.5\times10^{-2} \text{ J}$$

Work done in stretching the wire from 0.61 mm to 1.02 mm

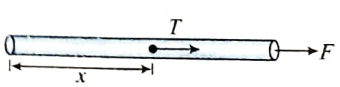
$$=(W_2-W_1)=2.5\times10^{-2} \text{ J}-0.9\times10^{-2} \text{ J}=1.6\times10^{-2} \text{ J}$$

ILLUSTRATION 7.32

A smooth uniform string of natural length l, cross-sectional area A and Young's modulus Y is pulled along its length by a force F on a horizontal surface. Find the elastic potential energy stored in the string.

Sol. As discussed, the tension T in the string at a distance x from its free end is given as:

$$T=F\left(\frac{x}{l}\right)$$



Hence $\rho=\dfrac{T}{A}=\dfrac{F}{Al}x$

Substituting (ρ) in the formula $U=\dfrac{1}{2Y}\int\rho^2 dV$

We have $U = \frac{1}{2Y}\int_0^l \frac{F^2}{A^2l^2}x^2\,dV$, where $dV = Adx$

This gives $U = \frac{F^2l}{6AY}$

CONCEPT APPLICATION EXERCISE 7.4

1. Assuming Young's modulus of aluminum to be 7×10^{10} N/m^2, find energy stored in an aluminium rod of cross section 10^{-4} m^2 and 0.1 m long when compressed by a load of 10 kg.
2. The work done in stretching a wire by 0.1 mm is 4 J. Find the work done in stretching another wire of same material, but with double the radius and half the length by 0.1 mm in joules.
3. A block of mass M is suspended from a wire of length L, area of cross section A and Young's modulus Y. Find the elastic potential energy stored in the wire.
4. A 40 kg boy whose leg bones are 4 cm^2 in area and 50 cm long falls through a height of 2 m without breaking his leg bones. If the bones can stand a stress of 0.9×10^8 N/m^2, calculate (i) Y for the material of the bone and (ii) k (force constant).
5. Identical springs of steel and copper are stretched by equal lengths. On which is more work done?
6. Identical springs of steel and copper are stretched by equal forces, F each. On which is more work done?

ANSWERS

1. 6.86×10^{-5} J 2. 32 J 3. $U = \frac{1}{2}\frac{M^2g^2L}{AY}$

4. (i) 2.1×10^9 N/m^2 (ii) 1.7×10^6 N/m

ELASTIC PROPERTY OF MATTER

Elasticity: The property of matter by virtue of which a body tends to regain its original shape and size after the removal of deforming force is called elasticity.

Plasticity: The property of matter by virtue of which it does not regain its original shape and size after the removal of deforming force is called plasticity.

Perfectly elastic body: If on the removal of the forces causing the deformation, the body regains its original configuration completely it is said to be perfectly elastic.

A quartz fibre and phosphor bronze (an alloy of copper containing 4% to 10% tin, 0.05% to 1% phosphorus) is the nearest approach to the perfectly elastic body.

Perfectly plastic body: If the body does not have any tendency to recover its original configuration, on the removal of deforming force, it is said to be perfectly plastic.

Paraffin wax and wet clay are the nearest approach to the perfectly plastic body.

Note: Practically there is no material which is either perfectly elastic or perfectly plastic and the behaviour of actual bodies lies between these two extremes.

Elastic limit: Elastic bodies show their property of elasticity upto a certain value of deforming force. If we go on increasing the deforming force then a stage is reached when on removing the force, the body will not return to its original state. "The maximum deforming force up to which a body retains its property of elasticity is called the elastic limit of the material of the body."

Elastic limit is the property of a body whereas elasticity is the property of material of the body.

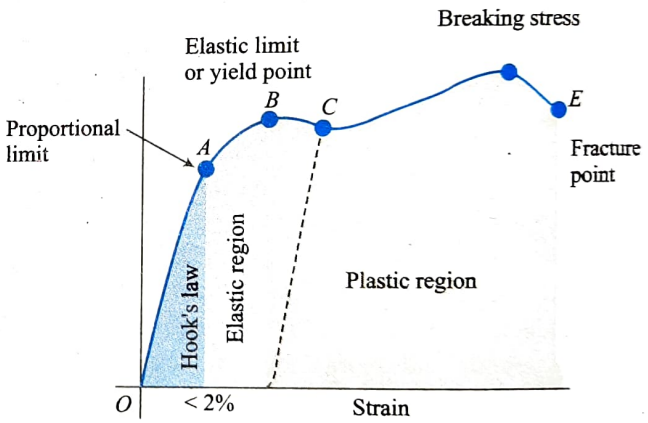
Elastic fatigue: The temporary loss of elastic properties because of the action of repeated altering deforming force is called elastic fatigue. Due to elastic fatigue:

(i) Bridges are declared unsafe after a long time of their use.
(ii) Spring balances show wrong readings after they have been in use for a long time.
(iii) We are able to break a wire by repeated bending.

Elastic after effect: The time delay in which the substance regains its original condition after the removal of deforming force is called elastic after effect. It is the time for which restoring forces are present after the removal of the deforming force, it is negligible for perfectly elastic substance, like quartz, phosphor bronze and large for others, like glass fibre.

STRESS–STRAIN CURVE

If we gradually increase the load on a vertically suspended metal wire and a graph is plotted between stress (or load) and longitudinal strain (or elongation) we will get the curve as shown in figure. From this curve it is clear that:



(1) When the strain is small (< 2%) (i.e., in region OA) stress is proportional to strain. This is the region where the Hooke's law is valid. The point A is called limit of proportionality and slope of line OA gives the Young's modulus Y of the material of the wire.

(2) If the strain is increased a little bit, i.e., in the region AB, the stress is not proportional to strain. However, the wire still regains its original length after the removal of stretching force. This behaviour is shown up to point B and is known as elastic limit or yield-point. In region OAB, body shows the elastic behaviour of the material of wire.

(3) If the wire is stretched beyond the elastic limit B, i.e., between BC, the strain increases much more rapidly and if the stretching force is removed the wire does not come back to its natural length. A permanent increase in length takes place.

(4) If the stress is increased further by a very small amount, a very large increase in strain is produced (region CD) and

after reaching point D, the strain increases even if the wire is unloaded and it gets ruptured at E. In the region DE the wire literally flows. The maximum stress corresponding to D after which the wire begins to flow, and breaks is called breaking or ultimate tensile strength. The region $BCDE$ represents the plastic behaviour of the material of wire.

IMPORTANT FACTS ABOUT ELASTICITY

(1) The body which requires a greater deforming force to produce a certain change in its dimension is said to be more elastic.

Example: Ivory and steel balls are more elastic than rubber.

(2) When equal deforming force is applied on different bodies then the body which shows less deformation is more elastic.

Example:

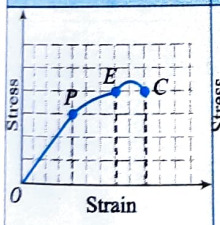
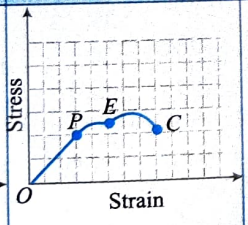
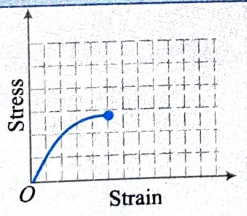
(i) For same load, more elongation is produced in rubber wire than in steel wire hence steel is more elastic than rubber.

(ii) Water is more elastic than air as volume change in water is less for same applied pressure.

(iii) Four identical balls of different materials are dropped from the same height then after collision, balls rise upto different heights. The order of their height can be given by $h_{ivory} > h_{steel} > h_{rubber} > h_{clay}$ because $Y_{ivory} > Y_{steel} > Y_{rubber} > Y_{clay}$.

(3) The value of moduli of elasticity is independent of the magnitude of the stress and strain. It depends only on the nature of material of the body.

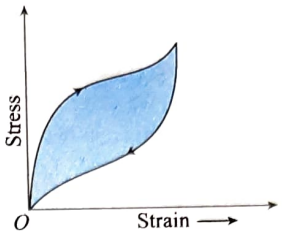
Stress–strain curve for different materials

Brittle material	Ductile material	Elastomers
		
The plastic region between E and C is small for brittle material and it will break soon after the elastic limit is crossed.	The material of the wire has a good plastic range and such materials can be easily changed into different shapes and can be drawn into thin wires.	Stress strain curve is not a straight line within the elastic limit for elastomers and strain produced is much larger than the stress applied. Such materials do not have a plastic range and the breaking point lies very close to elastic limit. Example rubber

ELASTIC HYSTERESIS

When a deforming force is applied on a body then the strain does not change simultaneously with stress rather it lags behind the stress. The lagging of strain behind the stress is defined as elastic hysteresis. This is the reason why the values of strain for same stress are different while increasing the load and while decreasing the load.

The area of the stress–strain curve is called the **hysteresis loop** and it is numerically equal to the work done in loading the material and then unloading it.

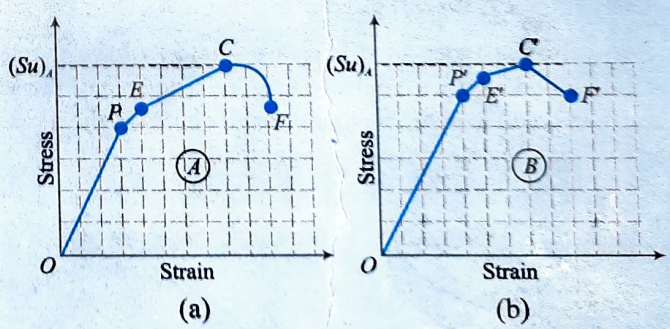


Important Points:

- The strain persists even when the stress is removed and thus lags behind the stress.
- Steel is highly elastic material. It is even more elastic than rubber.
- Rubber remains elastic even for a strain of the order of thousand.
- Young's modulus decreases with rise in temperature. A rise in temperature results, in general, in weakening of elastic properties and a fall in temperature results in their strengthening.
- Young's modulus is independent of length or radius of wire.
- Solids are the most elastic and gases are the least elastic.
- Young's modulus is defined for solids but not for liquids and gases. The value of Young's modulus increases on mixing the impurity in the solid.
- The area under the stress–strain graph represents the work done per unit volume.

ILLUSTRATION 7.33

The stress–strain graphs for two materials A and B are shown in figure (The graphs are drawn to the same scale).



(i) Which material has greater Young's modulus?
(ii) Which material is more ductile?
(iii) Which is more brittle?
(iv) Which of the two is the stronger material?

Sol.

(i) Young's modulus = stress/strain = slope of the straight portion of the stress–strain graph. Since slope of OP > slope of OP'. Y for material-A > Y for material-B.

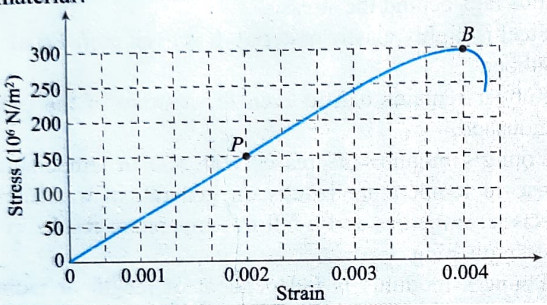
(ii) A material is ductile if there is a large plastic deformation, i.e., between the elastic limit (E) and the fracture point (F). Material-A is more ductile than material-B as $EF > E'F'$.

(iii) A material is brittle if the plastic region between the elastic limit and the fracture point is small. Material-B is more brittle than material-A as it has smaller plastic region ($E'F' < EF$).

(iv) Strength of a material is determined by the highest point on the stress-strain curve. The stress corresponding to this point is called the ultimate strength (S_u). Since this point (C) for material-A is higher than the corresponding point (C') for material-B, $(S_u)_A > (S_u)_B$. Hence, material-A is stronger than material-B.

ILLUSTRATION 7.34

Figure shows the stress–strain curve for a given material. What are **(a)** Young's modulus and **(b)** approximate yield strength of the material?



Sol.

(a) From the graph, at point P:

Stress $= 150 \times 10^6$ N/m^2

Strain $= 0.002 = 2 \times 10^{-3}$

Young's modulus,

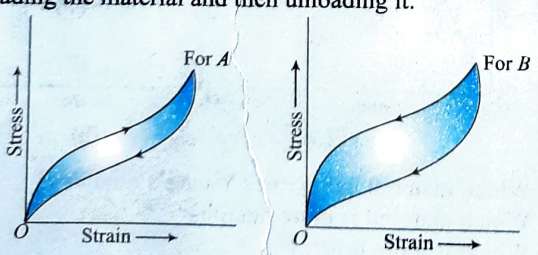
$$Y = \frac{\text{stress}}{\text{strain}} = \frac{150\times10^6 \text{ N/m}^2}{2\times10^{-3}} = 7.5\times10^{10} \text{ N/m}^2$$

(b) From graph, corresponding to point B (yield point), stress $\approx 300 \times 10^6$ N/m^2.

Thus, approximate yield strength $= 3 \times 10^8$ N/m^2.

ILLUSTRATION 7.35

Two different types of rubber are found to have the stress–strain curves as shown below in figure. The area of the stress–strain curve of a material is numerically equal to the work done in loading the material and then unloading it.



(a) Which of the two rubber materials would you choose for a car tyres?

(b) A heavy machine is to be installed in a factory. To absorb vibrations of the machine, a block of rubber is placed between the machine and the floor. Which of the two rubbers would you prefer to use for this purpose?

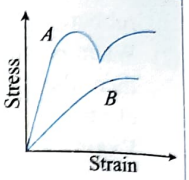
Sol.

(a) Rubber A should be used for making the car tyres. It is because of the reason that area of the curve i.e., work done in case of rubber A is lesser and hence the car tyres of this rubber will not get excessively heated.

(b) Rubber B should be used to absorb vibrations of the machinery. Because of the large area of the curve, a large amount of vibrational energy can be dissipated.

CONCEPT APPLICATION EXERCISE 7.5

1. The diagram shows stress v/s strain curve for the materials A and B. From the curves we infer that



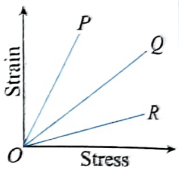
(a) A is brittle but B is ductile

(b) A is ductile and B is brittle

(c) Both A and B are ductile

(d) Both A and B are brittle

2. The strain stress curves of three wires of different materials are shown in the figure. P, Q and R are the elastic limits of the wires. The figure shows that



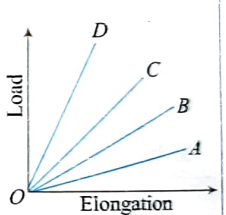
(a) Elasticity of wire P is maximum

(b) Elasticity of wire Q is maximum

(c) Tensile strength of R is maximum

(d) None of the above is true

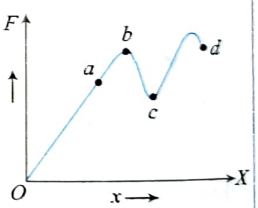
3. The load versus elongation graph for four wires of the same material is shown in the figure. The thickest wire is represented by the line



(a) OD (b) OC

(c) OB (d) OA

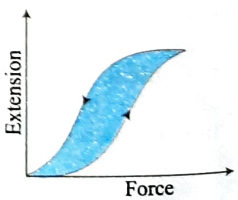
4. The graph is drawn between the applied force F and the strain (x) for a thin uniform wire. The wire behaves as a liquid in the part



(a) ab (b) bc

(c) cd (d) oa

5. The diagram shows a force-extension graph for a rubber band. Consider the following statements:



I. It will be easier to compress this rubber than expand it.

II. Rubber does not return to its original length after it is stretched.

III. The rubber band will get heated if it is stretched and released.

Which of these can be deduced from the graph?

(a) III only (b) II and III

(c) I and III (d) I only

ANSWERS

1. (b) **2.** (d) **3.** (a) **4.** (b) **5.** (a)

Solved Examples

EXAMPLE 7.1

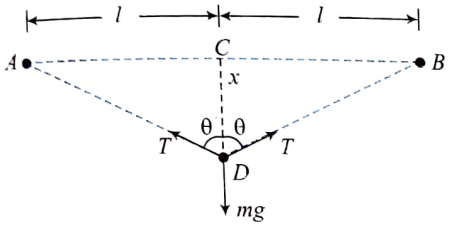
A mild steel wire of length 1.0 m and cross-sectional area 0.50×10^{-2} cm^2 is stretched well within its elastic limit horizontally between two pillars. A mass of 100 g is suspended from the midpoint of the wire. Calculate the depression at the midpoint. ($Y_{steel} = 200$ GPa)

Sol. Given $m = 100$ g $= 0.100$ kg

Refer figure, let x be the depression at the mid-point i.e., $CD = x$

In figure, $AC = CB = l = 0.5$ m

$$AD = BD = (l^2 + x^2)^{1/2}$$



Increase in length, $\Delta l = AD + DB - AB = 2AD - AB$

$$= 2(l^2 + x^2)^{1/2} - 2l = 2l\left(1 + \frac{x^2}{l^2}\right)^{1/2} - 2l$$

$$= 2l\left[1 + \frac{x^2}{2l^2}\right] - 2l = \frac{x^2}{l}$$

Strain $= x^2/2l^2$

If T is the tension in the wire, then

$$2T\cos\theta = mg \text{ or } T = \frac{mg}{2\cos\theta}$$

Here, $\cos\theta = \dfrac{x}{(l^2 + x^2)^{1/2}} = \dfrac{x}{l\left(1 + \dfrac{x^2}{l^2}\right)^{1/2}} = \dfrac{x}{l\left(1 + \dfrac{1}{2}\dfrac{x^2}{l^2}\right)}$

As $x << l$, so $1 >> \dfrac{1}{2}\dfrac{x^2}{l^2}$ and $1 + \dfrac{1}{2}\dfrac{x^2}{l^2} \approx 1$

$$\therefore \quad \cos\theta = \frac{x}{l}$$

Hence, $T = \dfrac{mg}{2(x/l)} = \dfrac{mgl}{2x}$

$$\text{Stress} = \frac{T}{A} = \frac{mgl}{2Ax}$$

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{mgl}{2Ax} \times \frac{2l^2}{x^2} = \frac{mgl^3}{Ax^3}$$

$$\therefore \quad x = l\left[\frac{mg}{YA}\right]^{1/3} = 0.5\left[\frac{0.1 \times 10}{2 \times 10^{11} \times 0.5 \times 10^{-6}}\right]^{1/3}$$

$$= 1.074 \times 10^{-2}\text{ m} = 1.074\text{ m}$$

EXAMPLE 7.2

The elastic limit and ultimate strength for steel is 2.48×10^8 Pa and 4.89×10^8 Pa respectively. A steel wire of 10 m length and 2 mm cross-sectional diameter is subjected to longitudinal tensile stress. [Young's modulus of steel is $Y = 2 \times 10^{11}$ Pa]

(a) Calculate the maximum elongation that can be produced in the wire without permanently deforming it. How much force is needed to produce this extension?

(b) Calculate the maximum stretching force that can be applied without breaking the wire.

Sol.

(a) Area of cross section $A = \pi r^2 = 3.14 \times (1 \times 10^{-3})^2$

The stress should not exceed the elastic limit otherwise the wire will suffer permanent deformation.

$$\therefore \quad \frac{F}{A} = 2.48 \times 10^8\text{ Pa}$$

$$\text{Strain} = \frac{\text{stress}}{Y} = \frac{2.48 \times 10^8}{2 \times 10^{11}} = 1.24 \times 10^{-3}$$

$$\frac{\Delta L}{L} = 1.24 \times 10^{-3}$$

$$\Rightarrow \quad \Delta L = 1.24 \times 10^{-3} \times 10\text{ m} = 1.24\text{ cm}$$

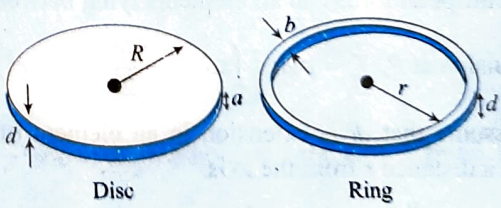
(b) The stress should not exceed the ultimate strength.

$$\frac{F}{A} = 4.89 \times 10^8\text{ Pa}$$

$$F = 4.89 \times 10^8 \times 3.14 \times 10^{-6} = 1535\text{ N}$$

EXAMPLE 7.3

A steel ring is to be fitted on a wooden disc of radius R and thickness d. The inner radius of the ring is r which is slightly smaller than R. The outer radius of the ring is $r + b$ and its thickness is d (same as the disc). There is no change in value of b and d after the ring is fitted over the disc; only the inner radius becomes R. If the Young's modulus of steel is Y, calculate the longitudinal stress developed in it. Also calculate the tension force developed in the ring. [Take $b << r$]



Sol. Increase in circumference of the ring $= 2\pi(R - r)$

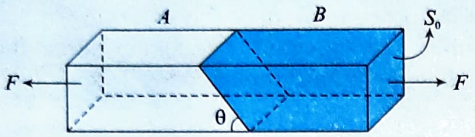
$$\therefore \quad \text{Strain} = \frac{2\pi(R - r)}{2\pi r} = \frac{R - r}{r}$$

$$\text{Stress} = Y(\text{strain}) = Y\left(\frac{R - r}{r}\right)$$

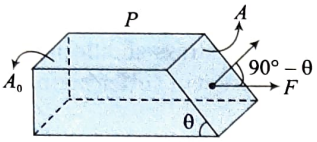
$$\therefore \quad \text{Tension} = (\text{stress})\,A = Ybd\left(\frac{R - r}{r}\right)$$

EXAMPLE 7.4

Two bars A and B are stuck using an adhesive. The contact surface of the bars make an angle θ with the length. Area of cross section of each bar is S_0. It is known that the adhesive yields if normal stress at the contact surface exceeds σ_0. Find the maximum pulling force F that can be applied without detaching the bars.



Sol. Normal force $F_n = F\sin\theta$



$$\text{Area } S = \frac{S_0}{\sin\theta}$$

$$\therefore \quad \sigma_n = \frac{F_n}{S} = \frac{F\sin\theta}{\frac{S_0}{\sin\theta}} = \frac{F}{S_0}\sin^2\theta$$

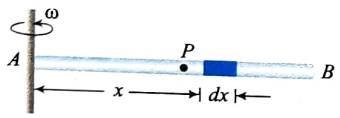
$$\Rightarrow \quad \frac{F_{max}\sin^2\theta}{S_0} = \sigma_0$$

$$\therefore \quad F_{max} = \frac{\sigma_0 S_0}{\sin^2\theta}$$

EXAMPLE 7.5

A uniform material rod of length L is rotated in a horizontal plane about a vertical axis through one of its ends. The angular speed of rotation is ω. Find increase in length of the rod if density and Young's modulus of the rod are ρ and Y respectively.

Sol. Consider an element of length dx at a distance x from the rotation axis. Centripetal force required for this element is $(\rho A dx)\omega^2 x$



Tension at a point P, at a distance r from the axis is equal to sum of centripetal forces on all elements lying between P and B.

$$\therefore \quad \text{Tension at } P,\ T_r = \rho A\omega^2 \int_r^L x\,dx = \frac{\rho A\omega^2}{2}(L^2 - r^2)$$

Now assume that dl is extension in an element of length dr located at a distance r from the axis.

$$\text{Strain} = \frac{dl}{dr}$$

$$\text{Stress} = \frac{T_r}{A} = \frac{1}{2}\rho\omega^2\,[L^2 - r^2]$$

$$\therefore \quad Y\frac{dl}{dr} = \frac{1}{2}\rho\omega^2\,[L^2 - r^2]$$

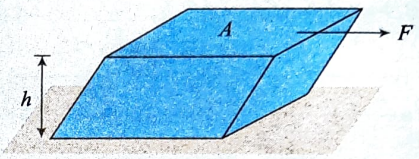
$$\Rightarrow \quad dl = \frac{1}{2}\frac{\rho\omega^2}{Y}[L^2 - r^2]dr$$

$\therefore$ Change in length of the entire rod,

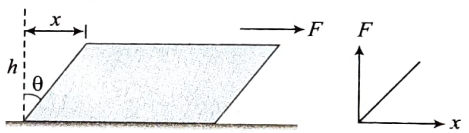
$$\Delta l = \int dl = \frac{\rho\omega^2}{2Y}\int_0^L (L^2 - r^2)\,dr = \frac{\rho\omega^2}{2Y}\left(L^3 - \frac{L^3}{3}\right) = \frac{1}{3}\frac{\rho\omega^2 L^3}{Y}$$

EXAMPLE 7.6

A rectangular bar is fixed to a hard floor. Height of the bar is h and its area in contact with the floor is A. A shearing force distorts the bar as shown. Prove that the work done by the shearing force is $W = \left(\frac{\sigma^2}{2\eta}\right) \times$ volume of the bar. (Here η is shear modulus of elasticity. Assume the deformation to be small.)



Sol.



Stress = η. Strain

$$\therefore \quad \sigma = \frac{F}{A} = \eta\frac{x}{h} \Rightarrow x = \frac{\sigma h}{\eta}$$

Since force changes linearly with x, we can write its work done as

$$W = \frac{1}{2}Fx = \frac{1}{2}(\sigma A)\left(\frac{\sigma h}{\eta}\right)$$

$$= \frac{\sigma^2}{2\eta}Ah = \frac{\sigma^2}{2\eta} \times \text{volume}$$

EXAMPLE 7.7

A horizontally oriented copper rod of length l is rotated about a vertical axis passing through its middle. Calculate the rotated frequency at which the rod ruptures. (Given: Breaking or rupture strength of copper is σ and density of copper is ρ.)

Sol. The stress is zero at the free ends and maximum at the axis. Therefore, the rod will rupture at the middle.

Let us consider the element of rod at a distance x from the axis, the mass of element $dm = \rho A dx$

Applying Newton's second law, $F - (F + dF) = (dm)\,a_n$

or $\quad -dF = (\rho A dx)\omega^2 x$

where ω is the rotation speed.

$$F = -\int dF = -\int(\rho A dx)\omega^2 x = -\rho A\omega^2\frac{x^2}{2} + C$$

at $\quad x = \frac{l}{2},\ F = 0$

$$\therefore \quad C = \frac{\rho A\omega^2 l^2}{8}$$

at $x = 0$, $F = \rho \dfrac{A\omega^2 l^2}{8}$

and Stress $= \dfrac{F}{A} = \dfrac{\rho\omega^2 l^2}{8}$

Rupture of rod will occur when Stress $= \sigma$

$$\sigma = \frac{\rho\omega^2 l^2}{8} \Rightarrow \omega = \sqrt{\frac{8\sigma}{\rho l^2}}$$

or $\nu = \dfrac{1}{2\pi}\sqrt{\dfrac{8\sigma}{\rho l^2}}$

EXAMPLE 7.8

The limiting stress of a typical human bone is 0.9×10^8 N/ m^2 while Young's modulus is 1.4×10^{10} N/ m^2. How much energy can be absorbed by two leg bones (without breaking) if each has a typical length of 50 cm and an average cross-sectional area of 5 cm^2?

Sol. We are given that limiting stress $= 0.9 \times 10^8$ N/ m^2,

$Y = 1.4 \times 10^{10}$ N/m^2

Length of two bones in a leg, $L = 2 \times 50$ cm $= 100$ cm $= 1.0$ m

Area of cross-section of the bone, $A = 5$ cm^2 $= 5 \times 10^{-4}$ m^2

As stress $= F/A$, $F =$ stress $\times A$

or $F = 0.9 \times 10^8 \times 5 \times 10^{-4} = 4.50 \times 10^4$ N

Further, as $Y = \dfrac{\text{stress}}{\text{longitudinal strain } (\Delta L / L)}$

or $\dfrac{\Delta L}{L} = -\dfrac{\text{stress}}{Y}$

or $\Delta L = \dfrac{\text{stress}}{Y} \times L = \dfrac{0.9 \times 10^8}{1.4 \times 10^{10}} \times 1 = 6.43 \times 10^{-3}$ m

Elastic potential energy, $U = \dfrac{1}{2} F \times \Delta L$

or $U = \dfrac{1}{2} \times (4.50 \times 10^4 \text{ N})(6.43 \times 10^{-3} \text{ m}) = 145$ J

EXAMPLE 7.9

A sphere of radius 10 cm and mass 25 kg is attached to the lower end of a steel wire which is suspended from the ceiling of a room. The point of support is 521 cm above the floor. When the sphere is set swinging as a simple pendulum, its lowest point just grazes the floor.
(Given: Young's modulus of steel $= 2 \times 10^{11}$ N/m^2,
Unstretched length of the wire = 500 cm,
Radius of the steel wire = 0.05 cm)

Sol. A wire can be considered as a spring of suitable force constant. For a wire,

$$Y = \frac{F/A}{\Delta l/l} = \frac{FL}{A\Delta l} \Rightarrow F = \frac{Y\Delta l A}{L}$$

Now $F = k \cdot \Delta l$

Therefore, k (force constant) $= \dfrac{F}{\Delta l} = \dfrac{YA}{l}$

$$\Rightarrow \quad k = \frac{20 \times 10^{10} \times \pi (0.05 \times 10^{-2})^2}{5} = \pi \times 10^4 \text{ N/m}^2$$

Now, we may treat the problem as if the mass were suspended by a spring of force constant $k = 10^4\ \pi$ N/m^2. Considering the dynamics of circular motion at the lowest point

$$T - mg = \frac{mv^2}{r}$$

Obviously, at the lowest point, the wire elongates by 1 cm

$$\therefore \quad T = k\Delta l = \pi \times 10^4 \times 0.01 = \pi \times 10^2 \text{ N}$$

$$\therefore \quad \pi \times 10^2 - 25 \times 9.8 = \frac{25 \times v^2}{5.21}$$

$$v^2 = \frac{(100\pi - 245) \times 5.21}{25} = 14.42$$

$$\Rightarrow \quad v = 3.8 \text{ ms}^{-1}$$

EXAMPLE 7.10

A circular ring of radius R and mass m made of a uniform wire of cross-sectional area A is rotated about a stationary vertical axis passing through its centre and perpendicular to the plane of the ring.

(a) If the breaking stress of the material of the ring is σ_0, then determine the maximum angular speed ω_{max} at which the ring may be rotated without failure.

(b) If Young's modulus of the material of ring is Y, then determine the increment in the radius ΔR of the ring.

Sol.

(a) Every element of the ring rotates in a circular path of radius R about the axis of rotation. The radial component of tension in the wire provides the centripetal forces.

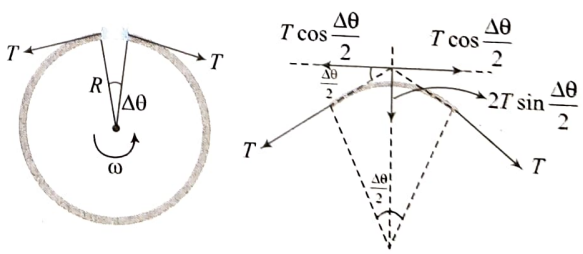


Figure shows a free-body diagram of a small element of mass

$$\Delta m = \frac{m\Delta\theta}{2\pi} \text{ of the ring}$$

Let the tension in the ring be T.

Its resolved component acting towards the centre of rotation is

$$2T \sin\left(\frac{\Delta\theta}{2}\right) = T\Delta\theta$$

The radial component of tension is

$$F_r = 2T\sin\left(\frac{\Delta\theta}{2}\right) = T\Delta\theta$$

Applying Newton's second law, we get

$$T\Delta\theta = \left(\frac{m\Delta\theta}{2\pi}\right)\omega^2 R \text{ or } T = \frac{m\omega^2 R}{2\pi}$$

If the breaking stress is σ_0, then the maximum value of F can be

$$T_{max} = \sigma_0 A$$

$$\frac{m\omega_{max}^2 R}{2\pi} = \sigma_0 A \Rightarrow \omega_{max} = \sqrt{\frac{2\pi\sigma_0 A}{mR}}$$

(b) Stress in wire, $\sigma = \frac{T}{A} = \frac{m\omega^2 R}{2\pi A}$

If ΔR is the increment in the radius of the ring, then the elongation in the length of the ring is

$$\delta = 2\pi(R + \Delta R) - 2\pi R = 2\pi(\Delta R)$$

$$\text{Strain } \in = \frac{\delta}{2\pi R} = \frac{\Delta R}{R}$$

$$\text{Since } Y = \frac{\sigma}{\in} \Rightarrow \in = \frac{\sigma}{Y}$$

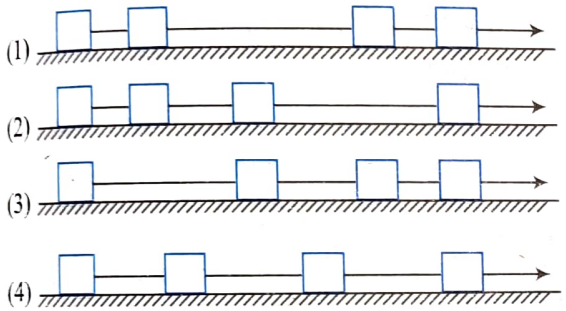
$$\Delta R = \frac{\sigma}{Y}R = \left(\frac{m\omega^2 R}{2\pi A}\right)\frac{R}{Y} \text{ or } \Delta R = \frac{m\omega^2 R^2}{2\pi AY}$$

Exercises

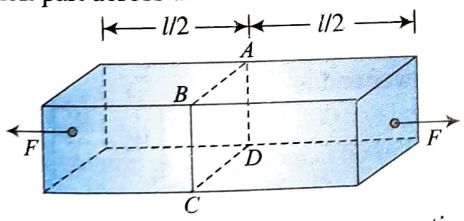
Single Correct Answer Type

1. A copper wire and a steel wire of the same cross-sectional area and length are joined end-to-end (at one end). Equal and opposite longitudinal forces are applied to the free ends giving a total elongation of l. Then the two wires will have
(1) same stress and same strain
(2) same stress and different strains
(3) different stresses and same strain
(4) different stresses and different strains

2. Each of the pictures shows four objects tied together with rubber bands being pulled to the right across a horizontal frictionless surface by a horizontal force F. All the objects have the same mass; all the rubber bands obey Hooke's law and have the same equilibrium length and the same force constant. Which of these pictures is drawn most correctly?



3. Two equal and opposite point forces are applied at mid-points of the ends of a rod of square cross section, as shown. Consider the dotted section $ABCD$. If the rod is cut across this cross section, the force exerted by the right part of the rod on left part across this cross section is



(1) acting at point passing through cross section $ABCD$
(2) acting at a point but not passing through the centre of cross section $ABCD$
(3) uniformly distributed across the cross section $ABCD$
(4) non-uniformly distributed across the cross section $ABCD$

4. On applying a stress of x N/m^2, the length of wire of some material gets doubled. Value of Young's modulus for the material of wire in N/m^2, is (assume Hooke's law to be valid and go for approx. results)
(1) x (2) $2x$
(3) $x/2$ (4) Insufficient information

5. The ratio of diameters of two wires of same material is n:1. The length of each wire is 4 m. On applying the same load, the increase in the length of the thin wire will be ($n > 1$)
(1) n^2 times (2) n times
(3) $2n$ times (4) $(2n + 1)$ times

6. A nylon rope 2 cm in diameter has a breaking strength of 1.5×10^5 N. The breaking strength of a similar rope 1 cm in diameter is
(1) 0.375×10^5 N (2) 2×10^5 N
(3) 6×10^5 N (4) 9×10^4 N

7. The dimensions of four wires of the same material are given below. In which wire the increase in the length will be maximum?
(1) Length 100 cm, diameter 1 mm
(2) Length 200 cm, diameter 2 mm
(3) Length 300 cm, diameter 3 mm
(4) Length 50 cm, diameter 0.5 mm

8. Two wires of the same material and length but diameters in the ratio 1:2 are stretched by the same force. The potential energy per unit volume for the two wires when stretched will be in the ratio
(1) 16:1 (2) 4:1
(3) 2:1 (4) 1:1

9. Two wires of the same material and length are stretched by the same force. Their masses are in the ratio 3:2. Their elongations are in the ratio
(1) 3:2 (2) 9:4
(3) 2:3 (4) 4:9

10. Two wires of the same length and same material but radii in the ratio of 1:2 are stretched by unequal forces to produce equal elongation. The ratio of the two forces is
(1) 1:1 (2) 1:2
(3) 1:3 (4) 1:4

11. When a weight of 5 kg is suspended from a copper wire of length 30 m and diameter 0.5 mm, the length of the wire increases by 2.4 cm. If the diameter is doubled, the extension produced is
(1) 1.2 cm (2) 0.6 cm
(3) 0.3 cm (4) 0.15 cm

12. The length of a wire is increased by 1 mm on the application of a given load. In a wire of the same material, but of length and radius twice that of the first, on application of the same load, extension is
(1) 0.25 mm (2) 0.5 mm
(3) 2 mm (4) 4 mm

13. A cube is shifted to a depth of 100 m is a lake. The change in volume is 0.1%. The bulk modulus of the material is nearly
(1) 10 Pa (2) 10^4 Pa
(3) 10^7 Pa (4) 10^9 Pa

14. An elastic material of Young's modulus Y is subjected to a stress S. The elastic energy stored per unit volume of the material is
(1) $\frac{SY}{2}$ (2) $\frac{S^2}{2Y}$
(3) $\frac{S}{2Y}$ (4) $\frac{2S}{Y}$

15. What amount of work is done in increasing the length of a wire through unity?

(1) $\frac{YL}{2A}$ (2) $\frac{YL^2}{2A}$

(3) $\frac{YA}{2L}$ (4) $\frac{YL}{A}$

16. Young's modulus of rubber is 10^4 N/m^2 and area of cross section is 2 cm^{-2}. If force of 2×10^5 dyn is applied along its length, then its new length becomes

(1) $3l$ (2) $4l$

(3) $2l$ (4) none of these

17. When a certain weight is suspended from a long uniform wire, its length increases by 1 cm. If the same weight is suspended from another wire of the same material and length but having a diameter half of the first one, the increase in length will be

(1) 0.5 cm (2) 2 cm

(3) 4 cm (4) 8 cm

18. Two wires of the same material have lengths in the ratio 1:2 and their radii are in the ratio $1:\sqrt{2}$. If they are stretched by applying equal forces, the increase in their lengths will be in the ratio

(1) $\sqrt{2}$:2 (2) 2:$\sqrt{2}$

(3) 1:1 (4) 1:2

19. A piece of copper wire has twice the radius of a piece of steel wire. Young's modulus for steel is twice that of the copper. One end of the copper wire is joined to one end of the steel wire so that both can be subjected to the same longitudinal force. By what fraction of its length will the steel have stretched when the length of the copper has increased by 1%?

(1) 1% (2) 2%

(3) 2.5% (4) 3%

20. The breaking stress for a substance is 10^6 N/m^2. What length of the wire of this substance should be suspended vertically so that the wire breaks under its own weight? (Given: density of material of the wire = 4×10^3 kg/m^3 and $g = 10$ m s^{-2})

(1) 10 m (2) 15 m

(3) 25 m (4) 34 m

21. When the load on a wire is slowly increased from 3 to 5 kg wt, the elongation increases from 0.61 to 1.02 mm. The work done during the extension of wire is

(1) 0.16 J (2) 0.016 J

(3) 1.6 J (4) 16 J

22. Two identical wires of iron and copper with their Young's modulus in the ratio 3:1 are suspended at same level. They are to be loaded so as to have the same extension and hence level. Ratio of the weight is

(1) 1:3 (2) 2:1

(3) 3:1 (4) 4:1

23. A wire of cross section A is stretched horizontally between two clamps located $2l$ apart. A weight W is suspended from the mid-point of the wire. If the mid-point sags vertically through a distance $x < 1$ the strain produced is

(1) $\frac{2x^2}{l^2}$ (2) $\frac{x^2}{l^2}$

(3) $\frac{x^2}{2l^2}$ (4) none of these

24. If in the above question, the Young's modulus of the material is Y, the value of extension x is

(1) $\left(\frac{Wl}{YA}\right)^{1/3}$ (2) $\left(\frac{YA}{Wl}\right)^{1/3}$

(3) $\frac{1}{l}\left[\frac{WA}{Y}\right]^{1/3}$ (4) $l\left[\frac{W}{YA}\right]^{1/3}$

25. A long wire hangs vertically with its upper end clamped. A torque of 8 Nm applied to the free end twists it through 45°. The potential energy of the twisted wire is

(1) π J (2) $\frac{\pi}{2}$ J

(3) $\frac{\pi}{4}$ J (4) $\frac{\pi}{8}$ J

26. Bulk modulus of water is 2×10^9 N/m^2. The change in pressure required to increase the density of water by 0.1% is

(1) 2×10^9 N/m^2 (2) 2×10^8 N/m^2

(3) 2×10^6 N/m^2 (4) 2×10^4 N/m^2

27. A wire of length L and radius r is fixed at one end. When a stretching force F is applied at free end, the elongation in the wire is l. When another wire of same material but of length $2L$ and radius $2r$, also fixed at one end is stretched by a force $2F$ applied at free end, then elongation in the second wire will be

(1) $l/2$ (2) l

(3) $2l$ (4) $l/4$

28. One end of uniform wire of length L and of weight W is attached rigidly to a point in the roof and a weight W_1 is suspended from its lower end. If s is the area of cross section of the wire, the stress in the wire at a height $(3L/4)$ from its lower end is

(1) $\frac{W_1}{s}$ (2) $\left[W_1 + \frac{W}{4}\right]/s$

(3) $\left[W_1 + \frac{3W}{4}\right]/s$ (4) $\frac{W_1 + W}{s}$

29. A wire is stretched 1 mm by a force of 1 kN. How far would a wire of the same material and length but of four times that diameter be stretched by the same force?

(1) $\frac{1}{2}$ mm (2) $\frac{1}{4}$ mm

(3) $\frac{1}{8}$ mm (4) $\frac{1}{16}$ mm

30. Young's modulus of brass and steel are 10×10^{10} N/m^2 and 2×10^{11} N/m^2, respectively. A brass wire and a steel wire of the same length are extended by 1 mm under the same force. The radii of the brass and steel wires are R_B and R_S, respectively. Then

(1) $R_S = \sqrt{2}\,R_B$ (2) $R_S = \dfrac{R_B}{\sqrt{2}}$

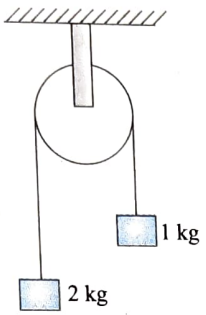
(3) $R_S = 4R_B$ (4) $R_S = \dfrac{R_B}{4}$

31. When the tension in a metal wire is T_1, its length is l_1. When the tension is T_2, its length is l_2. The natural length of wire is

(1) $\dfrac{T_2}{T_1}(l_1 + l_2)$ (2) $T_1 l_1 + T_2 l_2$

(3) $\dfrac{l_1 T_2 - l_2 T_1}{T_2 - T_1}$ (4) $\dfrac{l_1 T_2 + l_2 T_1}{T_2 + T_1}$

32. Two blocks of masses 1 kg and 2 kg are connected by a metal wire going over a smooth pulley as shown in figure. The breaking stress of the metal is $(40/3\pi) \times 10^6$ N/m^2. If g = 10 ms^{-2}, then what should be the minimum radius of the wire used if it is not to break?



(1) 0.5 mm (2) 1 mm

(3) 1.5 mm (4) 2 mm

33. A long elastic spring is stretched by 2 cm and its potential energy is U. If the spring is stretched by 10 cm, the PE will be

(1) 5 U (2) 25 U

(3) U/5 (4) U/20

34. A copper bar of length L and area of cross section A is placed in a chamber at atmospheric pressure. If the chamber is evacuated, the percentage change in its volume will be (compressibility of copper is 8×10^{-12} m^2/N and 1 atm = 10^5 N/m^2)

(1) 8×10^{-7} % (2) 8×10^{-5} %

(3) 1.25×10^{-4} % (4) 1.25×10^{-5} %

35. A small but heavy block of mass 10 kg is attached to a wire 0.3 m long. Its breaking stress is 4.8×10^7 N/m^2. The area of the cross section of the wire is 10^{-6} m^2. The maximum angular velocity with which the block can be rotated in the horizontal circle is

(1) 4 rad/s (2) 8 rad/s

(3) 10 rad/s (4) 32 rad/s

36. A ball falling in a lake of depth 200 m shows a decrease of 0.1% in its volume at the bottom. The bulk modulus of the elasticity of the material of the ball is (take g = 10 ms^{-2})

(1) 10^9 N/m^2 (2) 2×10^9 N/m^2

(3) 3×10^9 N/m^2 (4) 4×10^9 N/m^2

37. A massive stone pillar 20 m high and of uniform cross section rests on a rigid base and supports a vertical load of 5.0×10^5 N at its upper end. If the compressive stress in the pillar is not exceed 16×10^6 N/m^2, what is the minimum cross-sectional area of the pillar? (Density of the stone = 2.5×10^3 kg/m^3 take g = 10 N/kg.)

(1) 0.15 m^2 (2) 0.25 m^2

(3) 0.35 m^2 (4) 0.45 m^2

38. If the work done by stretching a wire by 1 mm is 2 J, the work necessary for stretching another wire of the same material but with the radius of cross section and half the length by 1 mm is

(1) $\frac{1}{4}$ J (2) 4 J

(3) 8 J (4) 16 J

39. Two wires of the same material and same mass are stretched by the same force. Their lengths are in the ratio 2:3. Their elongations are in the ratio

(1) 3:2 (2) 2:3

(3) 4:9 (4) 9:4

40. A wire is suspended vertically from a rigid support. When loaded with a steel weight in air, the wire extends by 16 cm. When the weight is completely immersed in water, the extension is reduced to 14 cm. The relative density of the material of the weight is

(1) 2 g/cm^3 (2) 6 g/cm^3

(3) 8 g/cm^3 (4) 16 g/cm^3

41. Two bars A and B of circular cross section, same volume and made of the same material, are subjected to tension. If the diameter of A is half that of B and if the force applied to both the rod is the same and it is in the elastic limit, the ratio of extension of A to that of B will be

(1) 16 (2) 8

(3) 4 (4) 2

42. A uniform cylindrical wire is subjected to a longitudinal tensile stress of 5×10^7 N/m^2. Young's modulus of the material of the wire is 2×10^{11} N/m^2. The volume change in the wire is 0.02%. The factional change in the radius is

(1) 0.25×10^{-4} (2) 0.5×10^{-4}

(3) 1.0×10^{-4} (4) 1.5×10^{-4}

43. A material has normal density ρ and bulk modulus K. The increase in the density of the material when it is subjected to an external pressure P from all sides is

(1) $P/\rho K$ (2) $K/\rho P$

(3) $\rho P/K$ (4) $\rho K/P$

44. A rubber rope of length 8 m is hung from the ceiling of a room. What is the increase in length of rope due to its own weight? (Given: Young's modulus of elasticity of rubber = 5×10^6 N/m and density of rubber = 1.5×10^3 kg/m^3. Take g = 10 ms^{-2})

(1) 1.5 mm (2) 6 mm

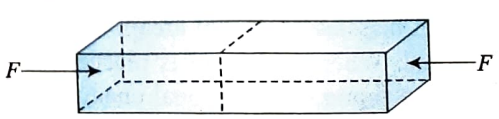
(3) 24 mm (4) 96 mm

45. A thick rope of density ρ and length L is hung from a rigid support. The increase in length of the rope due to its own weight is (Y is the Young's modulus)

(1) $\dfrac{0.1}{4Y}\rho L^2 g$ (2) $\dfrac{1}{2Y}\rho L^2 g$

(3) $\dfrac{\rho L^2 g}{Y}$ (4) $\dfrac{\rho L g}{Y}$

46. In the figure shown, forces of equal magnitude are applied to the two ends of a uniform rod. Consider A as the cross-sectional area of the rod. For this situation, mark out the incorrect statements.



(1) The rod is in compressive stress.
(2) The numerical value of stress developed in the rod is equal to F/A.
(3) The stress is defined as internal force developed at any cross section per unit area.
(4) None of the above.

47. The length of a steel cylinder is kept constant by applying pressure at its two ends. When the temperature of rod is increased by 100°C from its initial temperature, the increase in pressure to be applied at its ends is
($Y_{steel} = 2 \times 10^{11}$ N/m², $\alpha_{steel} = 11 \times 10^{-6}$/°C, 1 atm = 10^5 N/m².)
(1) 22×10^7 atm
(2) 2.2×10^3 atm
(3) zero
(4) 4.3×10^3 atm

48. A 5 kg rod of square cross section 5 cm on a side and 1 m long is pulled along a smooth horizontal surface by a force applied at one end. The rod has a constant acceleration of 2 m s^{-2}. Determine the elongation in the rod. (Young's modulus of the material of the rod is 5×10^9 N/m²).
(1) Zero, as for elongation to be there, equal and opposite forces must act on the rod
(2) Non-zero but cannot be determine from the given situation
(3) 0.4 μm
(4) 16 μm

49. The elastic limit of an elevator cable is 2×10^9 N/m². The maximum upward acceleration that an elevator of mass 2×10^3 kg can have when supported by a cable whose cross-sectional area is 10^{-4} m², provided the stress in cable would not exceed half of the elastic limit would be
(1) 10 m s^{-2}
(2) 50 m s^{-2}
(3) 40 m s^{-2}
(4) not possible to move up

50. A wire can sustain the weight of 20 kg before breaking. If the wire is cut into two equal parts, each part can sustain a weight of
(1) 10 kg
(2) 20 kg
(3) 40 kg
(4) 35 kg

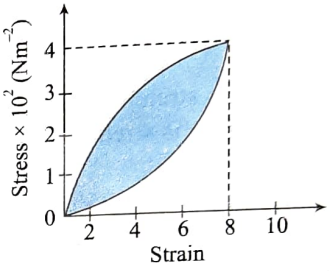
51. A steel wire of length 4.7 m and cross-sectional area 3×10^{-6} m² stretches by the same amount as a copper wire of length 3.5 m and cross-sectional area of 4×10^{-6} m² under a given load. The ratio of Young's modulus of steel to that of copper is
(1) 1.8
(2) 3.6
(3) 0.6
(4) 8.7

52. The edges of an aluminium cube are 10 cm long. One face of the cube is firmly fixed to a vertical wall. A mass of 100 kg is then attached to the opposite face of the cube. Shear modulus of aluminium is 25×10^9 Pa, the vertical deflection in the face to which mass is attached is
(1) 4×10^{-4} m
(2) 4×10^{-7} m
(3) 25×10^{-6} m
(4) 6×10^{-7} m

53. A solid sphere of radius R, made up of a material of bulk modulus K is surrounded by a liquid in a cylindrical container. A massless piston of area A floats on the surface of the liquid. When a mass M is placed on the piston to compress the liquid, the fractional change in the radius of the sphere is
(1) $\frac{Mg}{AK}$
(2) $\frac{Mg}{3AK}$
(3) $\frac{3Mg}{AK}$
(4) $\frac{Mg}{2AK}$

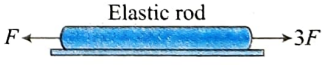
54. The density of water at the surface of ocean is ρ. If the bulk modulus of water is B, then the density of ocean water at depth, when the pressure is αp_0 and p_0 is the atmospheric pressure, is
(1) $\frac{\rho B}{B-(\alpha-1)p_0}$
(2) $\frac{\rho B}{B+(\alpha-1)p_0}$
(3) $\frac{\rho B}{B-\alpha p_0}$
(4) $\frac{\rho B}{B+\alpha p_0}$

55. A rubber of volume 2000 cc is alternately subjected to tension and released. The figure shows the stress–strain curve of rubber. Each curve is a quadrant of an ellipse. The amount of energy lost as heat per cycle per unit volume will be



(1) $\left(\frac{\pi}{2}-1\right)\times 16\times 10^2$ J
(2) $\left(\frac{\pi}{4}-1\right)\times 8\times 10^2$ J
(3) $\left(\frac{\pi}{4}-1\right)\times 32\times 10^2$ J
(4) $\left(\frac{\pi}{2}-1\right)\times 32\times 10^2$ J

56. A uniform elastic rod of cross-section area A, natural length L and Young's modulus Y is placed on a smooth horizontal surface. Now two horizontal forces (of magnitude F and $3F$) directed along the length of rod and in opposite direction act at two of its ends as shown. After the rod has acquired steady state, the extension of the rod will be



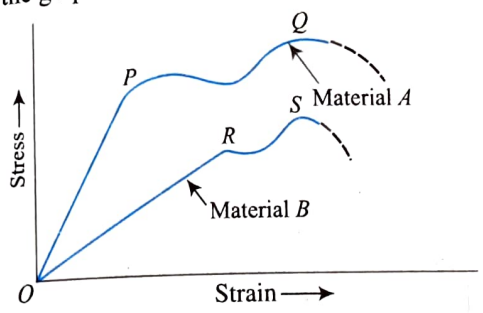
(1) $\frac{2F}{YA}L$
(2) $\frac{4F}{YA}L$
(3) $\frac{F}{YA}L$
(4) $\frac{3F}{2YA}L$

57. A light rod of length 2 m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross section 0.1 cm². The other wire is a brass of cross section 0.2 cm². A weight is suspended from a certain point of the rod such that equal stress are produced in both the wires. Which of the following are correct?
(1) The ratio of tension in the steel and brass wires is 0.5.
(2) The load is suspended at a distance of 400/3 cm from the steel wire.
(3) Both (1) and (2) are correct.
(4) Neither (1) nor (2) is correct.

58. Which of the following are correct?
(1) The product of bulk modulus of elasticity and compressibility is 1.
(2) A rope 1 cm in diameter breaks if the tension in it exceeds 500 N. The maximum tension that may be given to a similar rope of diameter 2 cm is 2000 N.
(3) Both (1) and (2) are correct.
(4) Neither (1) nor (2) is correct.

Multiple Correct Answers Type

1. Figure shows the stress–strain graphs for materials A and B. From the graph it follows that



(1) material A has a higher Young's modulus
(2) material B is more ductile
(3) material A can withstand greater stress
(4) material B can withstand greater stress

2. Two wires A and B have the same cross section and are made of the same material, but the length of wire A is twice that of B. Then, for a given load
(1) the extension of A will be twice that of B
(2) the extensions of A and B will be equal
(3) the strain in A will be half that in B
(4) the strains in A and B will be equal

3. Two wire A and B have equal lengths and are made of the same material, but the diameter of A is twice that of wire B. Then, for a given load
(1) the extension of B will be four times that of A
(2) the extension of A and B will be equal
(3) the strain in B is four times that in A
(4) the strains in A and B will be equal

4. In the previous question,
(1) Young's modulus of A is twice that of B
(2) Young's modulus is the same for both A and B
(3) A can withstand a greater load before breaking than B
(4) A and B will withstand the same load before breaking

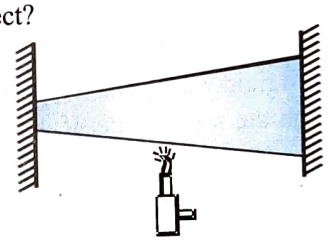
5. Choose the correct statements from the following:
(1) Steel is more elastic than rubber.
(2) The stretching of a coil spring is determined by the Young's modulus of the wire of the spring.
(3) The frequency of a tuning fork is determined by the shear modulus of the material of the fork.
(4) When a material is subjected to a tensile (stretching) stress the restoring forces are caused by interatomic attraction.

6. Which of the following are correct?
(1) For a small deformation of a material, the ratio (stress/strain) is constant.
(2) For a large deformation of a material, the ratio (stress/strain) is constant.
(3) Two wires made of different materials, having the same diameter and length are connected end to end. A force is applied. This stretches their combined length by 2 mm. Now, the strain is same in both the wire but stress is different.
(4) None of these is correct.

7. Which of the following are correct?
(1) The shear modulus of a liquid is infinite.
(2) Bulk modulus of a perfectly rigid body is infinity.
(3) According to Hooke's law, the ratio of the stress and strain remains constant.
(4) None of the above.

8. A uniform plank is resting over a smooth horizontal floor and is pulled by applying a horizontal force at its one end. Which of the following statements are not correct?
(1) Stress developed in plank material is maximum at the end at which force is applied and decrease linearly to zero at the other end.
(2) A uniform tensile stress is developed in the plank material.
(3) Since plank is pulled at one end only, plank starts to accelerate along direction of the force. Hence, no stress is developed in the plank material.
(4) None of the above.

9. A rod is made of uniform material and has non-uniform cross section. It is fixed at both the ends as shown and heated at mid-section. Which of the following statements are not correct?



(1) Force of compression in the rod will be maximum at mid-section.
(2) Compressive stress in the rod will be maximum at left end.
(3) Since rod in fixed at both the ends, its length will remain unchanged. Hence, no strain will be induced in it.
(4) None of the above.

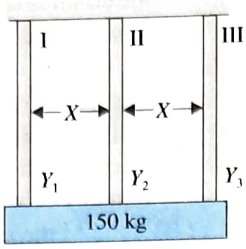
10. When a body of mass M is attached to lower end of a wire (of length L) whose upper end is fixed, then the elongation of the wire is l. In this situation, mark out the correct statement(s).
(1) Loss in gravitational potential energy of M is Mgl.
(2) Elastic potential energy stored in the wire is $\frac{Mgl}{2}$.
(3) Elastic potential energy stored in the wire is Mgl.
(4) Elastic potential energy stored in the wire is $\frac{Mgl}{3}$.

11. A metal wire length L, cross-sectional area A and Young's modulus Y is stretched by a variable force F. F is varying in such a way that F is always slightly greater than the elastic forces of resistance in the wire. When the elongation in the wire is l, up to this instant
(1) the work done by F is $\frac{YAl^2}{2L}$
(2) the work done by F is $\frac{YAl^2}{L}$

(3) the elastic potential energy stored in wire is $\frac{YAl^2}{2L}$

(4) no energy is lost during elongation

12. A heavy block of mass 150 kg hangs with the help of three vertical wires of equal length and equal cross-sectional area as shown in figure. Wire is attached to the mid-point (centre of mass) of block. Take $Y_2 = 2Y_1$. For this arrangement mark out the correct statement(s).



(1) The wire I and III should have same Young's modulus.

(2) Tension in I and III would be always equal.

(3) Tension in I and III would be different.

(4) Tension in II is 75g.

13. A composite rod consists of a steel rod of length 25 cm and area 2A and a copper rod of length 50 cm and area A. The composite rod is subjected to an axial load F. If the Young's moduli of steel and copper are in the ratio 2:1, then

(1) the extension produced in copper rod will be more

(2) the extension in copper and steel parts will be in the ratio 1:2

(3) the stress applied to copper rod will be more

(4) no extension will be produced in the steel rod

14. Four rods A, B, C and D of the same length and material but of different radii $r, r\sqrt{2}, r\sqrt{3}$ and $2r$, respectively, are held between two rigid walls. The temperature of all rods is increased through the same range. If the rods do not bend, then

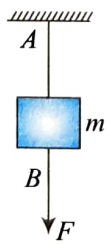
(1) the stress in the rods A, B, C and D is in the ratio 1:2:3:4

(2) the forces on them exerted by the wall are in the ratio 1:2:3:4

(3) the energy stored in the rods due to elasticity is in the ratio 1:2:3:4

(4) it is independent of area like surface tension while friction depends

15. The wires A and B shown in figure are made of the same material and have radii r_A and r_B, respectively. The block between them has a mass m. When the force F is $mg/3$, one of the wires breaks. Then



(1) A will break before B if $r_A = r_B$

(2) A will break before B if $r_A < 2r_B$

(3) either A or B may break if $r_A = 2r_B$

(4) the lengths of A and B must be known to predict which wire will break.

16. A wire of length L and cross-sectional area 'A', made of a material of Young's modulus Y, is pulled by a constant force F on a horizontal surface as shown in figure. Then

Coefficient of friction = μ

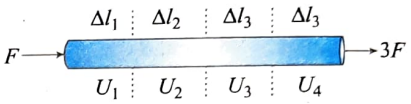
(1) The extension in the wire is $\frac{Fl}{YA}$.

(2) The extension in the wire is $\frac{Fl}{2YA}$.

(3) The tension in the rod at the middle point of the rod is $\frac{F}{2}$.

(4) The tension in the rod at the middle point of the rod is $\frac{F}{2} - \frac{\mu mg}{2}$.

17. If a rod of length l, very small area of cross-section A, Young's modulus of elasticity Y is acted upon by two parallel forces $3F$ and F respectively (as shown) and placed on a smooth horizontal plane. If the elastic limit is not crossed and then to study the change in length of rod (Δl) and it's elastic potential energy (U) the rod is segmented into four equal parts where magnitude of change in lengths are Δl_1, Δl_2, Δl_3, Δl_4 and elastic potential energy stored in each segment are U_1, U_2, U_3, U_4 respectively as shown then which is/are correct?



(1) $\Delta l_1 = \frac{\Delta l_3}{3} = \frac{\Delta l}{8}$ (2) $U_1 < \frac{U}{4} < U_4$

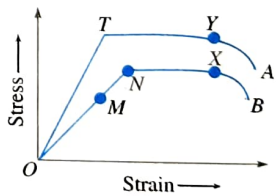
(3) $\Delta l_2 = \frac{\Delta l_4}{5} = \frac{\Delta l}{8}$ (4) $U_2 < U_3 < U_4$

Linked Comprehension Type

For Problems 1–3

Materials get deformed when force is applied. Some of them regain their status when the applied force is removed. They are termed as elastic. Those of which not regaining are called plastic. There may be delay in the regaining in some materials. They are said to have got elastic aftereffect, since they have gone beyond the elastic limit. Repeated application and removal of force leads to fatigueness in the material. Fatigued materials may break at any point of time and so are avoided.

The stress strain graph for two materials A and B is shown in the following figure:



1. If the intensity of A and B is E_A and E_B, respectively, then

(1) $E_A = E_B$ (2) $E_A > E_B$

(3) $E_A < E_B$ (4) $E_A \ll E_B$

2. The strength of the material A and B is S_A and S_B, respectively, while the longevity of plastic behaviour is L_A and L_B. Then

(1) $S_A > S_B, L_A < L_B$ (2) $S_A = S_B, L_A = L_B$

(3) $S_A > S_B, L_A > L_B$ (4) $S_A < S_B, L_A < L_B$

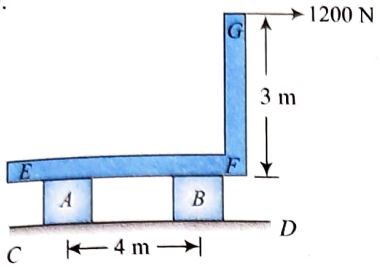
3. The time in which the two materials regain their original status is t_A and t_B related as $t_B = 2t_A$. Then the material under elastic aftereffect (relatively) is

(1) B (2) A

(3) both A and B (4) neither A nor B

For Problems 4–6

In the figure shown, A and B are two short steel rods each of cross-sectional area 5 cm^2. The lower ends of A and B are welded to a fixed plate CD. The upper end of A is welded to the L-shaped piece EFG, which can slide without friction on upper end of B. A horizontal pull of 1200 N is exerted at G as shown. Neglect the weight of EFG.



4. Mark out the correct statement(s).
   (1) Shearing stress in A is zero.
   (2) Shearing stress in B is zero.
   (3) Shearing stress in both A and B is zero.
   (4) Shearing stress in both A and B is non-zero.
5. Longitudinal stress in A is
   (1) tensile in nature and having magnitude 180 N/m^2
   (2) tensile in nature and having magnitude 240 N/m^2
   (3) compressive in nature and having magnitude 180 N/m^2
   (4) compressive in nature and having magnitude 240 N/m^2
6. Longitudinal stress in B is
   (1) tensile in nature and having magnitude 180 N/cm^2
   (2) tensile in nature and having magnitude 240 N/cm^2
   (3) compressive in nature and having magnitude 180 N/cm^2
   (4) compressive in nature and having magnitude 240 N/cm^2

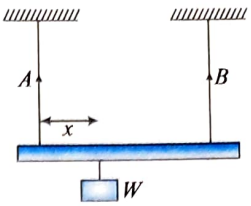
For Problems 7–9

According to Hooke's law, within the elastic limit stress/strain = constant. This constant depends on the type of strain or the type of force acting. Tensile stress might result in compressional or elongative strain; however, a tangential stress can only cause a shearing strain. After crossing the elastic limit, the material undergoes elongation and beyond a stage breaks. All modulus of elasticity are basically constants for the materials under stress.

7. Two wires of the same material have length and radius (l, r) and $\left(2l, \frac{r}{2}\right)$. The ratio of their Young's modulus is
   (1) 1:2 (2) 2:3
   (3) 2:1 (4) 1:1
8. Just on crossing the yield region, the material will have
   (1) reduced stress (2) increased stress
   (3) breaking stress (4) constant stress
9. If stress/strain is x in elastic region and y in the region of yield, then
   (1) $x = y$ (2) $x > y$
   (3) $x < y$ (4) $x = 2y$

For Problems 10–11

A light rod of length $L = 2$ m is suspended horizontally from the ceiling by two wires A and B of equal lengths. The wire A is made of steel with the area of cross section $A_S = 1 \times 10^{-5}$ m^2, while the wire B is made of brass of cross-sectional area $A_b = 2 \times 10^{-5}$ m^2. A weight W is suspended at a distance x from the wire A as shown in figure. Take, Young's modulus of steel and brass as $Y_s = 2 \times 10^{11}$ Nm^{-2} and $Y_b = 1 \times 10^{11}$ Nm^{-2}.



10. Determine the value of x so that equal stresses are produced in each wire.
    (1) 1.33 m (2) 2.5 m
    (3) 3.6 m (4) 2.1 m
11. Determine the value of x so that equal strains are produced in each wire.
    (1) 1 m (2) 2 m
    (3) 3 m (4) 2.2 m

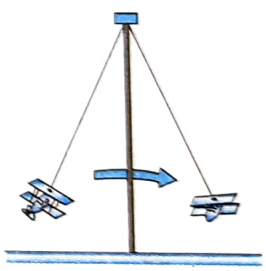
For Problems 12–13

A copper wire of negligible mass, 1m length and cross-sectional area 10^{-6} m^2 is kept on a smooth horizontal table with one end fixed. A ball of mass 1 kg is attached to the other end. The wire and the ball are rotating with an angular velocity of 20 rad/s. If the elongation in the wire is 10^{-3} m.

12. The Young's modulus of the wire is
    (1) 4.0×10^{11} N/m^2 (2) 2.5×10^{11} N/m^2
    (3) 2.0×10^{11} N/m^2 (4) 4.5×10^{11} N/m^2
13. If for the same wire as stated above the angular velocity is increase to 100 rad/s and the wire breaks down, the breaking stress is
    (1) 1.5×10^{10} N/m^2 (2) 10^{10} N/m^2
    (3) 2.0×10^{10} N/m^2 (4) 3.0×10^{10} N/m^2

For Problems 14–15

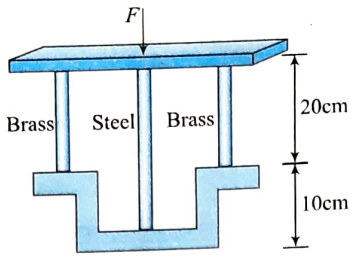
An amusement park ride consists of airplane shaped cars attached to steel rods. Each rod has a length of 20.0 m and a cross-sectional area of 8.00 cm^2. Young's modulus for steel is 2×10^{11} N/m^2. (Assume that each car plus two people seated in it has a total weight of 2000 N.)



14. How much is the rod stretched when the ride is at rest ?
    (1) 0.5 mm (2) 1.0 mm
    (3) 0.25 mm (4) 2.5 mm
15. When operating, the ride has a maximum angular speed of $\frac{\sqrt{19}}{5}$ rad/s. How much is the rod stretched then ?
    (1) 0.38 mm (2) 3.8 mm
    (3) 1.9 mm (4) 0.19 mm

For Problems 16–17

A steel rod of length $l_1 = 30$ cm and two identical brass rod of length $l_2 = 20$ cm each, support a light horizontal platform as shown in figure. Cross-sectional area of each of the three rods is $A = 1$ cm^2.



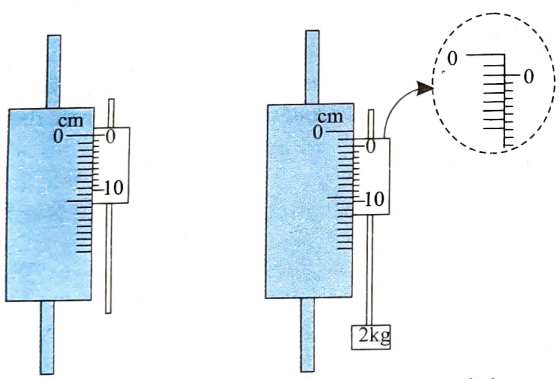
A vertically downward force $F = 5000$ N is applied on the platform. Given, Young's modulus of elasticity for steel $Y_S = 2\times10^{11}$ Nm^{-2} and brass $Y_b = 1\times10^{11}$ Nm^{-2}.

16. The stress developed in steel rod is
(1) 25 MPa (2) 20 MPa
(3) 10 MPa (4) 15 MPa

17. The stress developed in brass rod is
(1) 25 MPa (2) 20 MPa
(3) 10 MPa (4) 15 MPa

For Problems 18–19

In a experiment to determine the Young's modulus of a metal, two identical wires are suspended using a vernier arrangement as shown in figure. The left wire is held taut with a fixed mass. Different loads can be applied to the right wire and stretching of the wire can be measured by using the vernier scale attached to the wire on right. The test wire has a length of 2.2 m between support and vernier and a uniform diameter of 0.42 mm. Take $g = 10$ ms^{-2} and give your answer in appropriate significant digits.



(a) weight is not suspended **(b) 2kg weight is suspended**

18. The strain produced in the wire is
(1) 0.001 (2) 0.0010
(3) 0.091 (4) 0.0909

19. The Young's modulus of the material is
(1) 1.40×10^{11} Nm^{-2} (2) 1.50×10^{11} Nm^{-2}
(3) 1.70×10^{11} Nm^{-2} (4) 1.63×10^{11} Nm^{-2}

Matrix Match Type

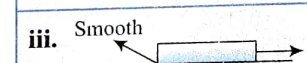
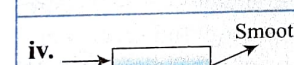
1. With regard to dependence of quantities given in Columns I and II, match the following:

Column I	Column II
i. Young's modulus of a substance	**a.** Depends on temperature
ii. Bulk modulus of a substance	**b.** Depends on length
iii. Modulus of rigidity of a substance	**c.** Depends on area of cross section
iv. Volume of a substance	**d.** Depends on the nature of material

2. Match the following:

Column I	Column II
i. longitudinal stress	**a.** Volume changes
ii. Shear stress	**b.** Shape changes
iii. Volume changes	**c.** Volume does not change
iv. Tensile stress	**d.** Shape does not change

3. In Column I, a uniform bar of uniform cross-sectional area under the application of forces is shown in the figure and in Column II, some effects/phenomena are given. Match the entries of Column I with the entries of Column II.

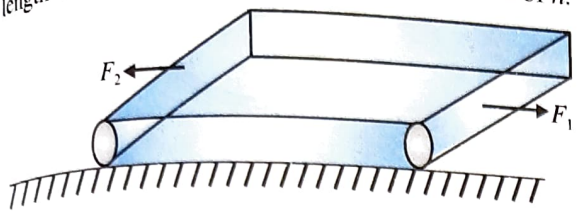
Column I	Column II
i.	**a.** Uniform stresses developed in the rod
ii.	**b.** Non-uniform stresses developed in the rod
iii. 	**c.** Compressive stress developed
iv. 	**d.** Tensile stresses developed

4. Consider a wire of length l, cross-sectional area A and Young's modulus Y and match Column I with Column II:

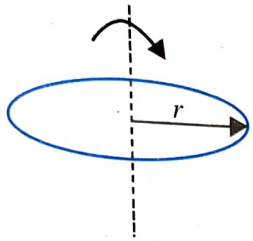
Column I	Column II
i. If the wire is pulled at its ends by equal and opposite forces of magnitude F so that it undergoes an elongation x, according to Hooke's law, $F = kx$, where (k) of the wire will depend on	**a.** Young's modulus Y
ii. Let us suspend the wire vertically from a rigid support and attach a mass m at its lower end. If the mass is slightly pulled down and released, it executes S.H.M. of a time period that will depend on	**b.** Elongation (x)
iii. If the given wire is fixed between two rigid supports and its temperature is increased thermal stress that develops in the rod will depend on	**c.** Length (l)
iv. Work done in stretching the wire to a section length $l + x$ will depend on	**d.** Area of cross-section (A)

Numerical Value Type

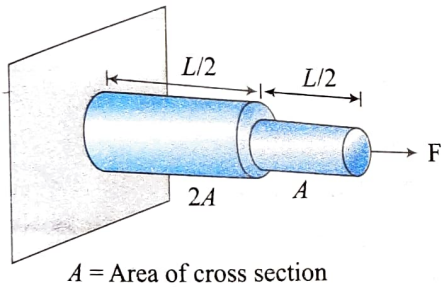
1. Two opposite forces $F_1 = 120$ N and $F_2 = 80$ N act on an elastic plank of modulus of elasticity $Y = 2 \times 10^{11}$ N/m^2 and length $l = 1$ m placed over a smooth horizontal surface. The cross-sectional area of the plank is $S = 0.5$ m^2. The change in length of the plank is $n \times 10^{-11}$ m. Find the value of n.



2. A ring of radius r made of wire of density ρ is rotated about a stationary vertical axis passing through its centre and perpendicular to the plane of the ring as shown in the figure. Determine the angular velocity (in rad/s) of ring at which the ring breaks. The wire breaks at tensile stress σ. Ignore gravity. Take $\sigma/\rho = 4$ and $r = 1$ m.

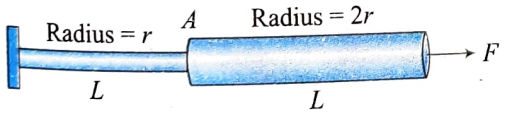


3. The bar shown in the figure is made of a single piece of material. It is fixed at one end. It consists of two segments of equal length $L/2$ each but different cross-sectional area A and $2A$. Find the ratio of total elongation in the bar to the elongation produced in thicker segment under the action of an axial force F. Consider the shape of joint to remain circular. (Given: Y is Young's modulus).



A = Area of cross section

4. Two steel wires of same length but radii r and $2r$ are connected together end to end and tied to a wall as shown.



The force stretches the combination by 10 mm. How far does the midpoint A move (in mm)?

5. A 0.05 m cube has its upper face displaced by 0.2 cm by a tangential force of 8 N. Calculate the modulus of rigidity (in $\times 10^4$ Nm^{-2}) of the material of the cube.

6. Two parallel and opposite forces, each of magnitude 4000 N, are applied tangentially to the upper and lower faces of a cubical metal block 25 cm on a side. Find the displacement of the upper surface relative to the lower surface (in $\times 10^{-5}$ cm). The shear modulus for the metal is 80 GPa.

7. A metal cube of side 10 cm is subjected to a shearing stress of 10^6 N/m^2. Calculate the modulus of rigidity (in $\times 10^8$ Nm^{-2}) if the top of the cube is displaced by 0.05 cm with respect to its bottom.

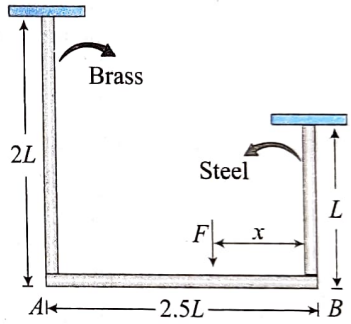
8. Calculate the density of water near the bottom of the ocean where the pressure is about 501 atm and density of water is 1025 kg/m^3 approximately. Take bulk modulus of water (in kg m^{-3}) as for water as 2.0×10^9 N/m^2 and 1 atm is 1000 kg/m^3

9. A hydraulic press contains 0.25 m^3 (250 L) of oil. Find the decrease in volume of the oil (in %) when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa. The bulk modulus of the oil is $B = 5.0 \times 10^9$ Pa.

10. Compressibility of water is 5×10^{-10} m^2/N. Find the decrease in volume (in mL) of 100 mL of water when subjected to a pressure of 15 MPa.

11. Bulk modulus for rubber is 9.8×10^8 N m^{-2}. To what depth (in m) should a rubber ball be taken in a lake so that its volume is decreased by 0.1%?

12. A very stiff bar (AB) of negligible mass is suspended horizontally by two vertical rods as shown in figure. Length of the bar is 2.5 L. The steel rod has length L and cross sectional radius of r and the brass rod has length $2L$ and cross sectional radius of $2r$. A vertically downward force F is applied to the bar at a distance x from the steel rod and the bar remains horizontal. Find the ratio of x/L if it is given that ratio of Young's modulus of steel and brass is $Y_S/Y_B = 2$.



13. A sphere of radius 0.1 m and mass 8π kg is attached to the lower end of a steel wire of length 5m and diameter 10^{-3} m. The wire is suspended from 5.22 m high ceiling of a room. When the sphere is made to swing as a simple pendulum, it just grazes the floor at its lowest point. Young's modulus of steel is 2×10^{11} Nm^{-2}. Find the velocity of the sphere at the lowest position in m/s. (Given: $g = 10$ m/s^2)

14. The breaking stress for a metal is 7.8×10^9 Nm^{-2}. The density of the metal is 7800 kg m^{-3}. If $g = 10$ N kg^{-1}, Find the maximum length of the wire made of this metal which may be suspended without breaking.

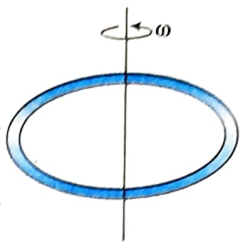
15. The average depth of indian ocean is about 3000 m. Find the fractional compression $\frac{\Delta V}{V}$ of water of the bottom of the ocean. Given, bulk modulus of water is 2.2×10^9 Nm^{-2} and density of water = 1000 kg m^{-3}.

16. A solid sphere of radius R made of a material of bulk modulus B is surrounded by a liquid in a cylindrical container. A massless piston of area A (the area of container is also A) floats on the surface of the liquid. When a mass m is placed on the piston to compress the liquid, find the fractional change in radius of the sphere. (Given: $\frac{Mg}{AB} = 0.3$)

17. A wire having a length L and cross-sectional area A is suspended at one of its ends from a ceiling. Density and Young's modulus of material of the wire are ρ and Y, respectively. Find its strain energy due to its own weight in μJ. (Given: $\frac{\rho^2 g^2 AL^3}{Y} = 12\times10^{-6}$ J)

18. Find the elastic potential energy per unit volume of water (in $\times 10^3$ Jm^{-3}) at a depth of 1km. Given, compressibility of water $= 5 \times 10^{10}$ SI units and density of water $= 1000$ kg m^{-3}.

19. A ring of radius r (made of wire of density ρ) placed on a smooth horizontal surface is rotated about a stationary vertical axis passing through its centre and perpendicular to the plane of the ring as shown in figure. Determine the angular velocity in rad/s at which the ring breaks. The wire breaks at a tensile stress σ. (given $\sigma/\rho = 9$ N-m/kg and $r = 1$ m)



Archives

JEE Main

Single Correct Answer Type

1. Two wires are made of the same material and have the same volume. However, wire 1 has cross-sectional area A and wire 2 has cross-sectional area $3A$. If the length of wire 1 increases by Δx on applying a force F, how much force is needed to stretch wire 2 by the same amount?
(1) F (2) $4F$
(3) $6F$ (4) $9F$ **(AIEEE 2009)**

2. The pressure that has to be applied to the ends of a steel wire of length 10 cm to keep its length constant when its temperature is raised by 100°C is
(For steel Young's modulus is 2×10^{11} N m^{-2} and coefficient of thermal expansion is 1.1×10^{-5} K^{-1})
(1) 2.2×10^7 Pa (2) 2.2×10^6 Pa
(3) 2.2×10^8 Pa (4) 2.2×10^9 Pa
(JEE Main 2014)]

3. A pendulum made of a uniform wire of cross sectional area A has time period T. When an additional mass M is added to its bob, the time period changes to T_M. If the Young's modulus of the material of the wire is Y, then $1/Y$ is equal to (g = gravitational acceleration)

(1) $\left[\left(\frac{T_M}{T}\right)^2 - 1\right]\frac{A}{Mg}$ (2) $\left[\left(\frac{T_M}{T}\right)^2 - 1\right]\frac{Mg}{A}$

(3) $\left[1 - \left(\frac{T_M}{T}\right)^2\right]\frac{A}{Mg}$ (4) $\left[1 - \left(\frac{T}{T_M}\right)^2\right]\frac{A}{Mg}$

(JEE Main 2015)

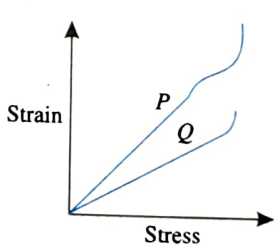
4. A man grows into a giant such that his linear dimensions increase by a factor of 9. Assuming that his density remains same, the stress in the leg will change by a factor of
(1) 81 (2) $\frac{1}{81}$
(3) 9 (4) $\frac{1}{9}$ **(JEE Main 2017)**

5. A solid sphere of radius r made of a soft material of bulk modulus K is surrounded by a liquid in a cylindrical container. A massless piston of area a floats on the surface of the liquid, covering entire cross section of cylindrical container. When a mass m is placed on the surface of the piston to compress the liquid, the fractional decrement in the radius of the sphere,
(1) $\frac{mg}{Ka}$ (2) $\frac{Ka}{mg}$
(3) $\frac{Ka}{3\,mg}$ (4) $\frac{mg}{3\,Ka}$
(JEE Main 2018)

JEE Advanced

Multiple Correct Answers Type

1. In plotting stress versus strain curves for the materials P and Q, a student by mistake puts strain on the y-axis and stress on the x-axis as shown in the figure. Then the correct statement(s) is(are)



(1) P has more tensile strength than Q
(2) P is more ductile than Q
(3) P is more brittle than Q
(4) The Young's modulus of P is more than that of Q.
(JEE Advanced 2015)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)	2. (2)	3. (3)	4. (1)	5. (1)
6. (1)	7. (4)	8. (1)	9. (3)	10. (4)
11. (2)	12. (2)	13. (4)	14. (2)	15. (3)
16. (3)	17. (3)	18. (3)	19. (2)	20. (3)
21. (2)	22. (3)	23. (3)	24. (4)	25. (1)
26. (3)	27. (2)	28. (3)	29. (4)	30. (2)
31. (3)	32. (2)	33. (2)	34. (2)	35. (1)
36. (2)	37. (4)	38. (4)	39. (3)	40. (3)
41. (1)	42. (1)	43. (3)	44. (4)	45. (2)
46. (4)	47. (2)	48. (3)	49. (3)	50. (2)
51. (1)	52. (2)	53. (2)	54. (1)	55. (4)
56. (1)	57. (3)	58. (3)		

Multiple Correct Answers Type

1. (1),(3)	2. (1),(4)	3. (1),(3)
4. (2),(3)	5. (1),(4)	6. (1),(2)
7. (1),(2),(3)	8. (2),(3)	9. (1),(3)
10. (1),(2)	11. (1),(3),(4)	12. (1),(2),(4)
13. (1),(2),(3)	14. (2),(3)	15. (1),(2),(3)
16. (2),(3)	17. (1),(2),(3),(4)	

Linked Comprehension Type

1. (2)	2. (3)	3. (1)	4. (2)	5. (1)
6. (3)	7. (4)	8. (1), (3)	9. (2)	10. (1)
11. (1)	12. (1)	13. (2)	14. (3)	15. (1)
16. (2)	17. (4)	18. (2)	19. (1)	

Matrix Match Type

1. i. → a.,d.; ii. → a.,d.; iii. → a.,d.; iv. → a.,b.,c.
2. i. → a.,d.; ii. → b.,c.; iii. → a.,d.; iv. → a.,d.
3. i. → a.,c.; ii. → a.,d.; iii. → b.,d.; iv. → b.,c.
4. i. → a.,c.,d.; ii. → a.,c.,d.; iii. → a.; iv. → a.,b.,c.,d.

Numerical Value Type

1. (1)	2. (2)	3. (3)	4. (8.00)	5. (8.00)
6. (2.00)	7. (2.00)	8. (1025)	9. (32.00)	10. (0.75)
11. (100.0)	12. (1.25)	13. (8.76)	14. (100)	15. (1.36)
16. (0.10)	17. (2.00)	18. (24.00)	19. (3.00)	

ARCHIVES

JEE Main

Single Correct Answer Type

1. (4)	2. (3)	3. (1)	4. (3)	5. (4)

JEE Advanced

Multiple Correct Answers Type

1. (1),(2)

8 Surface Tension and Viscosity

INTRODUCTION

In this chapter, we will discuss two important properties of liquid, i.e. (i) surface tension and (ii) viscosity. Surface tension is a special property of free surface of liquid. This is because of intermolecular forces of attraction by the virtue of which any free surface of a liquid behaves as a stretched membrane. This helps us to understand the walking of insects, floating of leaves etc., on water surface. Here we will also discuss the effect of surface tension in capillary action. The capillary action explains how the water rises from the inner core to the earth's surface and how the roots of the trees soak water from the soil, how the oil lamp glows by soaking oil by capillary action of the wick, how capillary action takes place in bloating paper, perspiration, water purifier/filter, etc.

Then the second property called viscosity arises from the haphazard (random) molecular motion between the liquid layers. This helps us to understand the real motion of liquid and gases in pipes, motion of water in channels and rivers, motion of rain drops in air, settlement of sediments in water, etc.

SURFACE TENSION

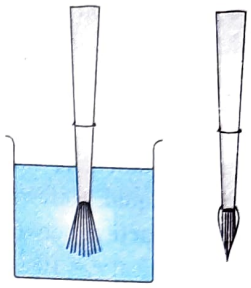
In gravity free space it is found that liquid always takes the shape of a spherical drop. It is because, for a given volume, a sphere has the least surface area. It means that the surface of every liquid has always a tendency to have least surface area and in this respect, it behaves like a stretched membrane having a tension in all directions parallel to the surface. This tension in the surface of a liquid is called surface tension.

Thus, the property of a liquid at rest by virtue of which the free surface of a liquid at rest behaves like a stretched elastic membrane and hence has a natural tendency to occupy as small an area as possible is called surface tension.

SOME OBSERVED PHENOMENA BASED ON SURFACE TENSION

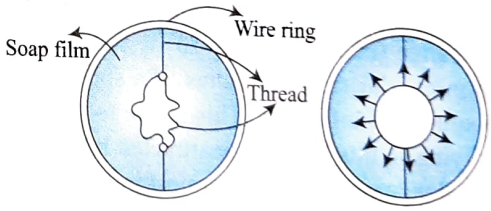
The following experiments demonstrate the property of surface tension.

1. Insects, like mosquitoes, move about freely on the surface of ponds without sinking. This is on account of the support they get from the stretched surface film of water.
2. If we gently place a greased needle on the free surface of water at rest. We shall find that the needle does not sink. A careful observation shows that the surface of water below the needle gets slightly depressed. This shows that the free surface of the liquid layer supports the needle in the same way as a stretched rubber membrane does.
3. Hair of a brush, when dipped in water spread out [Fig. (a)]; but as soon as it is taken out, its hair cling together as shown in Fig. (b).



(a) (b)

4. When a wet loop of cotton thread is placed on a film of soap solution. Let us trap a thin soap film on a circular wire by dipping it into a soap solution and taking it out carefully. On placing a wet loop of cotton thread on the film, it is found to lie as it is and has no tendency to change its form as shown in Fig. (a). Let the film inside the loop be now pricked with a hot needle end. The loop at once takes up a circular form and the film inside it disappears as shown in Fig. (b).



(a) (b)

The analogy of an elastic membrane to the free liquid surface is not wholly correct. In case of an elastic membrane, the tension increases with stretching and is directly proportional to it within elastic limits. But a liquid surface experiences a constant tension.

TYPES OF MOLECULAR FORCES

If we dip a clean glass rod in mercury bath, it is found that mercury does not stick to the glass rod. On the other hand, if we dip a glass rod into water, some of the water sticks to the rod. For explaining this we need to learn about intermolecular forces. The intermolecular forces of attraction may be classified as follows:

1. **Cohesion:** The force of attraction between molecules of the same substance is called the force of cohesion. The force of cohesion is maximum in solids, lesser in liquids and least in gases. It is for this reason that the solids have definite shape and resist all deforming forces.

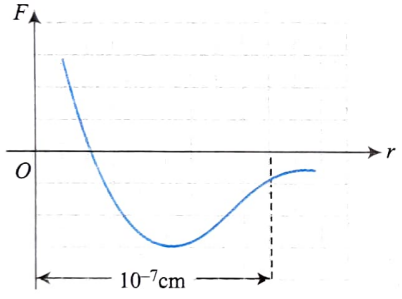
2. **Adhesion:** The force of attraction between the molecules of the different substances is called the force of adhesion. The examples of adhesion are the sticking of glue to the wood, paint to the wall, starch to the fingers, chalk particles to the black board, water to its container, etc.

Now, we can explain why mercury does not stick to the glass rod. The reason is that the force between two molecules of mercury (cohesive force) is greater than the force between a mercury molecule and the molecule of the glass (adhesive force). Also, the water wets the glass rod is explained by the fact that the force between two water molecules (cohesive force) is quite small as compared to the force between a water molecule and a glass molecule (adhesive force).

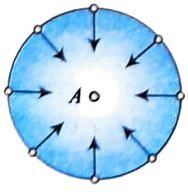
MOLECULAR THEORY OF SURFACE TENSION

For understanding the phenomenon of surface tension, first we need to have some idea about the intermolecular forces, which operate to keep the liquid molecules together.

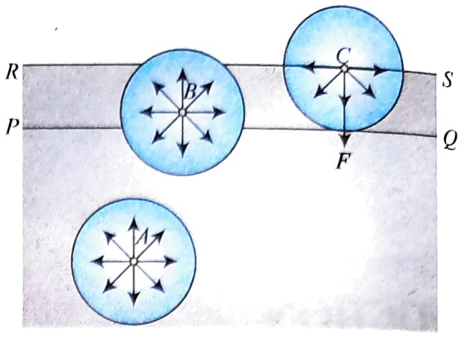
Sphere of influence: To understand the cause of surface tension we need to understand the molecular interaction of molecules at different places inside a liquid. Referring to the F-r diagram we find that the force of attraction between any two molecules decreases to a negligible value at $r = 10^{-7}$ cm.



The cohesive force is effective up to a certain distance called **molecular range** (of the order of 10^{-7}). With the molecule as centre and molecular range as radius, if a sphere is drawn, it is called the **sphere of influence**. This is due to the reason that all molecules lying in this sphere are attracted by the molecules at the centre and vice versa. The uppermost layer of the liquid with thickness equal to the molecular range is called **surface film**.



Force on different molecules: Let $PQRS$ be a surface film of a liquid contained in a glass trough as shown in the figure. Now, consider three molecules A, B, and C; the sphere of influence of the molecule A lies completely inside the liquid. Hence, all surrounding molecules within the sphere of influence attract the molecule equally in all directions. Consequently, the net force acting on the molecule A is zero. It means that the molecule A is free to move inside the liquid with its thermal speed. Now come to the molecule B which is situated little below the liquid surface such that its sphere of influence is mostly lying inside the liquid. There are few molecules more below the molecule B than above that molecule.



Hence, the net downward force is little bit more than the net upward force of the molecules lying inside the sphere of influence on the molecule B. Then, the molecule B experiences a little downward pull.

The molecule C lies on the top of the surface film. The upper half of its sphere of influence lies completely outside the liquid and contains no liquid molecules in it except a few gases or vapour molecules which exert a negligible upward force on C. The molecule C, therefore, experiences maximum downward force.

Hence, we conclude that no resultant force acts on a molecule so long as it lies below the surface film. But for all the molecules lying in the surface film, there is a resultant downward force of attraction. This force goes on increasing as we move from the bottom (PQ) of the surface film to its top (RS). This means that to take a molecule into the surface film from anywhere below it, some work is to be done. This work is stored in the film as the potential energy of the film Therefore, molecules on the free surface of a liquid have potential energy. Further, we know that any system, when free, tends to have minimum potential energy. Therefore, for minimum potential energy of the surface film, the number of molecules in it should be minimum. For this, the surface film should occupy minimum volume which would be so if the film has the least surface area as its thickness is fixed.

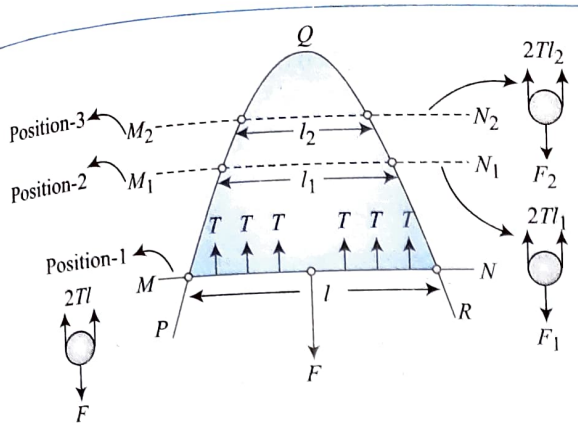
Thus, we observe that for a liquid at rest, the area of its surface film should be as small as possible. In trying to acquire least area, the surface film tends to contract and hence behaves as a stretched membrane.

Under the action of surface tension forces, the free surface of a liquid tends to have the least area for a given volume of liquid in order to minimize its potential energy. Since a sphere has minimum area for a given volume, therefore, rain drops and small droplets of mercury are approximately spherical in shape.

HOW TO DEFINE SURFACE TENSION

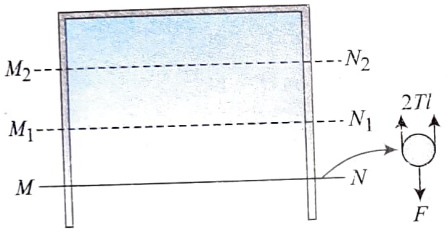
Let us consider a liquid film of the shape shown in the figure formed between a bent wire PQR and a metal wire MN.

The film has a tendency to contract and as such it exerts a pull on the wire MN tending to move it upwards. Obviously, to keep the wire MN in equilibrium, an equal force F has to be applied as shown in the figure. This clearly shows that there acts a force of tension at the boundary of the liquid film and it acts inwards along its surface.



If we repeat this experiment at constant temperature by changing the length of the wire MN in contact with the film. This can be done by changing the position of MN to M_1N_1, M_2N_2.... etc. We shall find that the force (F_1) required to keep the wire in equilibrium while at position M_1N_1 is greater than the force (F_2) required to keep it in equilibrium while at M_2N_2. Further, both F_1 and F_2 are less than F. Thus, it is obvious that F increases with the increase in length of the wire MN in contact with the film.

If we repeat the same experiment with a rectangular shape, we find that F is independent of the area of the soap film.



In the figure regardless of the position of the wire MN, F does not change. Thus, when the wire MN is moved to M_1N_1, though the area of the film increases, F remains the same. Thus, it should be borne in mind that F changes because of a change in length (l) and not on account of the change in the area of the film. In other words, F is directly proportional to l. Thus,

$$F \propto 2l \text{ or } F = T \times 2l \quad ...(i)$$

Factor 2 has been introduced because the film has two free surfaces.

Here, T is a constant of proportionality and is called the coefficient of surface tension or simply the surface tension of the liquid. Further, force of surface tension does not act only on the boundary line but on both sides of any other imaginary line, XY on the surface film, in a direction perpendicular to the line and tangential to the liquid surface.

As $F = T \times 2l$, $T = \dfrac{F}{2l}$ = force/length

Hence, the surface tension of a liquid can be measured as the force per unit length on an imaginary line drawn on the liquid surface, which acts perpendicular to the line on its either side at every point and tangentially to the liquid surface.

$$T = \frac{\text{Total force on either side of the imaginary line } (F)}{\text{Length of the line } (l)}$$

Since the direction of surface tension is perpendicular to any imaginary line, surface tension has no preferred direction. Hence, we can treat the surface tension T as a scalar, whereas force is a vector. SI unit of surface tension is N/m and its dimensional formula is $[MT^{-2}]$.

Important Points:

- Surface tension acts over the free surface of a liquid only and not within the interior of the liquid.
- At the boundary, the surface tension acts only on one side, i.e., the liquid side. But at any other point of the surface, it acts on either side of the imaginary line.
- The surface tension of a liquid is given as the linear density of surface tension force (cohesive force along the liquid surface) which is given as $T = \dfrac{F}{l}\left(\text{or } \dfrac{dF}{dl}\right)$.

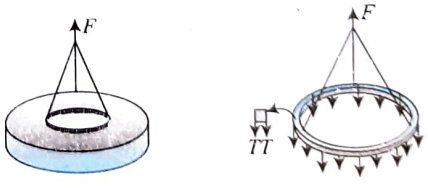
  T is constant at a given temperature.
- The magnitude of T depends on the temperature of the liquid and on the medium on the other side of the free surface. The surface tension of a liquid decreases by increasing temperature.
- The surface tension of a liquid decreases by adding impurities.

DETERMINATION OF COEFFICIENT OF SURFACE TENSION

One of the commonly used methods for the determination of coefficient of surface tension is shown in Fig. (a). This method involves the measurement of a force F needed to pull a platinum ring through the surface of a liquid. The ring forms one arm of a balance while weights are added to the other side. The total length of the liquid surface to be broken is twice the circumference of the ring as there are two surfaces to the film [Fig. (b)]. Thus, if r is the radius of the ring, the additional force needed for balance will be

$$F = 2(2\pi r)T = 4\pi rT \text{ or } T = \frac{F}{4\pi r} \quad ...(i)$$

Thus, knowing F and r; T can be calculated. It is to be noted that surface tension depends upon temperature and as such temperature must be kept constant throughout the experiment.



(a) (b)

ILLUSTRATION 8.1

A wire ring of 5 cm radius is rested on the surface of a liquid and then raised. The pull required is 3.14 g more before the film breaks than it is after. Find the surface tension of the liquid.

Sol. When the wire ring is resting on the liquid surface, a thin film of liquid is in contact with the ring. Force due to surface tension acts along the circumference. Therefore, the force required to lift the ring is equal to the sum of the weight of the ring and the force due to surface tension. Hence, the extra pull

required to break the film is equal to the force on the ring due to surface tension, i.e.

Pull required to break the film = Force on the ring due to surface tension

Here, pull required to break the film = 3.14 gf = 3.14×1000 dyne and radius of the ring, $r = 5$ cm

The liquid is in contact with the ring both along its inner and outer circumferences.

Therefore, force on the ring due to surface tension

$$= 2(2\pi rT) = 2\times2\pi\times5\times T = 20\pi T$$

Hence, $20\pi T = 3.14\times1000$

Or $$T = \frac{3.14\times1000}{20\pi} = 50.0 \text{ dyne cm}^{-1}$$

ILLUSTRATION 8.2

A U-shaped wire loop having a slider (of negligible mass) is dipped in a soap solution and then removed. A thin soap film is formed between the wire and the light slider. It is found that the film can support a weight of 0.006 N, before it will break. If the length of the slider is 0.1 m, what is the surface tension of the film?

Sol. Due to surface tension, the soap film tries to make its surface area minimum. As it has two free surfaces, the total force due to surface tension on the slider will be

$$F = 2(T\times l)$$

In equilibrium, total force due to surface tension on the slider will be equal to its weight, i.e.

$$2(T\times l) = Mg$$

Here, $Mg = 0.006$ N; $l = 0.1$ m

$$T = \frac{Mg}{2l} = \frac{0.006}{2\times0.1} = 0.03 \text{ N m}^{-1}$$

ILLUSTRATION 8.3

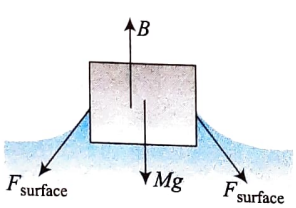
A cube of mass $m = 800$ g floats on the surface of water. Water wets it completely. The cube is 10 cm on each edge. By what additional distance is it buoyed up or down by surface tension? Surface tension of water = 0.07 N m^{-1}.

Sol. If surface tension is neglected the condition for floating gives

$$800\times10^{-3}g = (0.1^2 x\rho)g \text{ or } x = 0.08 \text{ m}$$

($\rho = 1000$ kg m^{-3} for water)

Since water wets the cube, the angle of contact is zero and force of surface tension acts vertically downwards. So, it is buoyed down by surface tension.



$$\therefore \quad 800\times10^{-3}g + 4\times0.1\times0.07 = (0.1^2 x'\rho)g$$

or $$x' = 0.08 + \frac{0.028}{98} = 0.08 + 2.8\times10^{-4}$$

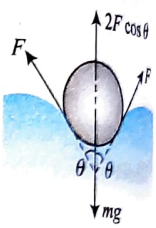
Therefore, the additional distance $= 2.8\times10^{-4}$ m

ILLUSTRATION 8.4

Find the maximum possible mass of a greased needle floating on water surface.

Sol. Let the mass of the needle be m. As the liquid surface is distorted, the surface tension forces acing on both sides of the needle make an angle θ, say, with vertical. Since the forces acting on the needle are F, F and mg, resolving the forces vertically for its equilibrium, we have

$$\sum F_y = F\cos\theta + F\cos\theta - mg = 0$$



This gives $$m = \frac{2F\cos\theta}{g}$$

where $F = Tl$

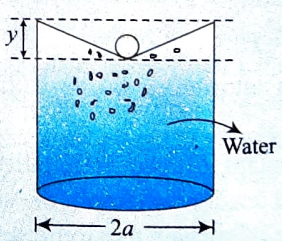
Then $$m = \frac{2Tl\cos\theta}{g}$$

For m to be maximum, $\cos\theta = 1$

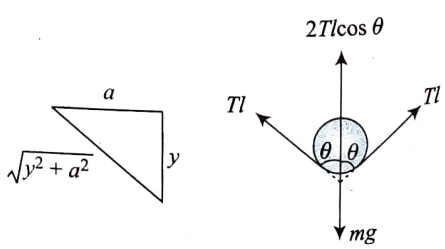
Hence, $$m_{max} = \frac{2Tl}{g}$$

ILLUSTRATION 8.5

A container of width $2a$ is filled with a liquid. A thin wire of weight per unit length k is gently placed over the liquid surface in the middle of surface as shown. As a result the liquid surface is depressed by a distance $y (y << a)$. Determine the surface tension of the liquid.



Sol. The free-body diagram of wire is shown in the figure. Let l be the length of wire. If λ is mass per unit length of wire, the weight of wire = $(l\lambda)g$ (acting vertically downward).



Vertical force due to surface tension $= 2Tl\cos\theta$ (upward)

Therefore, for vertical equilibrium $2Tl\cos\theta = (l\lambda)g$

$$T = \frac{\lambda g}{2\cos\theta}$$

From the figure, $\cos\theta = \dfrac{y}{\sqrt{y^2+a^2}} = \dfrac{y}{a}$ (for $y << a$ given)

$$T = \frac{\lambda g}{2\left(\frac{y}{a}\right)} = \frac{\lambda a g}{2y}$$

CONCEPT APPLICATION EXERCISE 8.1

1. A rigid ring A and a rigid thin disk B both made of same material, when gently placed on water, just manage to float due to surface tension as shown in the figure. Both the ring and the disk have same radius. What can you conclude about their masses?

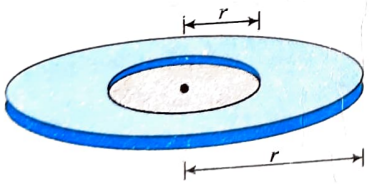
A B

(a) Both have the same mass.
(b) Mass of the ring is half of that of the disk.
(c) Mass of the ring is double to that of the disk.
(d) More information is needed to decide.

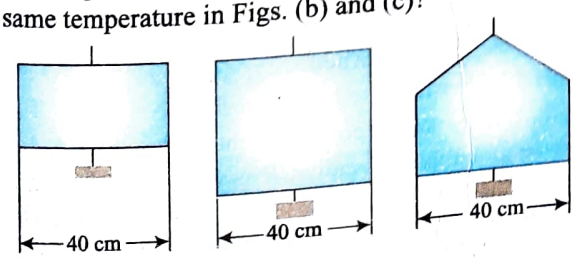
2. A plate of radius 10 cm is placed on a water surface. Calculate the force required to take away the plate from the surface of water. Surface tension of water $= 0.07\ \text{N m}^{-1}$.

3. A ring substitutes the circular plate in the previous problem. Calculate the force required to take away the plate from the surface of water.

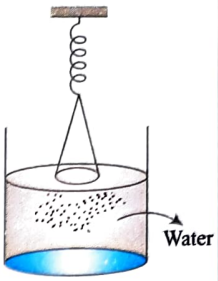
4. In the previous problem, if we take an annular disc of radius $r = 10\,\text{cm}$ and $R = 20\,\text{cm}$, Calculate the force required to take away the plate from the surface of water.



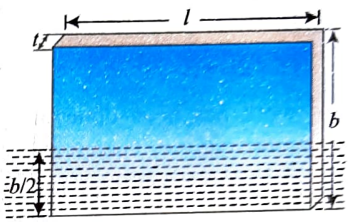
5. In the figure given below shows a thin liquid film supporting a small weight equal to 5.0×10^{-2} N. What is the weight supported by a film of the same liquid at the same temperature in Figs. (b) and (c)?



6. A circular ring has inner and outer radii equal to r and R respectively. Mass of the ring is m. It gently pulled out vertically from a water surface by a sensitive spring. When the spring is stretched 3.4 cm from its equilibrium position the ring is on verge of being pulled out from the water surface. If spring constant is k. Find the surface tension of water.



7. A glass plate of length 50 cm, breadth 40 cm and thickness 5 cm weighs 3 kg in air. If it is held vertically with long side horizontal and the plate half immersed in a liquid, what will be its apparent weight? Surface tension of the liquid $= 50 \times 10^{-2}$ N/m.



ANSWERS

1. (c) **2.** 0.44 N **3.** 0.88 N **4.** 1.32 N

6. $\dfrac{kx - mg}{2\pi(R-r)}$ **7.** 5.55 N

SURFACE ENERGY

As we know the free surface of a liquid always has a tendency to contract and possess minimum surface area. If it is required to increase the surface area of the liquid, work has to be done. This work done is stored in the surface film of the liquid as its potential energy.

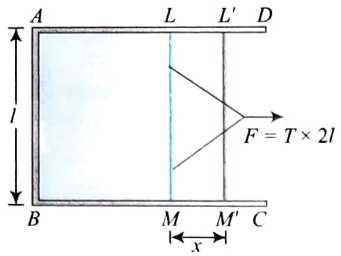
The potential energy per unit area of the surface film is called the surface energy.

It may also be defined as the amount of work done in increasing the area of the surface film through unity. Thus,

$$\text{Surface energy} = \frac{\text{work done in increasing the surface area}}{\text{increase in surface area}} \quad ...(i)$$

RELATION BETWEEN SURFACE ENERGY AND SURFACE TENSION

Consider a rectangular frame $ABCD$ made of a wire, such that the cross-piece LM can just slide. Dip the wire frame in the soap solution, so that a film is formed over the frame [see figure]. Due to the surface tension, the film will have a tendency to shrink and thereby, the cross-piece LM will be pulled in inward direction. However, the cross-piece can be held in this position under a force F, which is equal and opposite to the force acting on the cross piece LM all along its length due to surface tension in the soap film. If T is the force due to surface tension per unit length, then



$$F = T \times 2l$$

Here, l is length of the cross-piece LM. The length of the cross-piece has been taken as $2\,l$ for the reason that the film has got two free surfaces.

Suppose that the cross-piece LM is moved through a small distance x, so as to take the position $L'M'$. In this process, area of the film increases by $2l \times x$ (on the two sides) and to do so, the work done is given by

$$W = F \times x = (T \times 2l) \times x$$

This amount of work has been done in increasing area of surface film by an amount $2l \times x$. Therefore, by definition,

$$\text{Surface energy} = \frac{\text{work done in increasing the surface area}}{\text{increase in surface area}}$$

$$= \frac{T \times 2l \times x}{2l \times x}$$

Or T = surface energy ...(i)

Hence, the surface tension of a liquid is numerically equal to its surface energy.

ILLUSTRATION 8.6

Calculate the amount of energy evolved when 8 droplets of water (surface tension = 72×10^{-3} N/m) of radius 0.5 mm each combine into one.

Sol. Let R be the radius of the big drop formed by the combination of 8 small droplets.

As volume of the big drop = volume of 8 small droplets,

$$\frac{4}{3}\pi R^3 = 8\left(\frac{4\pi}{3}r^3\right)$$

or $R^3 = 8r^3$ or $R = 2r = 2(5\times10^{-4}\text{ m}) = 10^{-3}\text{ m}$

Surface area of the big drop $= 4\pi R^2$

$= 4\pi(10^{-3}\text{ m})^2 = 4\pi\times10^{-6}\text{ m}^2$

Surface area of 8 small droplets $= 8\times4\pi r^2$

$= 8\times4\pi(5\times10^{-4}\text{ m})^2 = 8\pi\times10^{-6}\text{ m}^2$

Decrease in surface area,

$= 8\pi\times10^{-6}\text{ m}^2 - 4\pi\times10^{-6}\text{ m}^2 = 4\pi\times10^{-6}\text{ m}^2$

Energy evolved,

$$W = T\times\Delta A = (72\times10^{-3}\text{ N/m})\times(4\pi\times10^{-6}\text{ m}^2)$$

$$= 9\times10^{-7}\text{ J}$$

ILLUSTRATION 8.7

Calculate the work done against surface tension in blowing a soap bubble from a radius of 10 cm to 20 cm, if the surface tension of soap solution in 25×10^{-3} N/m.

Sol. Original total surface area $= 2\times4\pi r_i^2 = 2\times4\pi\times(0.1)^2\text{ m}^2$ (as bubble has two surfaces).

Final total surface area $= 2\times4\pi r_2^2 = 2\times4\pi\times(0.2)^2\text{ m}^2$

Therefore, extension in area $= 2\times4\pi[(0.2)^2 - (0.1)^2] = 0.24\pi\text{ m}^2$

Now, work done W = surface tension × extension in area

$$= 25\times10^{-3}\times0.14\pi = 6\pi\times10^{-3}\text{ J}$$

ILLUSTRATION 8.8

A film of water is formed between two straight parallel wires each 10 cm long and at separation 0.5 cm. Calculate the work required to increase 1 mm distance between the wires. Surface tension of water = 72×10^{-3} N/m.

Sol. Initial surface area = 2 × length × separation

$$= 2\times10\text{ cm}\times0.5\text{ cm} = 10\text{ cm}^2 = 10\times10^{-4}\text{ m}^2$$

Final surface area $= 2\times10\text{ cm}\times(0.5+0.1)\text{ cm}$

$$= 2\times10\times0.6\text{ cm}^2 = 12\times10^{-4}\text{ m}^2$$

The required work,

$$W = T\Delta A = 72\times10^{-3}\times(12\times10^{-4} - 10\times10^{-4})\text{ J}$$

$$= 72\times10^{-3}\times2\times10^{-4} = 144\times10^{-7}\text{ J}$$

ILLUSTRATION 8.9

Two soap bubbles in vacuum having radii 3 cm and 4 cm respectively coalesce under isothermal conditions to form a single bubble. What is the radius of the new bubble?

Sol. Let r_1 and r_2 be the radii of two given soap bubbles and r be that of the coalesced bubble.

Here, $r_1 = 30$ cm and $r_2 = 4$ cm.

Since a soap bubble has two free surfaces, total surface area of the coalesced bubble $= 2\times4\pi r^2 = 8\pi r^2$

Potential energy of the coalesced bubble, i.e., $W = 8\pi r^2 T$

Similarly if W_1 and W_2 are the potential energies of the two given bubbles,

$$W_1 = 8\pi r_1^2 T \text{ and } W_2 = 8\pi r_2^2 T$$

Since the total energy of the system is conserved,

$$8\pi r^2 T = 8\pi r_1^2 T + 8\pi r_2^2 T$$

or $r^2 = r_1^2 + r_2^2 = (3)^2 + (4)^2 = 25$

or $r = 5$ cm

CONCEPT APPLICATION EXERCISE 8.2

1. Two droplets merge with each other and forms a large droplet. In this process
   (a) Energy is liberated
   (b) Energy is absorbed
   (c) Neither liberated nor absorbed
   (d) Some mass is converted into energy
2. A soap bubble of radius R is blown. After heating the solution a second bubble of radius $2R$ is blown. The work required to blow the second bubble in comparison to that required for the first bubble is

(a) Double
(b) Slightly less than double
(c) Slightly less than four times
(d) Slightly more than four times

3. If the work done in blowing a bubble of volume V is W, then find the work done in blowing the bubble of volume $2V$ from the same soap solution.
4. Calculate the work done in blowing up a soap bubble from an initial surface area of 0.50 cm^2 to a final surface area of 1.10 cm^2. The surface tension of soap solution is 30 dyne cm^{-1}.
5. A liquid drop of diameter D breaks into 27 tiny drops. Find the resultant change in energy. Take the surface tension of the liquid as T.

ANSWERS

1. (a) **2.** (c) **3.** $(4)^{1/3}\,W$ **4.** 36 erg **5.** $2\pi D^2 T$

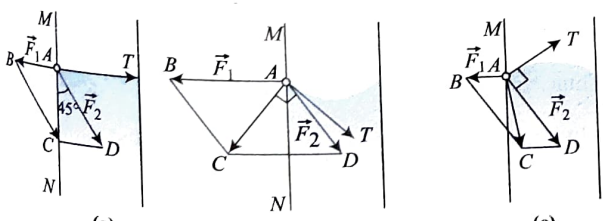
SHAPE OF LIQUID MENISCUS

If a liquid and a solid are in contact with each other, the liquid surface becomes curved near the point of contact. This curved surface is due to two forces, viz., cohesive force between the liquid molecules and the adhesive force between liquid and solid molecules. The nature of the curvature (i.e., whether it is plane, convex or concave) depends upon the relative magnitudes of these two forces.

Let us consider a thin glass capillary tube dipped in a liquid vertically in a liquid. Consider a molecule A on the liquid surface touching the wall MN of the tube. Neglecting the weight of the molecule, this liquid molecule is being acted upon by two forces:

1. Force F_1 due to force of adhesion because of the attraction of the glass molecules which acts outwards at right angle to the wall of the tube. This force is represented by AB.
2. Force F_2 due to force of cohesion because of the attraction of the liquid molecules which acts at an angle of 45° to the vertical. This force is represented by AD.

These two forces, AB and AD, are acting at an angle with each other and their resultant can be obtained by applying the parallelogram law of forces. It is represented in magnitude and direction by AC. The direction of the resultant force AC depends upon the relative magnitudes of AB and AD. Here three cases are possible:



(a) If $AB = AD\sin 45° = \dfrac{AD}{\sqrt{2}}$, the resultant AC acts vertically downwards as shown in Fig. (a).

(b) $AB > AD\sin 45°$, or if $AB > \dfrac{AD}{\sqrt{2}}$, the resultant AC lies outwards as shown in Fig. (b).

(c) $AB < AD\sin 45°$, i.e., $AB < \dfrac{AD}{\sqrt{2}}$, the resultant AC lies inwards as shown in Fig. (c).

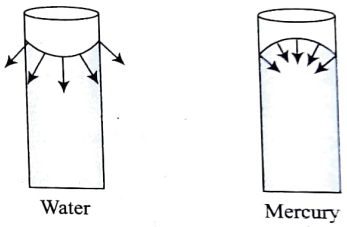
As liquid cannot permanently withstand any tangential force. As such when in equilibrium, its surface at every point sets itself perpendicular to the resultant force.

Special Cases

In case (a), $F_2 = \sqrt{2}\,F_1$, then the resultant force will act vertically downward and hence the meniscus will be plane or horizontal as shown in Fig. (a). For example, pure water contained in silver capillary tube.

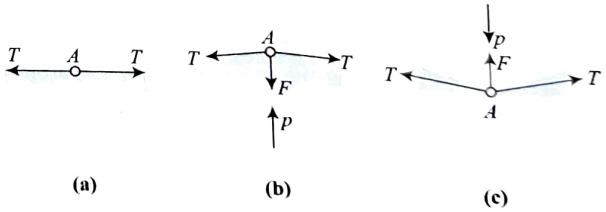
In case (b), $F_2 < \sqrt{2}\,F_1$, the resultant force AC is directed outwards, since the liquid surface has always to set itself perpendicular to this resultant force, it becomes concave upwards as in the case of water in a glass tube.

In case (c), $F_2 < \sqrt{2}\,F_1$, the resultant force AC is directed inwards, and here also the liquid surface has always to set itself perpendicular to the resultant force, it becomes convex upwards as in the case of mercury in a glass tube.



SURFACE OF A LIQUID

The surface of a liquid may be plane, convex or concave as shown in the figure. The force due to surface tension on molecule M in the surface of the liquid acts tangentially to the surface in all directions



Plane Surface

If the surface of the liquid is plane [as shown in Fig. (a)], the molecule on the liquid surface is attracted equally in all directions. The resultant force due to surface tension is zero. The pressure, therefore, on the liquid surface is normal.

Convex Surface

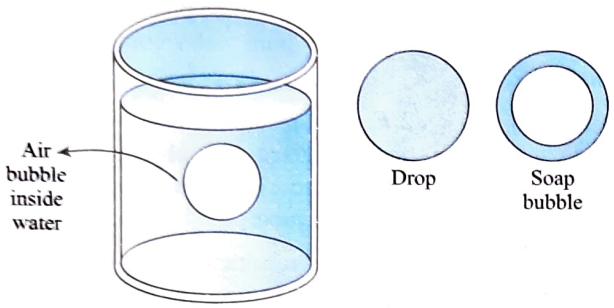
If the surface is convex [as shown in Fig. (b)], the resultant force due to surface tension acts in the downward direction. Since the molecules on the surface are in equilibrium, there must be an excess of pressure on the concave side of the surface acting in the upward direction to balance the downward resultant force of surface tension. Hence, there is always an excess of pressure on concave side of a curved surface over that on the convex side.

Concave Surface

If the surface is concave upwards [as shown in Fig. (c)], there will be upward resultant force due to surface tension acting on the molecule. Since the molecule on the surface is in equilibrium, there must be an excess of pressure on the concave side

Excess Pressure Inside a Curved Surface

Some examples of plane and curved surfaces (cavities, drops and bubbles) are shown in figure.



The bubbles, such as soap bubbles, are like blown-up balloons, air inside and air outside with a thin liquid film in between them. This thin film naturally has two free surfaces, one inside and the other outside. Cavities generally have air inside and liquid outside. They have one interface or free surface. Drops generally have water or some other liquid inside and air or some other gas outside them. They too have one exposed surface.

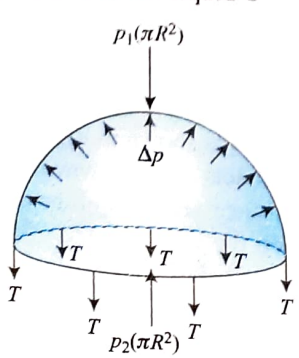
Excess Pressure Inside a Liquid Drop and a Bubble

Due to the property of the surface tension, a liquid surface always tends to have minimum surface area. The small drops and bubbles are found to possess spherical shapes. However, a big drop of liquid is not spherical in shape. The reason is that in case of a small drop of liquid, the effect of gravity is negligible. As the small drops and bubbles have spherical shapes, it implies that their spherical surfaces must be possessing minimum surface area. Further as a drop or a bubble does not collapse under the effect of the force of surface tension (which tends to minimise its surface area), it indicates that the pressure inside the drop or the bubble must be greater than that outside it.

Excess Pressure Inside a Liquid Drop

Figure shows one-half cross section of an air bubble formed inside liquid. It is in equilibrium under the action of three forces:

1. due to external pressure p_1
2. due to internal pressure p_2
3. due to surface tension of the liquid σ



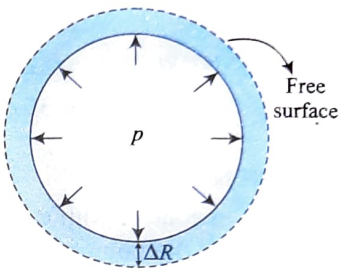
If R is the radius of the air bubble, then the forces due to external and internal pressures are $p_1(\pi R^2)$ and $p_2(\pi R^2)$, respectively. Since the surface tension acts around the circumference of the bubble, therefore, the force of surface tension is $T(2\pi R)$.

Thus, from the condition of equilibrium,

$$p_2(\pi R^2) = p_1(\pi R^2) + T(2\pi R)$$

or $$p_2 - p_1 = \frac{2T}{R}$$

Alternate approach: Let us consider a liquid drop of radius R and let T be surface tension of the liquid forming the drop. As already discussed, there is an excess pressure (p) inside the drop and it acts normally outwards as shown in the figure.



Let this excess pressure increase the radius of the drop from R to $(R + \Delta R)$.

Obviously, work done by the excess pressure, i.e.,

W = force × distance

= excess pressure × surface area × distance

$= p \times 4\pi R^2 \times \Delta R$

i.e., $W = 4\pi pR^2\Delta R$ ...(i)

Increase in the surface area of the drop, i.e.,

ΔA = final surface area – initial surface area

$= 4\pi(R + \Delta R)^2 - 4\pi R^2$

$= 4\pi[R^2 + (\Delta R)^2 + 2R\Delta R - R^2]$ ...(ii)

$= 8\pi R\Delta R$ [neglecting (ΔR^2) as it is very small]

We know that $T = \frac{W}{\Delta A}$ ...(iii)

From Eqs. (i), (ii) and (iii),

$$T = \frac{4\pi pR^2\Delta R}{8\pi R\Delta R} = \frac{pR}{2}$$

Thus, $p = \frac{2T}{R}$ ...(iv)

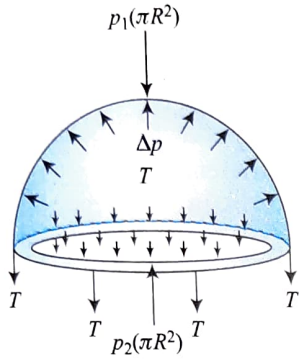
Obviously, excess pressure increases as the radius of drop decreases.

Note:

- A curved surface always maintains a pressure difference, which depends on the radius of the curved surface and surface tension of the liquid.
- The concave side of the surface always possesses greater pressure. If the angle of contact is zero, then the excess pressure is given by $\Delta p = \frac{2T}{R}$

Excess Pressure in Soap Bubble

A soap bubble forms two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble. Figure shows one-half cross section of the soap bubble. By considering its equilibrium, we get



$$p_2(\pi R^2) = p_1(\pi R^2) + T[2(2\pi R)]$$

$$\Rightarrow \quad p_2 - p_1 = \frac{4T}{R}$$

Alternate approach: Let us consider a liquid bubble (say a soap bubble) of radius R and let T be the surface tension of the liquid. We know that there is an excess pressure (p) inside the bubble and it acts normally outwards. Let the excess pressure increase the radius of the bubble from R to $(R + \Delta R)$ as shown in figure.

Work done by the excess pressure, i.e.,

W = force × distance

= excess pressure × surface area × distance

$= p \times 4\pi R^2 \times \Delta R$

i.e., $W = 4\pi pR^2 \Delta R$ ...(i)

The soap bubble has two free surfaces, one inside the bubble and the other outside it.

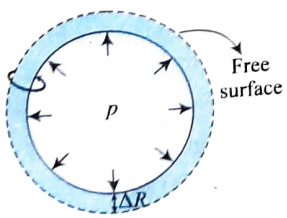
Total increase in the surface area of the bubble, i.e.,

$$\Delta A = 2[4\pi(R + \Delta R)^2 - 4\pi R^2]$$

$$= 8\pi[R^2 + (\Delta R)^2 + 2R \times \Delta R - R^2]$$

$= 16\pi R\Delta R$ [neglecting (ΔR) as it is very small ...(ii)

We know that $T = \dfrac{W}{\Delta A}$ ...(iii)



From Eqs. (i), (ii) and (iii), $T = \dfrac{4\pi pR^2 \Delta R}{16\pi R\Delta R} = \dfrac{pR}{4}$

Or $p = \dfrac{4T}{R}$ ...(iv)

Since the excess pressure is inversely proportional to the radius of the bubble, it means smaller the bubble, greater the excess pressure inside it

Inside an air bubble: In a drop of liquid, air is outside the drop and the liquid is inside it. In an air bubble, liquid is outside the spherical core of air. Thus, as in the case of a drop, an air bubble also has one free surface. As before, it can be easily proved that excess of pressure inside an air bubble,

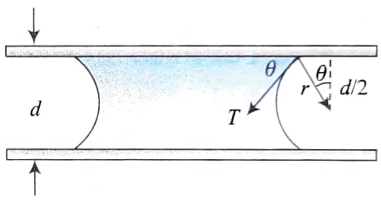
$$p_i - p_0 = \frac{4T}{R} \quad ...(i)$$

Note: The above discussion can be summarised in the form of a general result that there is an excess of pressure on the concave side of the liquid–air interface and the excess of pressure is given by

$$p_i - p_0 = \frac{2T}{R}$$

where the letters have their usual meanings.

Liquid between two plates: When a small drop of water is placed between two glass plates put face to face, it forms a thin film which is concave outward along its boundary. Let 'R' and 'r' be the radii of curvature of the enclosed film in two perpendicular directions.



Hence, the pressure inside the film is less than the atmospheric pressure outside it by an amount p given by

$$p = T\left(\frac{1}{r} + \frac{1}{R = \infty}\right)$$

and we have $p = \dfrac{T}{r}$ ...(i)

If d be the distance between the two plates and θ be the angle of contact for water and glass, then from the figure,

$$\cos\theta = \frac{\frac{1}{2}d}{r} \quad \text{or} \quad \frac{1}{r} = \frac{2\cos\theta}{d}$$

Substituting for $1/r$ in Eq. (i), we get $p = \dfrac{2T}{d}\cos\theta$

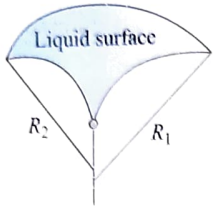
θ can be taken zero for water and glass, i.e., $\cos\theta = 1$. Thus, the upper plate is pressed downward by the atmospheric pressure minus $2T/d$. Hence, the resultant downward pressure acting on the upper plate is $2T/d$. If A be the area of the plate wetted by the film, the resultant force F pressing the upper plate downward is given by F = resultant pressure × area = $2TA/d$. For very nearly plane surface, d will be very small and hence the pressing force F is very large. Therefore, it will be difficult to separate the two plates normally.

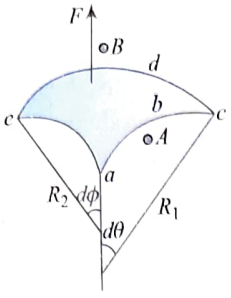
Relation Between Surface Tension, Radii of Curvature and Excess Pressure on a Curved Surface

Let us consider a small element $ABCD$ (see figure) of a curved liquid surface which is convex on the upper side. R_1 and R_2 are

the maximum and minimum radii of curvature, respectively. They are called the 'principal radii of curvature' of the surface. Let p be the excess pressure on the concave side. Then

$$p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$





If instead of a liquid surface, we have a liquid film, the above expression will be

$$p = 2T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

because a film has two surfaces.

EXCESS PRESSURE ON CURVED SURFACES IN GENERAL

The pressure on the concave side (whatever be on its left or right) is always greater than the pressure on the convex side, such that

$$p_{\text{concave}} - p_{\text{convex}} = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where σ = surface tension.

This formula is same as what we derived earlier, i.e.,

$$p_i - p_0 = \frac{2T}{R} = T\left(\frac{1}{R} + \frac{1}{R}\right)$$

1. For a single spherical surface, e.g. drops, cavities, etc.,

   $R_1 = R_2 = R$ (say)

   Then $p_i - p_0 = T\left(\frac{1}{R} + \frac{1}{R}\right) = \frac{2T}{R}$

2. For spherical film, e.g., soap bubble,

   $$p_i - p_0 = (2T)\left(\frac{1}{R} + \frac{1}{R}\right) = \frac{4T}{R}$$

   because it has two free surfaces.

3. For a single cylindrical surface, $R_1 = R, R_2 = \infty$

   $$\therefore\ p_i - p_0 = (T)\left(\frac{1}{R} + \frac{1}{\infty}\right) = \frac{T}{R}$$

ILLUSTRATION 8.10

What should be the pressure inside a small air bubble of 0.1 mm radius situated just below the surface of water? Surface tension of water = 72×10^{-3} N/m and atmospheric pressure = 1.013×10^5 N/m^2.

Sol. The excess pressure for air bubble is $p = \frac{2T}{r}$

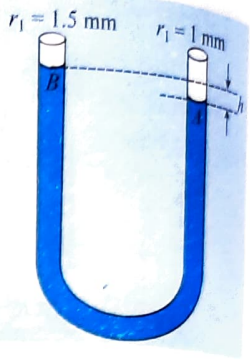
or $p = \frac{2 \times 72 \times 10^{-3}}{0.1 \times 10^{-3}} = 1440$ N/m^3

Since the bubble is just below the water surface, the external pressure on it is equal to the atmospheric pressure P. Hence the pressure inside the bubble is

$$P + p = 1.013 \times 10^5 + 1440$$
$$= 1.013 + 10^5 + 0.0144 \times 10^5$$
$$= 1.274 \times 10^5 \text{ N/m}^2$$

ILLUSTRATION 8.11

Calculate the difference (h) in water levels in two communicating capillary tubes of radius 1 mm and 1.5 mm. Surface tension of water = 0.07 Nm^{-1}.



Sol. Pressure at $A = p_0 + \frac{2T}{r_1}$

(Since pressure on concave side is greater than that on convex side)

Pressure at $B = p_0 - \frac{2T}{r_2}$

Therefore, pressure difference $= 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

Let this pressure difference correspond to h units of the liquid. Then,

$$2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \rho g h \quad \Rightarrow h = \frac{2T}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\therefore\ h = \frac{2 \times 0.07}{1000 \times 9.8}\left(\frac{1}{1 \times 10^{-3}} - \frac{1}{1.5 \times 10^{-3}}\right) = 4.76 \text{ mm}$$

ILLUSTRATION 8.12

A glass tube of 1 mm bore is dipped vertically into a container of mercury, with its lower end 2 cm below the mercury surface. What must be the gauge pressure of air in the tube to blow a hemispherical bubble at its lower end? Given that density of mercury = 13,600 kg m^{-3} and surface tension of mercury = 0.465 N m^{-1}.

Sol. Here, surface tension of mercury, $T = 0.465$ Nm^{-1}; density of mercury, $\rho = 13{,}600$ kg m^{-3}

and radius of the tube, $R = \frac{1}{2} = 0.5 \text{ mm} = 5 \times 10^{-4}$ m

Now, the excess of pressure inside the air bubble blown at the lower end of the tube,

$$p_i - p_0 = \frac{2T}{R} = \frac{2 \times 0.465}{5 \times 10^{-4}} = 1{,}860 \text{ N m}^{-2}$$

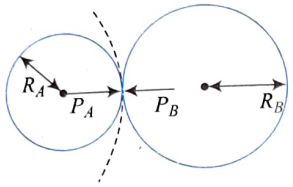
$$= \frac{1{,}860}{13{,}600 \times 9.8} = 0.014 \text{ m of Hg} = 1.4 \text{ cm of Hg}$$

ILLUSTRATION 8.13

Two separate air bubbles (radii 0.002 cm and 0.004 m) formed of the same liquid (surface tension 0.07 N/m) come together to form a double bubble. Find the radius and the sense of curvature of the internal film surface common to both the bubbles.

Sol. Let R be the radius of curvature of common surface when bubbles A and B of radii R_A and R_B coalesce. The excess pressure in A and B are $4TR_A$ and $4TR_B$, respectively.

$$\therefore \quad p_A = \frac{4T}{R_A} \quad \text{and} \quad p_B = \frac{4T}{R_B}$$



If R is radius of common interface, then we must have

$$p_A - p_B = \frac{4T}{R} \quad \text{or} \quad \frac{4T}{R_A} - \frac{4T}{R_B} = \frac{4T}{R}$$

$$\text{This gives } R = \frac{R_A R_B}{R_B - R_A} = \frac{0.002 \times 0.004}{0.004 - 0.002} = 0.004 \text{ m}$$

As the excess pressure is always towards concave surface and pressure in smaller bubble is greater than that in larger bubble, the common surface is concave towards the centre of the smaller bubble.

ILLUSTRATION 8.14

A small hollow sphere which has a small hole in it is immersed in water to a depth 40 cm before any water penetrates into it. Find the radius of the hole. Surface tension of water = 75×10^{-3} N m^{-1} and density of water = 10^3 kg m^{-3} and $g = 10$ m/s^2.

Sol. The depth of the water column, when the water just enters the hollow sphere through the hole,

$h = 40$ cm

Let R be the radius of the hole in the hollow sphere. When the hollow sphere having hole is immersed in water, an air bubble is formed at the hole. The pressure due to water at the depth of 40 cm will try to force water into the hollow sphere, while the excess of pressure inside the air bubble will opposite it. The water will enter the hollow sphere just when the excess of pressure inside the air bubble becomes equal to the pressure exerted by 40 cm column of water.

Now, $p_i - p_0 = h\rho g = 0.40 \times 1000 \times 10$

or $p_i - p_0 = \Delta p = 4.0 \times 10^3$ N m^{-2} ...(i)

As the radius of the air bubble formed will be equal to the radius of the hole,

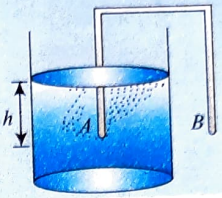
$$p_i - p_0 = \Delta p = \frac{2T}{R} \qquad \text{...(ii)}$$

From Eq. (ii), we have

$$R = \frac{2T}{\Delta p} = \frac{2 \times 75 \times 10^{-3} \text{ N/m}}{4.0 \times 10^3 \text{ N/m}^2} = 3.75 \times 10^{-6} \text{ m}$$

ILLUSTRATION 8.15

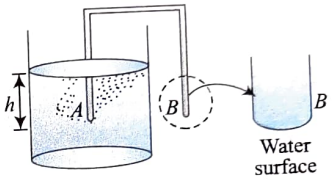
In the siphon shown in the figure the ends A and B of the tube are at same horizontal level. Water fills the entire tube but it does not flow out of the end B. With the help of a diagram show how the water surface at end B changes if the end B were slightly lower than the position shown.



Sol. The pressure at end B inside the siphon tube is

$P_B = P_0 + \rho gh$ [P_0= atmospheric pressure]

$\therefore$ Gauge pressure $P_B - P_0 = \rho gh$

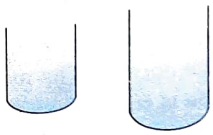


We know the pressure at concave side is always greater

$$\therefore \quad \frac{2T}{r} = \rho gh \quad [r = \text{radius of curvature of the surface}]$$

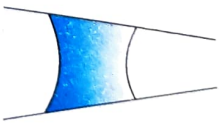
With increase in h, the curvature of water surface increases (i.e., r decreases) so as to prevent water from flowing out.

The figure shows the change in surface as B is lowered.



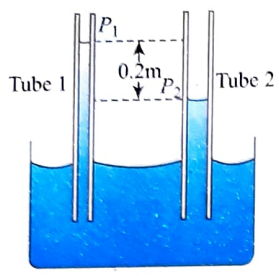
CONCEPT APPLICATION EXERCISE 8.3

1. A conical pipe shown in figure has a small water drop. In which direction does the drop will tend to move?

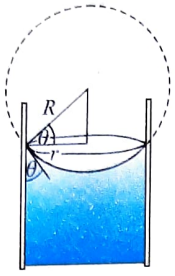


2. There is an air bubble of radius 1.0 mm in a liquid of surface tension 0.075 N/m and density 1000 kg/m^3. The bubble is at a depth of 10 cm below the free surface. By what amount is the pressure inside the bubble greater than the atmospheric pressure? Take $g = 10$ m/s^2.
3. Two soap bubbles have radii in the ratio 1 : 2. Compare the excess of pressure inside these bubbles. Also compare the work done in blowing these bubbles
4. Consider a rain drop falling at terminal speed. For what radius (R) of the drop can we disregard the influence of gravity on its shape? Surface tension and density of water are T and ρ respectively.

5. A barometer contains two uniform capillaries of radii 5.0×10^{-4} m and 1.0×10^{-3} m. If the height of liquid in narrow tube is 0.2 m more than that in the wide tube, calculate the true pressure difference. (Given: Density of liquid = 10^3 kg m^{-3}, surface tension = 75×10^{-3} N m^{-1}, angle of contact = 0° and g = 10 m s^{-2}.



(a)



(b)

ANSWERS

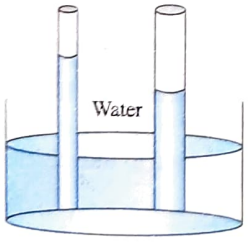
1. Towards the tapered end. **2.** 1150 Pa

3. 1 : 4 **5.** 1850 N/m^2

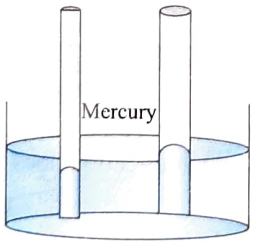
CAPILLARITY

If a narrow glass tube open at both ends is pushed in water as shown in the figure, water rises in the tube to a height above its surface. The narrower the tube, the greater is the height to which water rises. This phenomenon is known as capillarity. When the same capillary tubes are placed in mercury, the liquid is depressed below the outside level. The depression increases as the diameter of the capillary tube decreases. A close observation of the capillary phenomenon reveals the following facts.

The free surface (meniscus) of the liquid which rises in the capillary tube is concave upward. The meniscus of liquid which falls in the capillary tube is convex upward.



(a)



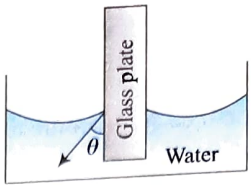
(b)

ANGLE OF CONTACT

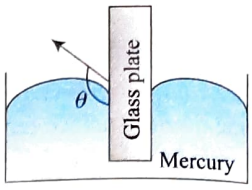
It is seen that when a liquid comes in contact with a solid surface, it is generally curved.

The angle θ, which the tangent to the liquid surface at the point of contact makes with the solid surface inside the liquid, is called the angle of **contact** or the capillary angle.

The angle of contact is acute (less than 90°) in the case of liquids, which wet the walls of the container [Fig. (a)] and is obtuse (greater than 90°) for the liquids which do not wet the walls of the container as shown in Fig. (b).



(a)



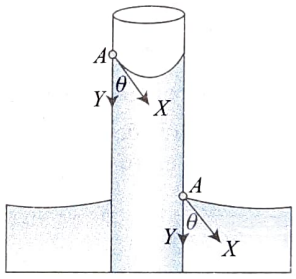
(b)

The angle of contact for pure water and clean glass surface is zero.

Those liquids, which wet the walls of the container (say in case of water and glass), have meniscus concave upwards and their value of angle of contact is less than 90° (also called acute angle). However, those liquids, which do not wet the walls of the container (say in case of mercury and glass), have meniscus convex upwards and their value of angle of contact is greater than 90° (also called obtuse angle).

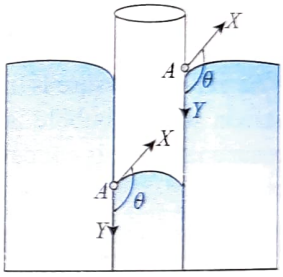
Consider two tubes dipping in two different liquids (water and mercury) as shown in Figs. (a) and (b).

Let A represent the point of contact of the liquid with the tube. Through A, draw the tangent AX to the liquid surface and another tangent AY to the solid surface inside the liquid. Then $\angle XAY = \theta$ (say) is called the angle of contact between the liquid and the given solid. Thus,



Water

(a)



Mercury

(b)

RISE OF A LIQUID IN A CAPILLARY TUBE (ASCENT FORMULA)

Consider a capillary tube of radius r open at both ends and dipped into a liquid, which wets the walls of the tube. Such a liquid rises in the tube forming a concave meniscus as already discussed.

The liquid surface is in contact with the walls of the tube all along the circle of circumference $2\pi r$, where r is the radius of the tube. Due to surface tension of the liquid, force T per unit length acts along the tangent to the meniscus and in inward direction. Since the liquid surface has a tendency to contract, it pulls the tube inward all along the circumference of the circle of contact. Thus, the liquid exerts a force T per unit length on the glass tube and in accordance with Newton's third law of motion, the tube gives an equal and opposite reaction to the liquid in a direction as shown in the figure.

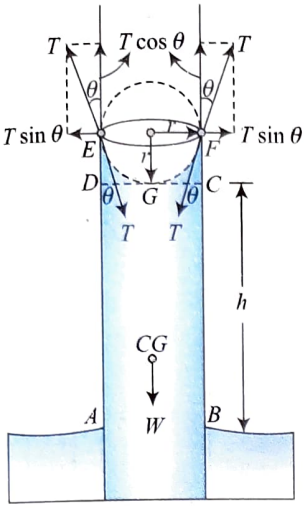
This reaction can be resolved into two rectangular components:

(i) $T \cos \theta$, acting upward along the walls of the tube,

(ii) $T \sin \theta$, acting perpendicular to the walls of the tube. This component, therefore, plays no part in raising the liquid in the tube.

Here, θ = angle of contact between the liquid and the glass; Let h = height to which the liquid rises in the tube; and ρ = density of

the liquid at room temperature. The only component responsible for raising the liquid in the tube is $T\cos\theta$.



The component $T\cos\theta$ acts all along the circumference of the circle of contact of the liquid with the tube.

The total upward force on the liquid $= 2\pi r \times T\cos\theta$ ...(i)

Under the effect of this upward force, the liquid rises in the tube to a height, say h; till the weight of the liquid above the free surface becomes just equal to the vertical upward force.

Volume of the liquid in the cylindrical column $ABCD$ of height h and radius r, $V = \pi r^2 h$.

If the tube is narrow, the radius of the curvature of the meniscus is approximately equal to the radius (r) of the capillary tube.

Volume of the liquid under the meniscus

= volume of the cylinder $CDEF$ of height r and radius r

– volume of the hemisphere EGF of radius r

$$= \pi r^2 \times r - \frac{1}{2}\times\frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

Total volume of the liquid in the capillary tube

$$= \pi r^2 h + \frac{1}{3}\pi r^3 = \pi r^2 (h + r/3)$$

Hence, mass of liquid in the tube.

$$m = \text{volume}\times\text{density} = \pi r^2\left(h + \frac{r}{3}\right)\rho \quad \text{...(ii)}$$

The weight of the liquid in the tube.

$$W = mg = \pi r^2\left(h + \frac{r}{3}\right)\rho g$$

This weight acts at the CG (centre of gravity) of liquid column in the tube.

In equilibrium, total upward force, F = weight of the liquid in the tube, W. From Eqs. (i) and (ii), we have

$$2\pi r T\cos\theta = \pi r^2\left(h + \frac{r}{3}\right)\rho g \quad \text{...(iii)}$$

or $$2T\cos\theta = r\left(h + \frac{r}{3}\right)\rho g$$

or $$h + \frac{r}{3} = \frac{2T\cos\theta}{r\rho g}$$

or $$h = \frac{2T\cos\theta}{r\rho g} - \frac{r}{3} \quad \text{...(iv)}$$

This relation is known as the ascent formula or Jurin's equation.

When the tube is of a very fine bore, $r/3$ may be neglected as compared to h and the value of surface tension T is given by [From Eq. (iv)]

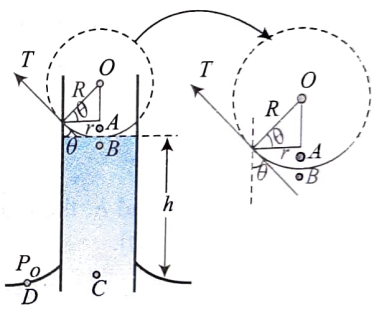
$$h = \frac{2T\cos\theta}{r\rho g} \quad \text{...(v)}$$

This relation is also called the **ascent formula**.

As, r, θ, ρ and g are constants, therefore, for tubes of different radii, $h \propto \frac{1}{r}$. Thus, the liquid rises more in a narrow tube and less in a wider tube. When pure water is contained in a glass tube, $\theta = 0°, \cos\theta = \cos 0° = 1$.

From Eq. (v), $$h = \frac{2T}{r\rho g} \quad \text{...(vi)}$$

Alternative method: Let us take four points A, B, C and D as shown in the figure. A and B are the points just above and just below the meniscus, respectively.



Now, we have $P_A = P_B + \frac{2T}{R}$

where R = radius of curvature of liquid meniscus. Substituting $P_A = P_0$, we have

$$P_B = P_0 - \frac{2T}{R} \quad \text{...(vii)}$$

If C is the point at the bottom of the excess liquid column, we have

$$P_C = P_B + \rho g h$$

where h = height of the excess liquid column.

Substituting $P_C = P_D = P_0$, we have

$$P_B = P_0 - \rho g h \quad \text{...(viii)}$$

Using Eqs. (vii) and (viii), we have

$$P_0 - \rho g h = P_0 - \frac{2T}{R}$$

This gives $h = \frac{2T}{\rho g R}$ where $R = \frac{r}{\cos\theta}$

Then $$h = \frac{2T\cos\theta}{\rho g r}$$

Important Points:

- In case of liquids which do not wet the container, θ is obtuse (i.e. $\theta > 90°$) and $\cos\theta$ is negative. From Eq. (v), h is negative, i.e., the level of the liquid in the tube is below the liquid level outside the tube.
- Equation (v) can be obtained in a much simpler way as given below. Let us neglect the volume of the liquid below the meniscus. Weight of the liquid in the cylinder $ABCD$, i.e., W = volume of cylinder $ABCD$ × density of the liquid × g

or $W = \pi r^2 h\rho g$ ...(ix)

As we have already derived in Eq. (i), total upward force due to surface tension, i.e.,

$F = 2\pi rT\cos\theta$ ...(x)

In equilibrium, from Eqs. (ix) and (x),

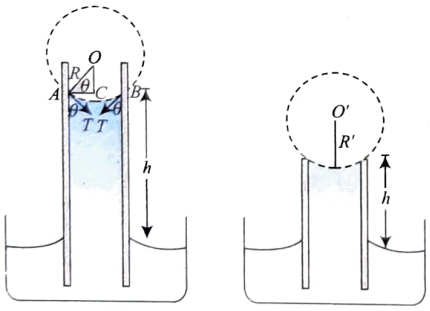
$$\pi r^2 h\rho g = 2\pi rT\cos\theta \text{ or } h = \frac{2T\cos\theta}{\rho gr}$$

Rise of Liquid in a Capillary Tube of Insufficient Length

Consider that a capillary tube of radius r is dipped in a liquid of surface tension T. The height to which the liquid will rise in the tube is given by

$$h = \frac{2T\cos\theta}{r\rho g} \quad ...(i)$$

Let O be the centre of curvature and R, the radius of curvature of the meniscus of the liquid [Fig. (a)].



(a) (b)

From the right angled ΔACO, we have

$$\cos\theta = \frac{r}{R} \text{ or } r = R\cos\theta$$

Substituting for r in the Eq. (i), we have

$$h = \frac{2T\cos\theta}{(R\cos\theta)\rho g} = \frac{2T}{R\rho g}$$

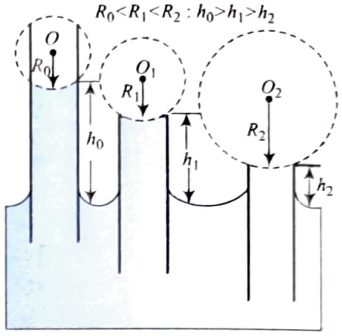
or $Rh = \frac{2T}{\rho g} = a$ (constant) ...(ii)

When tube is of insufficient length. If the length of the tube h' is less than h [Fig. (b)], it is found that the liquid does not overflow. In a tube of insufficient length, the liquid rises up to the top of the tube and increases the radius of curvature of its meniscus to a value R', so that

$$R'h' = Rh \quad ...(iii)$$

i.e., smaller the length (h') of the tube, greater will be the radius of curvature (R') of the meniscus, but the liquid will never overflow.

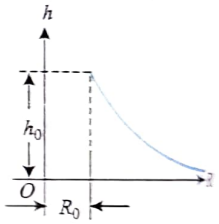
It should be clearly noted that whereas the radius of curvature of the meniscus changes, angle of contact remains constant as it depends upon the nature of the liquid and solid.



If we slowly push the capillary tube vertically down (into) the liquid, the height of the liquid column remains the same till the length of the tube above the liquid surface is equal to $2T\cos\theta/(\rho gr)(= h)$. Therefore, the liquid meniscus gets flattened gradually by increasing its radius of curvature to obey the law hR = constant.

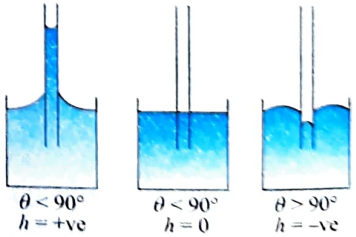
It means that $hr = h_1 r_1 = h_2 r_2 = \cdots$

When the capillary tube just sinks, i.e., $h = 0$, we can see that the meniscus becomes flat, radius of curvature $r \to \infty$.

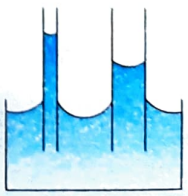


Important Points

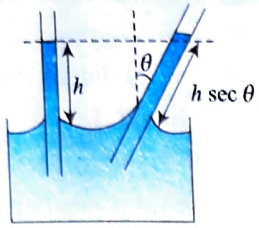
- h depends on surface tension T, angle of contact θ, radius of the tube, density ρ of liquid and acceleration due to gravity (or effective g). Hence the capillary action depends on the nature of liquid and solid (capillary tube) in contact, temperature of the surroundings.
- If $\theta < 90°$, h is positive (wetting) which signifies capillary rise and concave meniscus; if $\theta = 90°$, $h = 0$ (neither wetting nor non-wetting): plane meniscus; if $\theta > 90°$, h = negative (non-wetting) which signifies capillary fall and convex meniscus.



- Since T, ρ and θ are constant at a given temperature, for any given liquid and capillary tube, $hr = c$; this tells us that capillary rise (or fall) will be greater in thinner tubes and vice versa.



- Since T, ρ, θ, g and R are constant for a given set of capillary and liquid, we have h = constant. It means that if we change the angle β of orientation from vertical, the capillary rise h (or capillary fall) remains constant even though the length of the excess liquid in the tube increases from h to $h \sec \beta$.



- If the radius of the liquid meniscus remains constant, the weight (but not length) of the liquid column remains the same in capillary tubes of different shapes and sizes.

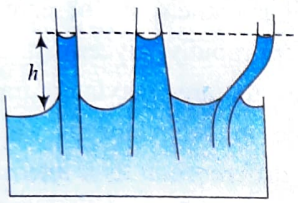


ILLUSTRATION 8.16

A vertical capillary with inside diameter 0.50 mm is submerged into water so that the length of its part emerging outside the water surface is equal to 25 mm. Find the radius of curvature of the meniscus. Surface tension of water is 73×10^{-3} N/m.

Sol. In the capillary tube, the water should rise to a height

$$h = \frac{2T}{r\rho g}$$

Here $T = 73 \times 10^{-3}$ Nm, $r = \frac{0.50 \text{ mm}}{2} = 0.25 \times 10^{-3}$ m,

$\rho = 10^3$ kg/m^3

$$\therefore \quad h = \frac{2 \times 73 \times 10^{-3}}{0.25 \times 10^{-3} \times 10^3 \times 9.8} = 59 \times 10^{-3} \text{ m} = 59 \text{ mm}$$

Now $h > h'$, i.e., length is outside water surface.

Therefore, radius of meniscus > radius of capillary r.

If R is the radius of meniscus, then we have

$$\frac{2T}{R} = h'\rho g \quad \text{or} \quad R = \frac{2T}{h'\rho g}$$

Here $h' = 25$ mm $= 25 \times 10^{-3}$ m

$$\therefore \quad R = \frac{2 \times 73 \times 10^{-3}}{25 \times 10^{-3} \times 10^3 \times 9.8} = 0.6 \times 10^{-3} \text{ m} = 0.6 \text{ mm}$$

ILLUSTRATION 8.17

A glass rod of diameter $d_1 = 1.5$ mm is inserted symmetrically into a glass capillary with inside diameter $d_2 = 2.0$ mm. Then the whole arrangement is vertically oriented and brought in contact with the surface of water. To what height will the liquid rise in the capillary? Surface tension of water = 73×10^{-3} N/m.

Sol. If R is radius of meniscus, then $\frac{2T}{R} = h\rho g$

Here $R = \frac{r_2 - r_1}{\cos\theta}$

θ being angle of contact, r_1 = radius of glass rod, r_2 = radius of capillary.

$$\frac{2T\cos\theta}{r_2 - r_1} = h\rho g \text{ or } h = \frac{2T\cos\theta}{(r_1 - r_1)\rho g}$$

Here $r_1 = d_1/2, r_2 = \frac{d_2}{2}$

$$\therefore \quad h = \frac{4T\cos\theta}{(d_2 - d_1)\rho g}$$

Substituting given values and $\theta \cong 0°$ for water–glass interface, we have

$$h = \frac{4 \times 73 \times 10^{-3} \cos 0°}{(2.0 - 1.5) \times 10^{-3} \times 10^3 \times 9.8} = 60 \times 10^{-3} \text{ m} = 6 \text{ cm}$$

ILLUSTRATION 8.18

The end of a capillary tube with a radius r is immersed in water. Is mechanical energy conserved when the water rises in the tube? The tube is sufficiently long. If not, calculate the energy change.

Sol. In the equilibrium position ($\theta = 0°$ for water and glass)

$$2\pi rT \cos 0° = \pi r^2 h\rho g$$

$$\text{or} \quad h = \frac{2T}{\rho g r}$$

Work done by surface tension $= (2\pi rT) \times h = \frac{4\pi T^2}{\rho g}$

The potential energy of water in the tube, $U = (\pi r^2 h\rho)gh/2$; it is multiplied by $h/2$ because the centre of gravity of the water and the capillary tube is at a height $h/2$

$$\therefore \quad U = \frac{2\pi T^2}{\rho g}$$

Thus, it is seen that the mechanical energy is not conserved.

Therefore, mechanical energy loss $= \frac{4\pi T^2}{\rho g} - \frac{2\pi T^2}{\rho g} = \frac{2\pi T^2}{\rho g}$

This energy is converted into heat.

ILLUSTRATION 8.19

A glass capillary tube of internal radius r = 0.25 mm is immersed in water. The top end of the tube projects by 2 cm above the surface of water. At what angle does the liquid meet the tube? Surface tension of water = 0.7 N m^{-1}.

Sol. Water wets glass and so the angle of contact is zero. Neglecting the small mass in the meniscus, for full rise,

$$2\pi rT = \pi r^2 h\rho g$$

$$\text{or} \quad h = \frac{2T}{r\rho g} = \frac{2 \times 0.07}{0.25 \times 10^{-3} \times 1000 \times 9.8} = 0.057 \text{ m} = 5.7 \text{ cm}$$

But here the tube is only 2 cm above the water and so water will rise by 2 cm and meet the tube at an angle such that

$$2\pi rT\cos\theta = \pi r^2 h'\rho g$$

$$\Rightarrow \quad 2T\cos\theta = h'r\rho g \quad \Rightarrow \cos\theta = \frac{h'r\rho g}{2T}$$

$$\therefore \quad \cos\theta = \frac{2\times10^{-2}\times0.25\times10^{-3}\times1000\times9.8}{2\times0.07} \quad \Rightarrow \theta = 70^\circ$$

Concept Application Exercise 8.4

1. A false statement is
   (a) Angle of contact $\theta < 90^\circ$, if cohesive force < adhesive force
   (b) Angle of contact $\theta > 90^\circ$, if cohesive force > adhesive force
   (c) Angle of contact $\theta = 90^\circ$, if cohesive force = adhesive force
   (d) If the radius of capillary is reduced to half, the rise of liquid column becomes four times
2. When a capillary tube is dipped in water it rises up to 8 cm in the tube. What happens when the tube is pushed down such that its end is only 5 cm above the outside water level
   (a) The radius of the meniscus increases and therefore water does not overflow
   (b) The radius of the meniscus decreases and therefore water does not overflow
   (c) The water forms a droplet on top of the tube but does not overflow
   (d) The water start overflowing
3. A glass capillary tube of internal diameter 0.6 mm is held vertically with the lower end in water and with 80 mm of the tube above the surface of water. How much high does the water rise? If the tube is now lowered until 30 mm of its length is above the surface of water, what happens? Surface tension of water is 75 dyne cm^{-1} and $g = 1000\ cm/s^2$
4. A long capillary tube of radius 0.1 cm open at both ends is filled with water and placed vertically. What will be the height of the column of water left in the capillary? Given, surface tension of water = 72 dyne cm^{-1} and density of water = 1 g cm^{-3}.
5. Water rises in a capillary tube to a height 2 cm. In an other capillary tube, whose radius is one third of it, how much water will rise? If the first capillary tube is inclined at an angle of 60° with the vertical, then what will be the position of water in the tube?

Answers

1. (d) **2.** (a) **3.** 5.0 cm, 0.05 cm
4. 2.94 cm **5.** 6 cm, 12 cm

VISCOSITY

We are familiar with the property of friction in case of solids. When a solid body slides over another solid body, a frictional force begins to act between them. This force opposes the relative motion of the bodies. Similarly, when a layer of a liquid slides over another layer of the same liquid, a frictional force acts between them which opposes the relative motion between the layers. This force is called 'internal frictional force' or 'viscosity'. Because of this property a backward dragging force (viscous drag) acts tangentially on the layers of the liquid in motion and it tries to destroy the motion of the liquid. Let us consider the following examples.

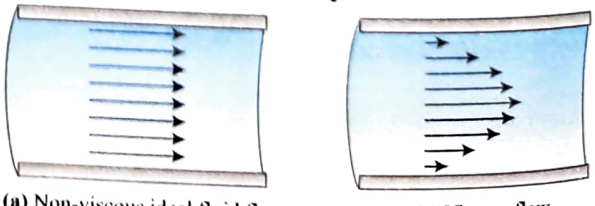
(i) If you move in a boat on a gentle river, you might have noticed that your boat moved faster in the middle of the river than very close to the banks. Why would this happen?

(ii) Let us pour equal quantities of water, glycerine and honey into three similar funnels. We observe that whereas water flows down very rapidly, glycerine trickles down drop by drop and honey requires maximum time to trickle down. This difference in behaviour of various liquids is on account of the difference in their internal frictional forces. This internal frictional force opposing the flow of liquid is maximum in case of honey, lesser in case of glycerine and least in the case of water.

(iii) When we stir a liquid contained in a beaker with a glass rod, it starts rotating in coaxial cylindrical layers. If the stirring is stopped, the liquid continues rotating for some time and then comes to rest. According to the Newton's first law of motion, the liquid can stop only if an opposing force acts on it. Since no force has been applied from outside, this opposing force, called the internal frictional force, must be arising from within the liquid.

In an ideal fluid there is no viscosity to hinder the fluid layers as they slide past one another. Within a pipe of uniform cross section, every layer of an ideal fluid moves with the same velocity, even the layer next to the wall, as Fig. (a) shows. When viscosity is present, the fluid layers have different velocities, as Fig. (b) of the drawing illustrates. The velocity profile for the streamlines in viscous flow in a tube is sketched in Fig. (b). The profile is parabolic, with the velocity approaching zero at the walls and reaching its maximum value in the centre. This flow is still laminar, with the streamlines all flowing parallel to one another.

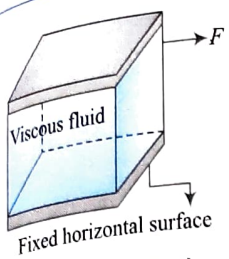
Velocity profiles in a cylindrical tube



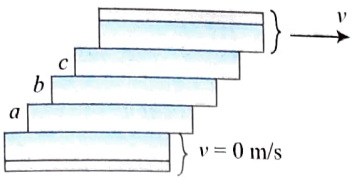
(a) Non-viscous ideal fluid flow (b) Viscous flow

How Viscous Force Acts

Suppose a liquid is flowing in a streamlined motion on a fixed horizontal surface.



(a) A force F is applied to the top plate, which is in contact with a viscous fluid



(b) Because of the force, the top plate and the adjacent, layer of fluid move with a constant velocity

The layer of the liquid which is in contact with the surface is at rest, while the velocity of other layers increases with a distance from the fixed surface. Thus, there is a relative motion between adjacent layers of the liquid. Let us consider three parallel layers—a, b and c. Their velocities are in an increasing order. The layer a tends to retard the layer b, while b tends to retard c.

Thus, each layer tends to decrease the velocity of the layer above it. Similarly, each layer tends to increase the velocity of the layer below it. This means that in between any two layers of the liquid, internal tangential forces act which try to destroy the relative motion between the layers. These forces are called 'viscous forces'. If the flow of the liquid is to be maintained, an external force must be applied to overcome the dragging viscous forces. In the absence of the external force, the viscous forces would soon bring the liquid to rest. The property of the liquid by virtue of which it opposes the relative motion between its adjacent layers is known as 'viscosity'.

Viscosity comes into play only when there is a relative motion between the layers of the same material. This is why it does not act in solids.

Similarities and Differences Between Viscosity and Sliding Friction

Similarities:

1. Both viscosity and sliding friction oppose relative motion. Whereas viscosity opposes the relative motion between two adjacent liquid layers, sliding friction opposes the relative motion between two solid layers.
2. Both come into play whenever there is a relative motion between layers of liquid or solid surfaces as the case may be.
3. Both are due to molecular attractions.

Dissimilarities:

	Viscosity	Sliding friction
1.	Viscosity (or viscous drag) between layers of liquid is directly proportional to the area of the liquid layers.	Friction between two solids is independent of the area of solid surfaces in contact.
2.	Viscous drag is proportional to the relative velocity between two layers of liquid.	Friction is independent of the relative velocity between two surfaces.
3.	Viscous drag is independent of normal reaction between two layers of liquid.	Friction is directly proportional to the normal reaction between two surfaces in contact.

Some Applications of Viscosity

Knowledge of viscosity of various liquids and gases has been put to use in daily life. Some applications of its knowledge are discussed as follows:

1. As the viscosity of liquids varies with temperature, proper choice of lubricant is made depending upon season.
2. Liquids of high viscosity are used in shock absorbers and buffers at railway stations.
3. The phenomenon of viscosity of air and liquid is used to damp the motion of some instruments.
4. The knowledge of the coefficient of viscosity of organic liquids is used in determining the molecular weight and shape of the organic molecules.
5. It finds an important use in the circulation of blood through the arteries and veins of human body.

Comparing the Viscosity of Two Fluids

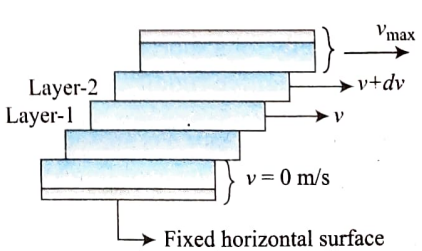
Viscosity is a measure of frictional forces between the adjacent layers of a fluid with which it opposes the relative motion between them. The frictional resistance of fluids is also observed when a solid body moves within the body of a fluid. Viscous forces are present both in liquids and gases. Obviously, the liquids are more viscous than gases.

The viscosity of two liquids can be compared in the following two ways:

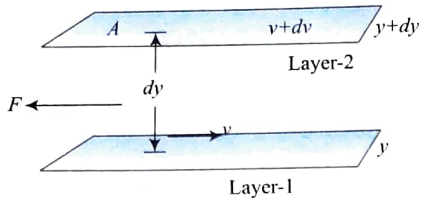
1. By allowing them to flow out of identical vessels. For example, a liquid of low viscosity such as water flows rapidly out of it, and on the contrary, a highly viscous liquid such as glue or glycerine makes a slow movement out of it.
2. By dropping steel ball bearings through each liquid. The balls fall more slowly in a liquid of high viscosity.

Measurement of Viscosity

Consider a liquid flowing over a fixed horizontal solid surface in the form of parallel layers. The layer in contact with the solid surface is found to be at rest and as we move up, the velocity of the layers goes on increasing and the layer at the top possesses maximum velocity. Let us consider two parallel layers '1' and '2' at distances y, $y + dy$ from the fixed surface and moving with the velocities v and $v + dv$ respectively. Then, dv/dy denotes the rate of change of velocity with the distance in the direction of increasing distance and it is called the velocity gradient. The relative motion between the two layers can take place only, if some external force acts between them. Due to viscosity, a force F acts in opposite direction to destroy the relative motion.



(a)



(b)

According to Newton, the viscous force F depends upon the following factors:

- It is directly proportional to the area of the layers in contact i.e. $F \propto A$
- It is directly proportional to the velocity gradient between the layers i.e. $F \propto \frac{dv}{dy}$

Combining these two factors, we have

$$F \propto A\frac{dv}{dy} \text{ or } F = -\eta A\frac{dv}{dy} \quad ...(i)$$

where η is a constant of proportionality and is called the **coefficient of viscosity** or simply the viscosity of the liquid. Its value depends upon the nature and physical conditions of the liquid. The negative sign in Eq. (i) indicates that the viscous drag acts in a direction opposite to the liquid flow.

Definition of η

Let the area of the layer $A = 1$ and velocity gradient $dv/dy = 1$, then from Eq. (i), $\eta = F$ (neglecting negative sign). Thus, 'coefficient of viscosity of a liquid is defined as the tangential viscous drag acting per unit area between two liquid layers moving with unit velocity gradient.'

From Eq. (i), neglecting negative sign,

$$\eta = \frac{(F/A)}{(dv/dy)} = \frac{\text{tangential stress}}{\text{velocity gradient}} \quad ...(ii)$$

Also, we know that modulus of rigidity,

$$\eta = \frac{F/A}{\Delta x/L} = \frac{\text{tangential stress}}{\text{displacement gradient}} \quad ...(iii)$$

Comparing Eqs. (ii) and (iii), we can observe that coefficient of viscosity in liquids is analogous to the modulus of rigidity in solids.Thus viscosity of a liquid is the limiting case of rigidity in solids when they (i.e., solids) break down under the applied stress. A liquid develops a shearing strain (i e., shear) for a short time after which it breaks down. This shear is, however, formed again and the process of breaking down and reformation of shear goes on till the shearing stress is removed. Thus, a liquid offers resistance to shearing stress only for a short time and as such it is said to possess fugitive (i.e., transient, short lived) rigidity i.e., fluids possess rigidity only for a short time. It is due to this reason that viscosity is termed fugitive rigidity.

Note: The coefficient of viscosity η is analogous to the modulus of rigidity of solids. The shear stress of fluids is proportional to the rate of shear strain and that in case of solids is proportional to the shear strain.

Dimensional Formula for η

From Eq. (i), $\eta = \frac{F}{A(dv/dy)}$ (neglecting –ve sign)

$$[\eta] = \frac{[MLT^{-2}]}{[L^2][LT^{-1}/L]} = [ML^{-1}T^{-1}]$$

Units of Coefficient of Viscosity

(i) **In cgs system**, the unit of η is dyne s/cm^2 and is called poise (P). The coefficient of viscosity of a liquid is said to be one poise if a tangential viscous drag of 1 dyne/cm acts between two liquid layers moving with a velocity gradient of 1 (cm/s) per cm (i.e., s^{-1}).

(ii) **SI unit** of viscosity is Ns / m^2 and is called a **pascal** second (Pa s) as

$$1\frac{\text{N.s}}{\text{m}^2} = 1\left(\frac{\text{N}}{\text{m}^2}\right).\text{s} = 1 \text{ Pa.s}$$

The coefficient of viscosity of a liquid is said to be one Pa s if a tangential drag of 1 N/m acts between two liquid layers moving with a velocity gradient of 1 (m/s) per m (i.e., s^{-1}).

Also,

$$1 \text{ Pa}\cdot\text{s} = 1\frac{\text{N.s}}{\text{m}^2} = 1\left(\frac{10^5 \text{ dyne}}{10^4 \text{ cm}^2}\right).\text{s} = 10\left(\frac{\text{dyne.s}}{\text{cm}^2}\right)$$

$\Rightarrow 1 \text{ Pa}\cdot\text{s} = 10 \text{ poise} = 1 \text{ decapoise}$

Effect of Temperature on Viscosity

Experiments show that η of a liquid decreases sharply with increase in its temperature and becomes zero at its boiling point. On the other hand, η of gases increase with temperature.

The viscosity of liquids decreases with increase in temperature and increases with the decrease in temperature. That is, $\eta \propto 1/\sqrt{T}$. On the other hand, the value of viscosity of gases increases with the increase in temperature and vice versa. That is, $\eta \propto \sqrt{T}$.

Illustration 8.20

A plate of area 100 cm^2 is placed on the upper surface of castor oil, 2 mm thick. Taking the coefficient of viscosity to be 15.5 poise, calculate the horizontal force necessary to move the plate with a velocity 3 cm s^{-1}.

Sol. The (horizontal tangential) viscous force is given by

$$F = -\eta A\frac{dv}{dy}$$

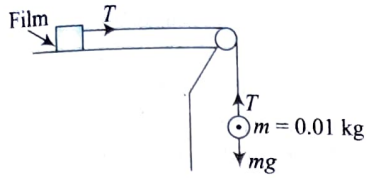
Given η = 15.5 poise, A = 100 cm^2

$$\frac{dv}{dx} = \frac{v_2 - v_1}{x_2 - x_1} = \frac{(0-3)\text{ cm/s}}{(2-0)\text{ mm}} = -\frac{3}{0.2}\text{s}^{-1} = -15\text{ s}^{-1}$$

$$F = -15.5 \times 100 \times (-15) = 2.235 \times 10^4 \text{ dyn} = 0.2325 \text{ N}$$

ILLUSTRATION 8.21

A metal plate of area 0.10 m^2 is connected to a 0.01 kg mass via a string that passes over an ideal pulley (considered to be frictionless), as shown in the figure. A liquid with a film thickness of 3.0 mm is placed between the plate and the table. When released, the plate moves to the right with a constant speed of 0.085 $m s^{-1}$. Find the coefficient of viscosity of the liquid.



Sol. As the metal plate moves with constant velocity,

$mg - T = 0 \Rightarrow T = mg$

If F is the viscous force on the plate, then

$F = T = mg = 0.01 \times 9.8 = 9.8 \times 10^{-2}$ N

By Newton's law of viscosity, $F = \eta A \frac{dv}{dy}$

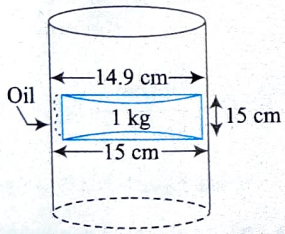
where $\frac{dv}{dy} = \frac{0.085}{0.3 \times 10^{-3}} = 28.3$ s

$\Rightarrow 9.8 \times 10^{-2} = \eta \times 0.10 \times 28.3$

$\Rightarrow \eta = 3.4 \times 10^{-3}$ N s/m^2

ILLUSTRATION 8.22

A sliding fit cylindrical body of mass of 1 kg drops vertically down at a constant velocity of 5 cm s^{-1}. Find the viscosity of the oil.



Sol. As the body moves with constant velocity,

$mg = F_v \Rightarrow 1g = \eta A \frac{dv}{dy}$

$\Rightarrow 1g = \eta(2\pi l)\frac{dv}{dy}$

$\Rightarrow 1g = \eta(2\pi \times 7.5 \times 10^{-2} \times 15 \times 10^{-2})\left(\frac{0.05 - 0}{0.05 \times 10^{-2}}\right)$

After solving, we get $\eta = 1.4$ N s/m^2.

REYNOLDS NUMBER

Reynolds was able to show that flow of a liquid in a tube is laminar or turbulent depending upon the values of four physical parameters:

(i) average velocity of the liquid, v
(ii) viscosity of the liquid, η
(iii) density of the liquid, ρ and
(iv) diameter of the tube, d

In order to find a relation between them, let

$v \propto \eta^a \rho^b d^c$ or $v = k\eta^a \rho^b d^c$ ...(i)

Writing the dimensions of various quantities on both sides, we get,

$[M^0L^1T^{-1}] = [ML^{-1}T^{-1}]^a [ML^{-3}]^b [L]^c$

Equating the powers of M, L and T on both sides,

$a + b = 0, -a - 3b + c = 1$ and $-a = -1$

Solving these equations, we get

$a = 1, b = -1$ and $c = -1$

From Eq. (i),

$v = k\eta\rho^{-1}d^{-1}$ or $v = k\frac{\eta}{\rho d}$ ...(ii)

Equation (ii) is called **Reynolds formula** and the dimensionless number k is called the **Reynolds number** and is denoted by R_e.

Replacing k by R_e, Eq. (ii) becomes,

$v = R_e \frac{\eta}{\rho d}$ or $R_e = \frac{\rho v d}{\eta}$ ...(iii)

Experiments have shown that if

(i) $R_e < 2000$, liquid flow is laminar.
(ii) $R_e > 3000$, liquid flow is turbulent.
(iii) For values of R_e between 2000 and 3000, the flow is unstable and may change back and forth from one type to the other.

Equation (iii) is a general relation and is valid for all velocities. For fixed values of ρ, d and η, R_e is constant for a fixed velocity. Its value changes with the velocity of liquid flow.

The value of R_e when $v = v_c$ (critical velocity, the velocity of flow above which the flow becomes turbulent), is called the critical Reynolds number and is denoted by $(R_e)_c$. From Eq. (iii),

$v_c = (R_e)_c \frac{\eta}{\rho d}$ or $(R_e)_c = \frac{v_c \rho d}{\eta}$ ...(iv)

ILLUSTRATION 8.23

Calculate the Reynolds number for blood flowing at 30 cm/s through an aorta of radius 1.0 cm. Assume that the blood has a viscosity of 4 mPa s and a density of 1060 kg/m^3. What is the nature of flow?

Sol. Here, $v = 30$ cm/s $= 3 \times 10^{-1}$ m/s, $\rho = 1060$ kg/m^3,

$d = 2 \times 1.0$ cm $= 20 \times 10^{-2}$ m

$\eta = 4$ mPa s $= 4 \times 10^{-3}$ Pa s $= 4 \times 10^{-3}$ Ns/m^2

Thus, $R_e = \frac{v\rho d}{\eta}$

$= \frac{(3 \times 10^{-1}\ \text{m/s})(1060\ \text{kg/m}^3)(2.0 \times 10^{-2}\ \text{m})}{(4 \times 10^{-3}\ \text{Ns/m}^2)} = 1590$

Since R_e is less than 2000, this flow will be laminar.

ILLUSTRATION 8.24

What should be the maximum average velocity of water in a tube of diameter 2 cm so that the flow is laminar? The viscosity of water is 10^{-3} Ns/m^2.

Sol. Here, $d = 2\text{ cm} = 2 \times 10^{-2}\text{ m}$, $\eta = 10^{-3}\text{ Ns/m}^2$, $\rho = 10^3\text{ kg/m}^3$

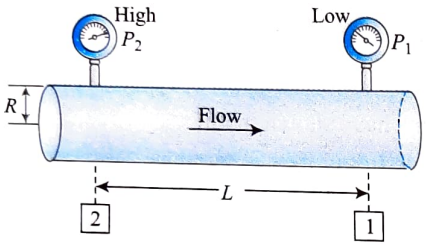
For laminar flow through the tube, the maximum value for R_e is 2000. If v is the maximum average velocity of water, then

$$R_e = \frac{v\rho d}{\eta} \quad \text{or} \quad v = \frac{R_e \eta}{\rho d}$$

$$\text{or} \quad v = \frac{(2000)(10^{-3}\text{ Ns/m}^2)}{(10^3\text{ kg/m}^3)(2\times10^{-2}\text{ m})} = 0.1\text{ m/s}$$

Poiseuille's Formula for Flow of Liquid Through a Capillary Tube

When a liquid flows 'in a tube', the viscous forces oppose the flow of the liquid. Hence a pressure difference is applied between the ends of the tube which maintains the flow of the liquid.



Poiseuille derived a formula for the volume of liquid flowing out per second through a narrow horizontal tube by assuming the flow of liquid to be streamlined and parallel to axis of the tube.

Poiseuille found that volume of liquid flowing per second (Q) is as follows:

- Directly proportional to $P_2 - P_1 = \Delta P$, a greater pressure difference leading to a larger flow rate, i.e. $Q \propto \Delta P$.
- A long pipe offers greater resistance to the flow than a short pipe does, and Q is inversely proportional to the length L, i.e. $Q \propto \frac{1}{L}$
- High-viscosity fluids flow less readily than low-viscosity fluids, and Q is inversely proportional to the viscosity η, i.e. $Q \propto \frac{1}{\eta}$
- Finally, the volume flow rate is larger in a pipe of larger radius, other things being equal. The dependence on the radius R is a surprising one, Q being proportional to the fourth power of the radius, or R^4, i.e. $Q \propto R^4$. If, for instance, the pipe radius is reduced to one-half of its original value, the volume flow rate is reduced to one-sixteenth of its original value.

Combining all these factors, we have

$$Q \propto \frac{R^4 \Delta p}{\eta L} \quad \text{or} \quad Q = \frac{kR^4 \Delta p}{\eta L}$$

Here, k is constant of proportionality and is found to be equal to $\pi/8$. Therefore, above relation becomes

$$Q = \frac{\pi R^4 \Delta p}{8\eta L}$$

This expression is known as **Poiseuille's equation** for the flow of a liquid through a narrow tube. The volume of liquid flowing out per second from a tube is also known as rate of flow or rate of discharge of the liquid. Poiseuille's equation holds good only under the following assumptions:

1. The liquid flow is steady and the streamlines are parallel to the axis of the tube.
2. As there is no radial flow of liquid, the pressure is constant over any cross section of the tube.
3. The liquid in contact with the walls of the tube is at rest.
4. The tube is narrow and of uniform area of cross section.
5. The tube is held horizontally, so that there is no effect of gravitational force on the liquid flow.
6. The liquid is capable of sustaining a small shearing stress.

Illustration 8.25

An engineer wants to have the same flow rate of water and light machine oil from the pipes of the same length and with the same pressure head. What should be ratio of the radii of the two pipes? Given that viscosity of water = 0.01 poise and that of light machine oil = 11 poise.

Sol. Let L and R be the length and radius of the pipe. Let Δp be the pressure difference between the two ends of the pipe. If η is viscosity of the water, then the rate of flow of the water through the pipe,

$$Q = \frac{\pi \Delta p R^4}{8\eta L} = \frac{\pi \Delta p R^4}{8\times0.01\times L} \quad \text{...(i)}$$

Suppose that the radius of the pipe through which light machine oil flows has to be taken as R', so as to keep the rate of flow same in the two cases. If η' is viscosity of the light machine oil, then the rate of flow of the oil through the pipe,

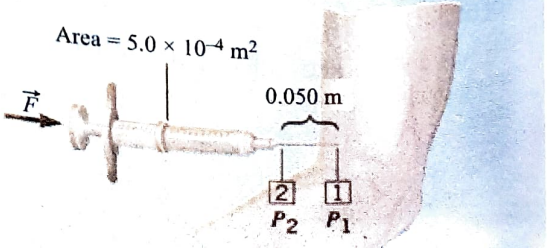
$$Q = \frac{\pi p R'^4}{8\eta' L} \quad \text{or} \quad Q = \frac{\pi \Delta p R'^4}{8\times11\times L} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we have

$$\frac{\pi \Delta p R'^4}{8\times11\times L} = \frac{\pi \Delta p R^4}{8\times0.01\times L} \quad \text{or} \quad \frac{R'}{R} = \left(\frac{11}{0.01}\right)^{1/4} = (1100)^{1/4} = 5.76$$

Illustration 8.26

A hypodermic syringe is filled with a solution whose viscosity is 3.14×10^{-5} Pa. As figure shows, the plunger area of the syringe is $5.0\times10^{-4}\text{ m}^2$, and the length of the needle is 5.0 cm. The internal radius of the needle is 2.0×10^{-4} m. The gauge pressure in the vein that is injected is 2000 Pa. What force must be applied to the plunger, so that $8.0\times10^{-3}\text{ m}^3$ of solution can be injected in 4.0 s?



Sol. The force required here is the pressure applied to the plunger times the area of the plunger. In this case the viscous flow is occurring, the pressure is different at different points along the syringe. The barrel of the syringe is wide enough that little pressure difference is required to sustain the flow up to point 2, where the fluid encounters the narrow needle. It means the pressure applied to the plunger is nearly equal to the pressure P_2 at point 2.

To find this pressure, we apply Poiseuille's law to the needle. According to Poiseuille's law

$$P_2 - P_1 = \frac{8\eta LQ}{\pi R^4} \qquad ...(i)$$

Here pressure P_1 is given as a gauge pressure, which, in this case, is the amount of pressure in excess of atmospheric pressure. And we are interested to find the amount of force in excess of the force applied to the plunger by the atmosphere.

The volume flow rate $Q = \dfrac{8.0\times10^{-6}\ \text{m}^3}{4.0\ \text{s}} = 2.0\times10^{-6}\ \text{m}^3/\text{s}$

Hence from (i)

$$P_2 - P_1 = \frac{8(3.14\times10^{-5}\ \text{Pa s})(0.05\ \text{m})(2.0\times10^{-6}\ \text{m}^3/\text{s})}{\pi(2.0\times10^{-4}\ \text{m})^4} = 5000\ \text{Pa}$$

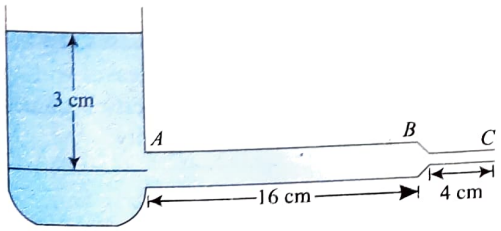
Since P_1 = 2000 Pa, the pressure P_2 = 5000 Pa + 2000 Pa = 7000 Pa. The force that must be applied to the plunger is this pressure times the plunger area:

$$F = (7000\ \text{Pa})(5.0\times10^{-4}\ \text{m}^2) = 3.5\ \text{N}$$

ILLUSTRATION 8.27

Two capillary tubes AB and BC are joined end to end at B. AB is 16 cm long of diameter 4 mm, tube BC is 4 cm long and of diameter 2 mm. The composite tube is held horizontally as in Poiseuille's experiment with A connected to a vessel of water giving a constant head of 3 cm and C is open to the air. Calculate the pressure difference between B and C.

Sol. AB and BC are two capillary tubes which are joined end to end at B as shown in the figure.



Let the atmospheric pressure be h cm of water column. Pressure at $A = (h + 3)$ and pressure at $C = h$

Further, let h' be the pressure at B.

Clearly, pressure difference across AB, i.e., $P_1 = (h + 3 - h')$ and pressure difference across BC, i.e., $P_2 = (h' - h)$

Since the volume of the liquid flowing (per second) through AB = volume of the liquid flowing (per second) through BC,

$$\frac{\pi P_1 r_1^4}{8\eta l_1} = \frac{\pi P_2 r_2^4}{8\eta l_2} \text{ or } \frac{P_1 r_1^4}{l_1} = \frac{P_2 r_2^4}{l_2}$$

Or $$\frac{P_1}{P_2} = \frac{l_1}{l_2}\left(\frac{r_2}{r_1}\right)^4$$

Or $$\frac{(h+3-h')}{(h'-h)} = \frac{16}{4}\times\left(\frac{0.1}{r_1}\right)^4$$

(l_1 = 16 cm, l_2 = 4 cm, r_1 = 0.2 cm, and r_2 = 0.1 cm)

Or $$\frac{h+3-h'}{h'-h} = \frac{16}{4}\times\frac{1}{16} = \frac{1}{4}$$

Or $h' - h = 4h + 12 - 4h'$

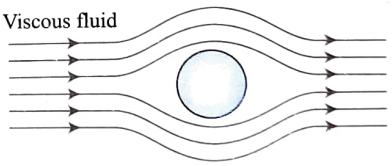
Or $5h' = 5h + 12$

Or $h' = h + 24$

Pressure difference across $BC = h' - h$ = 2.4 cm of water column.

STOKES' LAW

If you move a spherical body through a fluid, it experiences a viscous force. An expression for the viscous force experienced by the spherical object was derived by Sir George Gabriel Stokes and after his name, this expression is called Stokes' law. This law deals with the motion of a body through a viscous medium. When a solid body moves through a fluid, the layer of the fluid immediately in contact with the body moves with it. This causes a relative motion amongst different layers of the fluid, giving rise to a viscous drag which opposes the motion of the body. The streamlines for a fluid flowing slowly past a stationary solid sphere are shown in the figure.



When the sphere moves slowly as compared to the fluid, the pattern is similar but the streamlines then flow in such a way that the apparent motion of the fluid particles is as seen by someone on the moving sphere. In this latter case, it is known that the layer of fluid in contact with the sphere moves with it, thus creating a velocity gradient between this layer and other layers of fluid. Viscous forces are thereby brought into play and constitute the resistance experienced by the moving sphere. If we make the plausible assumption that the viscous retarding force F depends on the size of the body, the velocity with which it moves, the viscosity of the fluid and mass density of the fluid, then an expression can be derived for F by the method of dimensions. Thus,

$$F = k\eta^x v^y r^z$$

where x, y and z are the indices to be found and k is a dimensionless constant. The dimensional equation is

$$[MLT^{-2}] = [ML^{-1}T^{-1}]^x[LT^{-1}]^y[L]^z$$

Equating indices of M, L and T on both sides, we have

$1 = x$

$1 = -x + y + z$

$-2 = -x - y$

Solving, we get $x = 1, y = 1$ and $z = 1$. Hence $F = k\eta vr$

A detailed treatment, first done by Stokes, gives $k = 6\pi$ and so

$$F = 6\pi\eta vr$$

The above relation is called the Stokes' law. According to this law: **Whenever a spherical body of radius r moves in a fluid of viscosity η with a velocity v, it experiences a viscous drag, F which is given by** $F = 6\pi\eta vr$.

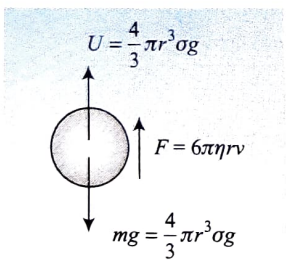
The Stokes' law is valid under the following assumptions:

- The medium through which the body moves is of infinite extent.
- The moving body is perfectly smooth and rigid.
- Though the size of the moving body is small, it is much larger than the distance between the molecules of the medium and as such the medium may be considered as continuous.
- There is no slip between the moving body and the medium.
- The velocity of the moving body is less than the critical velocity of the medium and as such the motion is streamlined.

TERMINAL VELOCITY

When a body is dropped in a viscous fluid, it is first accelerated and then its acceleration becomes zero and it attains a constant velocity called terminal velocity. Consider a small ball of radius r, which is gently dropped into the liquid of infinite extent.

As the body falls, the following three forces act on the body [see figure].



(i) Weight of the body acting vertically downwards. It is given by

$$mg = \frac{4}{3}\pi r^3\rho g$$

(ii) Upward thrust equal to the weight of the fluid displaced acting vertically upwards. It is given by

$$U = \frac{4}{3}\pi r^3\sigma g$$

Therefore, effective or apparent weight of the body acting vertically downwards,

$$mg' = mg - U = \frac{4}{3}\pi r^3\rho g - \frac{4}{3}\pi r^3\sigma g$$

Or $$mg' = \frac{4}{3}\pi r^3(\rho - \sigma)g$$

(iii) The viscous force acting vertically upwards (in a direction opposite to the motion of the body). The viscous force goes on increasing as the velocity of the falling body increases. A stage comes, when the weight of the body becomes just equal to the sum of the die upthrust and viscous force. At that stage, the body falls with constant maximum velocity, called the terminal velocity. If v is terminal velocity attained, then according to Stokes' law, the viscous force on the body is given by $F = 6\pi\eta rv$

In equilibrium, the effective weight mg' of the body is equal to the viscous force F acting on the body on attaining the terminal velocity [see figure] i.e.

$$\frac{4}{3}\pi r^3(\rho - \sigma)g = 6\pi\eta rv$$

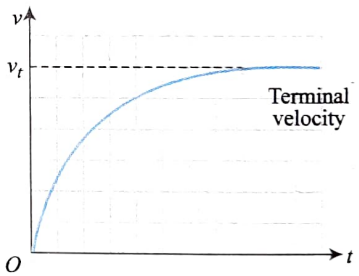
Or $$v = \frac{2}{9} \times \frac{r^2 \times (\rho - \sigma)g}{\eta} \quad \text{...(i)}$$

The following results follow from Eq. (i):

(i) The terminal velocity varies directly as the square of the radius of the body (i.e., $v_t \propto r^2$) and inversely as the coefficient of viscosity of the fluid, i.e., $v_t \propto 1/\eta$. It also depends upon the densities of the body and the fluid.

(ii) If the density of the body (ρ) is less than the density of the fluid (σ), the value of v_t is negative. Therefore, the body will move up with uniform velocity. It is on account of this that gas bubbles rise up through soda water.

(iii) By knowing the radius (r) of the body, densities (ρ, σ) of the body and the fluid and terminal velocity (v_t) [by noting the time taken by the body to cover a known distance in the fluid], the coefficient of viscosity of the fluid can be calculated.



If we plot the variation of velocity of the falling sphere with time, we obtain a graph as shown in the figure. The following points can be noted:

1. Initially the velocity of the ball increases at a fast rate.
2. The rate of increase of velocity decreases with time.
3. Finally, the rate of increase of velocity becomes zero and the sphere acquires a terminal velocity.

Important Point:

Air bubble in water always goes up. It is because density of air (ρ) is less than the density of water (σ). So the terminal velocity for air bubble is negative, which implies that the air bubble will go up. Positive terminal velocity means the body will fall down.

EXAMPLES OF TERMINAL VELOCITY

Some examples of terminal velocity are discussed as follows:

1. **Determining the electronic charge by Millikan's experiment:** Stokes' formula is used in Millikan's method for determining the electronic charge. In this method, the

formula is applied for finding out the radii of small oil drops by measuring their terminal velocities in air.

2. **Velocity of rain drops:** Rain drops are formed by the condensation of water vapour on dust particles. When they fall under gravity, their motion is opposed by the viscous drag in air. As the velocity of their fall increases, the viscous drag also increases and finally becomes equal to the effective force of gravity. The drops then attain a (constant) terminal velocity which is directly proportional to the square of the radius of the drops. In the beginning, the rain drops are very small in size and so they fall with such a small velocity that they appear floating in the sky as cloud. As they grow in size by further condensation, they reach the earth with appreciable velocity.
3. **Falling down with the help of a parachute:** When a person jumps down from a flying aeroplane, his parachute is closed. Hence, initially his velocity increases very rapidly while the viscosity of air tries to reduce his velocity. When the parachute opens, the viscosity of air exerts greater viscous force in upward direction (since viscous force is directly proportional to surface area), due to which the velocity of the person begins to decrease and finally he attains the terminal velocity and reaches safely on ground.
4. **Formation of clouds:** When water vapours present in the atmosphere condense, small droplets are formed. The weight of these droplets in air is very small. Therefore, they attain the terminal velocity very soon due to viscosity of air. Because the value of their terminal velocity is very small, they appear to float in the sky as clouds.

ILLUSTRATION 8.28

A powder comprising particles of various sizes is stirred up in a vessel filled to a height of 10 cm with water. Assuming the particles to be spherical, find the size of the largest particle that will remain in suspension after 1 h (density of powder = 4 g/cm^3, viscosity of water = 0.01 poise).

Sol. Terminal velocity of the largest particle which is just about to settle at the bottom of the vessel is

$$v_t = \frac{10\times10^{-2}}{3600}\ \text{m/s}$$

Let r be the radius of that particle. Then

$$v_t = \frac{2}{9}r^2\frac{(\sigma-\rho)}{\eta}g$$

where $\sigma = 4\times10^3$ kg/m^3 and $\eta = \frac{0.01}{10}$ Ns/m^3

After solving, we get $r = 2.0\times10^{-6}$ m.

ILLUSTRATION 8.29

Spherical particles of pollen are shaken up in water and allowed to settle. The depth of water is 2×10^{-2} m. What is the diameter of the largest particles remaining in suspension one hour later? Density of pollen = 1.8×10^3 kg m^{-3}, viscosity of water = 1×10^{-2} poise and density of water = 1×10^{-5} kg m^{-3}.

Sol. Terminal velocity, $v = \frac{2r^2(\rho-\sigma)g}{9\eta}$ ...(i)

But we know $v = \frac{s}{t}$

$$\therefore\quad \frac{s}{t} = \frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta} \quad\Rightarrow r^2 = \frac{9s}{2t}\frac{\eta}{(\rho-\sigma)g}$$

Given $s = 2\times10^{-2}$ m, $t = 1$ h $= 3600$ s

$$\therefore\quad \eta = 1\times10^{-2}\ \text{poise} = 1\times10^{-3}\ \text{kg m}^{-1}\text{s}^{-1}$$

Substituting given values, we get

$$r^2 = \frac{9}{2}\times\frac{2\times10^{-2}}{3600}\times\frac{1\times10^{-3}}{(1.8\times10^3-1\times10^3)\times10} = \frac{9}{36}\times\frac{1}{8}\times10^{-10}$$

$$= \frac{1}{32}\times10^{-10}$$

$$\therefore\quad r = \sqrt{\frac{100}{32}}\times10^{-6}\ \text{m} = 1.77\times10^{-6}\ \text{m}$$

Diameter $D = 2r = 2\times1.77\ \mu\text{m} = 3.54\ \mu\text{m}$

ILLUSTRATION 8.30

A small sphere falls from rest in a viscous liquid. Due to friction, heat is produced. Find the relation between the rate of production of heat and the radius of the sphere at terminal velocity.

Sol. Viscous force on a falling sphere in a liquid $F = \pi\eta r v_T$ where $v_T = (2/9)r^2(\rho-\sigma)g/\eta$ is terminal velocity, ρ = density of sphere, σ = density of liquid.

Rate of production of heat (= power) = $F\,v_T = 6\pi\eta r v_T^2$

$$= 6\pi\eta r\left[\frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}\right]^2 = \frac{8}{27}\frac{\pi g^2(\rho-\sigma)^2}{\eta}r^5$$

Clearly, $\frac{dQ}{dt} \propto r^5$

ILLUSTRATION 8.31

With what terminal velocity will an air bubble 0.8 mm in diameter rise in a liquid of viscosity 0.15 N m^{-2} s and specific gravity 0.9? Specific gravity of air = 0.001293. What is the terminal velocity of same bubble in water, if coefficient of viscosity of water is 10^{-3} N m^{-2} s?

Sol. Here, radius of the air bubble, $r = 0.8/2 = 0.4$ mm $= 4\times10^{-4}$ m; coefficient of viscosity of the liquid, $\eta = 0.15$ N m^{-2} s; and specific gravity of the liquid = 0.9.

Therefore, density of the liquid (medium) = 0.9 g cm^{-3} = 0.9×10^3 kg m^{-3} = 900 kg m^{-3}. Specific gravity of air = 0.001293.

Therefore, density of air (spherical object),

$$\rho = 0.001293\times10^3\ \text{kg m}^{-3} = 1.293\ \text{kg m}^{-3}$$

Now, terminal velocity of the air bubble,

$$v = \frac{2r^2(\rho-\sigma)g}{9\eta} = \frac{2\times(4\times10^{-4})^2\times(1.293-900)\times9.8}{9\times0.15}$$

$$= -2.09\times10^{-3}\ \text{m s}^{-1}$$

The negative sign indicates that the air bubble will rise up.

Terminal velocity of air bubble in water:

Density of water, $\sigma = 1000\ \text{kg m}^{-3}$;

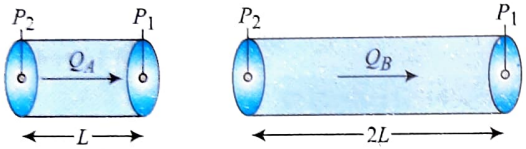
Coefficient of viscosity of water, $\eta = 10^{-3}\ \text{N m}^{-2}\ \text{s}$

Now, $v = \dfrac{2r^2(\rho-\sigma)g}{9\eta} = \dfrac{2\times(4\times10^{-4})^2(1.293-1000)\times9.8}{9\times10^{-3}}$

$= -0.348\ \text{m s}^{-1}$

CONCEPT APPLICATION EXERCISE 8.5

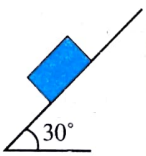
1. A viscous fluid is flowing through two horizontal pipes. The pressure difference $P_2 - P_1$ between the ends of each pipe is the same. The pipes have the same radius, although one is twice as long as the other. How does the volume flow rate Q_B in the longer pipe compare with the rate Q_A in the shorter pipe?



   (a) Q_B is the same as Q_A.
   (b) Q_B is twice as large as Q_A.
   (c) Q_B is four times as large as Q_A.
   (d) Q_B is one-half as large as Q_A.

2. A man is rowing a boat with a constant velocity 'v_0' in a river. The contact area of boat is 'A' and coefficient of viscosity is η. The depth of river is 'D'. Find the force required to row the boat.

3. A cubical block (of side 2 m) of mass 20 kg slides on inclined plane lubricated with the oil of viscosity $\eta = 10^{-1}$ poise with constant velocity of 10 m s^{-1}. Find out the thickness of the layer of liquid (take $g = 10\ \text{m s}^{-2}$).



4. A drop of water of radius 0.0015 mm is falling in air. If the coefficient of viscosity of air is 1.8×10^{-5} kg/(m s), what will be the terminal velocity of the drop? Density of water = $1.0 \times 10^3\ \text{kg/m}^2$ and $g = 9.8$ N/kg. Density of air can be neglected.

5. A metallic sphere of radius 1.0×10^{-3} m and density $1.0 \times 10^4\ \text{kg/m}^3$ enters a tank of water, after a free fall through a distance of h in the earth's gravitational field. If its velocity remains unchanged after entering water, determine the value of h. Given: coefficient of viscosity of water = $1.0 \times 10^{-3}\ \text{N s/m}^2$, $g = 10\ \text{m s}^{-2}$ and density of water = $1.0 \times 10^3\ \text{kg/m}^3$.

6. Find the minimum force required to drag a hard polythene plate of area 2 m^2 on a thin film of oil of thickness 0.25 cm and $\eta = 15$ poise. Assume the speed of the plate is 10 cm s^{-1}.

7. A force of 3.14 N is required to drag a sphere of radius 4 cm with a speed of 5 m s^{-1} in a medium in gravity free space. Find the coefficient of the viscosity of the medium.

8. A square plate of 10 cm side moves parallel to another plate with a velocity of 10 cm s^{-1}, both plates immersed in water. If the viscous force is 200 dyne and viscosity of water is 0.01 poise, what is their distance apart?

9. In a hospital, a patient receives a 500 cm^3 blood transfusion through a needle with a length of 5 cm and inner radius of 0.03 cm. If the blood bag is kept 85 cm above the needle, how long the transfusion takes place? Given that the viscosity of blood is 0.017 poise and the density of blood is 1.02 g cm^{-3}.

ANSWERS

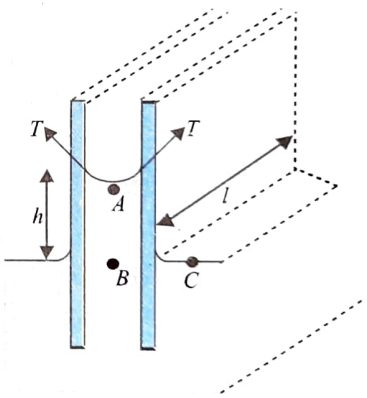
1. (d)	3. 4 mm	4. 2.72×10^{-4} m/s	5. 20 m
6. 1.2×10^7 dyne	7. 8.3×10^{-5} poise	8. 0.05 cm	
9. 1529 s			

Solved Examples

EXAMPLE 8.1

Derive an expression for the height of capillary rise between two parallel plates dipping in a liquid of density σ separated by a distance d. The surface tension of the liquid is T.

Sol.



The meniscus between two plates is cylindrical in shape. Pressure at A (the lowest point of the meniscus)

$p_A = p_0 - T/r$

Pressure at B = pressure at $C = p_0$ = pressure at $A + \rho gh$

$\therefore \quad p_0 = p_0 - \dfrac{T}{r} + \rho gh \quad \times h = \dfrac{T}{\rho gr} = \dfrac{2T}{\rho gd}$

Alternative method:

Force upward = $2lT\cos\theta = 2lT \quad (\because \theta = 0°)$

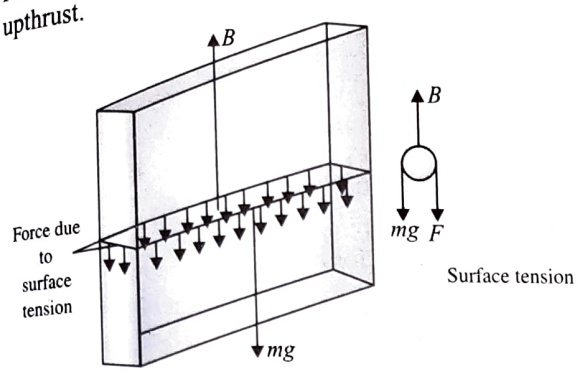
Gravitational pull = (Volume × Density)g = $lhd\rho g$

$\therefore \quad 2lT = lhd\rho g \quad \Rightarrow h = \dfrac{2T}{d}$

EXAMPLE 8.2

A glass plate of length 10 cm, breadth 1.54 cm and thickness 0.20 cm weighs 8.2 g in air. It is held vertically with the long side horizontal and the lower half under water. Find the apparent weight of the plate. Surface tension of water = 7.3×10^{-2} N/m and $g = 9.8\ \text{m s}^{-2}$.

Sol. The forces acting on the plate are
(a) Weight of plate (mg) acting vertically downward.
(b) Force of surface tension (TL) acting vertically downward.
(c) upthrust.



Weight of glass plate = 8.2 g = 8.2×10^{-3} kg

Total length of plate in contact in with water surface is

$$l = 2(10 + 0.2)\text{cm} = 20.4 \times 10^{-2} \text{ m}$$

Therefore, downward pull upon the plate due to surface tension

$$TL = 7. \times 10^{-2} \times 20.4 \times 10^{-2} = 14.892 \times 10^{-3} \text{ N}$$

$$= \frac{14.892 \times 10^{-3}}{9.8} = 1.5196 \times 10^{-3} \text{ kg wt}$$

Volume of plate immersed in water is

$$v = 10 \times \frac{1}{2}(1.54) \times 0.2 = 1.54.10^{-6} \text{ m}^3$$

Therefore, upthrust = $v\sigma g$ = 1.54 × 10^{-6} × 10^3 × 9.8 N

$$= \frac{1.54 \times 10^{-3} \times 9.8}{9.8} \text{ kg wt} = 1.54 \times 10^{-3} \text{ kg wt}$$

Therefore, apparent weight of plate in water = Weight of plate in air + Force of surface tension – Upthrust

$$= 8.2 \times 10^{-3} + 1.5196 \times 10^{-3} - 1.54 \times 10^{-3}$$

$$= 8.1796 \times 10^{-6} \text{ kg wt} = 8.1796 \text{ kg wt}$$

EXAMPLE 8.3

A barometer contains two uniform capillaries of radii 1.44×10^{-3} m and 7.2×10^{-4} m. If the height of liquid in narrow tube is 0.2 m more than that in wide tube, calculate the true pressure difference. Density of liquid = 10^3 kg/m^3, surface tension = 72×10^{-3} N/m and $g = 9.8$ m s^{-2}.

Sol. Let the pressures in wide and narrow limbs be P_1 and P_2, respectively. If R_1 and R_2 be the radii of meniscus in wide and narrow limb pressure just below the meniscus of wide tube

$$= P_1 - \frac{2T}{R_1}$$

and pressure just below the meniscus of narrow limb $= P_2 - \frac{2T}{R_2}$.

Therefore, difference of these pressures

$$\left(P_1 - \frac{2T}{R_1}\right) - \left(P_2 - \frac{2T}{R_2}\right) = h\rho g$$

Therefore, true pressure difference,

$$P_1 - P_2 = h\rho g - 2T\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

For the water and glass surface, taking the angle of contact θ to be zero, we have $R_1 = \frac{r_1}{\cos\theta} \approx r_1$ and $R_2 = \frac{r_2}{\cos\theta} \approx r_2$ where r_1 and r_2 are radii of wide and narrow limbs, respectively.

$$\therefore \quad P_1 - P_2 = h\rho g - 2T\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$= 0.2 \times 10^3 \times 9.8 - 2 \times 72 \times 10^{-3} \times \left(\frac{1}{7.2 \times 10^{-4}} - \frac{1}{1.44 \times 10^{-3}}\right)$$

$$= 1.96 \times 10^3 - 0.10 \times 10^3 = 1.86 \times 10^3 \text{ N/m}^2 = 1860 \text{ N/m}^2$$

EXAMPLE 8.4

An open capillary tube contains a drop of water. When the tube is in its vertical position, the drop forms a column with a length of **(a)** 2 cm, **(b)** 4 cm and **(c)** 2.98 cm. The internal diameter of the capillary tube is 1 mm. Determine the radii of curvature of the upper and lower meniscuses in each case. Consider the wetting to be complete. Surface tension of water = 0.0075 N/m.

Sol. When capillary is in vertical position, the upper meniscus is concave and pressure due to surface tension

$$p_1 = \frac{2T}{R_1}$$

is directed vertically upward, R_1 being radius of curvature of upper meniscus. When wetting is complete

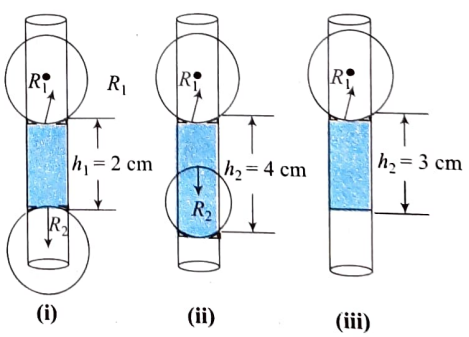
$$p_1 = \frac{2T}{r}$$

r being radius of capillary tube $= \frac{D}{2} = \frac{1.0}{2}$ mm

$R_1 = r = 0.5$ mm in each case.

We have $p_1 = \frac{2T}{r} = \frac{2 \times 75 \times 10^{-3}}{0.5 \times 10^{-3}} = 300 \text{ N/m}^2$

The hydrostatic pressure $p_2 = h\rho g$ is always directed downwards.



(a) If h = 2 cm [see Fig. (i)]

$$p_2 = h\rho g = 2 \times 10^{-2} \times 10 \times 10^3 = 200 \text{ N/m}^2$$

Clearly $p_1 > p_2$. That is resulting pressure is directed upward. For equilibrium, the pressure due to lower meniscus should be downwards. This makes the lower meniscus concave downwards. Then

$$p_1 - p_2 = \frac{2T}{R_2}$$

or $300-200=\dfrac{2\times75\times10^{-3}}{R_2}$

$$R_2=\frac{2\times75\times10^{-3}}{100}=1.50\times10^{-3}\,\text{m}=1.50\ \text{mm}$$

(b) If $h = 4$ cm [see Fig. (ii)]

$$p_2=h\rho g=4\times10^{-2}\times10^{3}\times10=392\,\text{N/m}^2$$

In this case, $p_1 < p_2$, hence the pressure due to lower meniscus should be directed upwards. For this meniscus should be concave upwards.

Then $p_2-p_1=\dfrac{2T}{R_2}$

$$400-300=\frac{2\times75\times10^{-3}}{R_2}$$

$$R_2=\frac{2\times75\times10^{-3}}{100}=1.50\times10^{-3}\,\text{m}=1.50\ \text{mm}$$

(c) If $h = 3.00$ cm [see Fig. (iii)]

$$p_2=h\rho g=3.00\times10^{-2}\times10^{3}\times10=300\ \text{N/m}^2$$

In this case $p_1=p_2$

$$p_1-p_2=\frac{2T}{R_2}$$

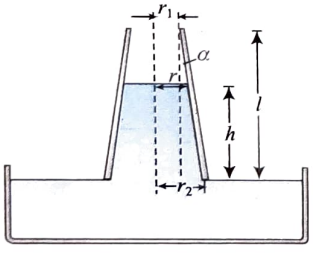
or $\dfrac{2T}{R_2}=0$ or $R_2=\infty$

i.e., lower surface will be flat.

EXAMPLE 8.5

A tube of conical bore of length 10 cm is just dipped inside the water. The diameters of upper and lower ends are 0.04 cm and 0.06 cm respectively. Find the height to which the liquid rises in the tube. Given surface tension of water is 70 dyne cm^{-1} and angle of contact 0°.

Sol. Figure shows a conical tube of length l diped inside water.



Let r_1 and r_2 be the radii of upper and lower ends of the tube. Suppose that water rises to a height h in the tube. Let r be the radius of the tube at the meniscus of the water.

Now, $h=\dfrac{2T\cos\theta}{r\rho g}$

Here, $T = 70$ dyne cm^{-1} $= 0°$; $\rho = 1$ g cm^{-3}; $g = 980$ cm s^{-1}

$$h=\frac{2\times70\times\cos0°}{r\times1\times980}\ \text{or}\ h=\frac{1}{7r}\ \text{or}\ r=\frac{1}{7}h$$

Suppose that the wall of the conical tube makes an angle α with the vertical. Then,

$$\tan\alpha=\frac{r_2-r_1}{l}=\frac{r-r_1}{l-h}$$

Here, $r_1=\dfrac{0.04}{2}=0.02$ cm;

$$r_2=\frac{0.06}{2}=0.03\ \text{cm};\ l=10\ \text{cm}$$

$$\frac{0.03-0.02}{10}=\frac{r-0.02}{10-h}$$

or $(10-h)\times0.001=r-0.02$

or $0.001h+r-0.03=0$ ...(i)

In Eq. (i), substituting for r, we have

$$0.001h+\frac{1}{7h}-0.03=0$$

or $7h^2-210h+1000=0$

or $h=\dfrac{210\pm\sqrt{(-210)^2-4\times7\times1000}}{14}$

$$=\frac{210\pm126.9}{14}=24.06\ \text{or}\ 5.94\ \text{cm}$$

The water cannot rise up to 24.06 cm. It can be checked that in a tube of uniform bore r_1 ($< r$), water can rise up to a height of 7.35 cm only. Hence, in the conical tube, water will rise up to a height of 5.94 cm.

EXAMPLE 8.6

The lower end of a capillary tube of diameter 2 mm is dipped 8 cm below the surface of water in a beaker. What is the pressure required in the tube in order to blow a hemispherical bubble at its end in the water. The surface tension of water is 7.3×10^{-2} N m^{-1}, density of water $= 10^3$ kg m^{-3}, $g = 9.8$ m s^{-2} and 1 atmospheric pressure $= 1.013\times10^5$ N m^{-2}. Also calculate the excess of pressure inside the bubble.

Sol. Here, $T = 7.3\times10^{-2}$ N m^{-1}; $\rho = 10^3$ kg m^{-3};

$$r=\frac{2}{2}=1\ \text{mm}=10^{-3}\ \text{m}$$

1 atmospheric pressure, $p = 1.013\times10^5$ Nm^{-2}; $g = 9.8$ ms^{-2};

Length of the tube inside water, $h = 8$ cm $= 8\times10^{-2}$ m

Let p_i be the pressure inside the hemispherical air bubble blown at the lower end of the tube and p_0, just outside it. Then,

$$p_i-p_0=\frac{2T}{r}\ \text{or}\ p_i=p_0+\frac{2T}{r}$$

Now, pressure outside the hemispherical air bubble,

p_0 = atmospheric pressure + pressure exerted by the water column

or $p_0=p+h\rho g$

The pressure required to blow the bubble is equal to pressure inside the bubble. Therefore, pressure required to blow the bubble,

$$p_i=p_0+\frac{2T}{r}=p+h\rho g+\frac{2T}{r}$$

$$=1.013\times10^5+8\times10^{-2}\times10^3\times9.8+\frac{2\times7.3\times10^{-2}}{10^{-3}}$$

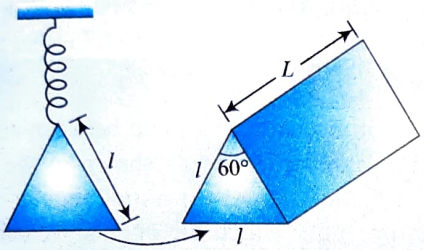
$$=1.013\times10^5+7.84\times10^2+1.46\times10^2=1.0223\times10^5\ \text{Nm}^{-2}$$

The excess of pressure,

$$p_i - p_0 = \frac{2T}{r} = \frac{2\times 7.3\times 10^{-2}}{10^{-3}} = 1.46\times 10^{2}\ \text{Nm}^{-2}$$

EXAMPLE 8.7

A glass prism has its principal section in the form of an equilateral triangle of side length l. The length of the prism is L (see figure). The prism, with its base horizontal, is supported by a vertical spring of force constant k. Half the slant surface of the prism is submerged in water. Surface tension of water is T and contact angle between water and glass is 0°. Density of glass is d and that of water is ρ ($<d$). Calculate the extension in the spring in this position of equilibrium.



Sol. Volume of the prism, $V = \left(\frac{1}{2}l\times\frac{\sqrt{3}}{2}l\right)\times L = \frac{\sqrt{3}}{4}l^2L$

The volume of the prism submerged in water,

$$V' = \frac{\sqrt{3}}{4}l^2L - \frac{\sqrt{3}}{4}\left(\frac{l}{2}\right)^2 L = \frac{3\sqrt{3}}{16}l^2L$$

The forces acting on prism are as follows:

(i) Weight $W = \frac{\sqrt{3}}{4}l^2Ldg$

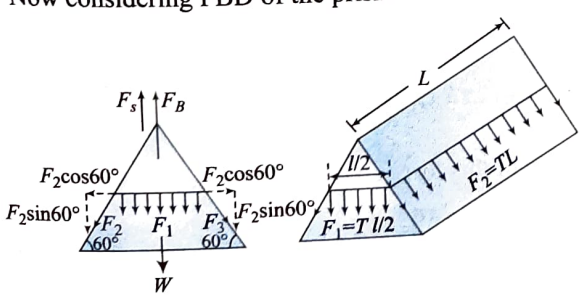
(ii) Buoyancy force $F_B = \left(\frac{3\sqrt{3}}{16}l^2L\right)\rho g$

(iii) Surface tension force

$$F_{ST} = 2(F_1 + F_2) = \left(T\frac{l}{2} + TL\sin 60°\right)$$

(iv) Spring force (F_s) $F_s = kx$

Now considering FBD of the prism



In equilibrium,

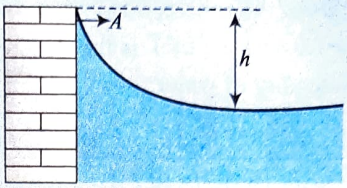
Spring force (F_s) + Buoyancy (F_B) = Weight (W) + Surface tension force (F_{ST})

$$kx + \frac{3\sqrt{3}}{16}l^2L\rho g = \frac{\sqrt{3}}{4}l^2Ldg + 2TL\cos 30° + 2T\frac{l}{2}$$

$$x = \frac{1}{k}\left[\frac{\sqrt{3}}{4}l^2Ldg - \frac{3\sqrt{3}}{16}l^2L\rho g + \sqrt{3}TL + Tl\right]$$

EXAMPLE 8.8

A liquid having surface tension T and density ρ is in contact with a vertical solid wall. The liquid surface gets curved as shown in the figure. At the bottom the liquid surface is flat.



The atmospheric pressure is P_0.

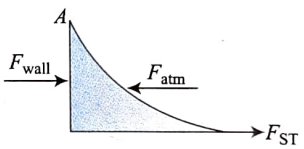
(i) Find the pressure in the liquid at the top of the meniscus (i.e. at A)

(ii) Calculate the difference in height (h) between the bottom and top of the meniscus.

Sol.

(i) Pressure at the lowest level of meniscus is P_0 [$\because$ liquid surface is flat].
Pressure in liquid at A is $P_A = P_0 - \rho gh$

(ii) We will consider the horizontal equilibrium of the liquid in the meniscus. We will consider a depth L perpendicular to the plane of the figure.



Force due to wall,

$$F_{Wall} = p_{av}hL$$

$$= \left(\frac{P_0 + P_0 - \rho gh}{2}\right)hL = \left(P_0 - \frac{\rho gh}{2}\right)hL$$

Force due to atmosphere $F_{atm} = P_0hL$

Force due to surface tension $F_{ST} = T\cdot L$

$$F_{Wall} + F_{ST} = F_{atm}$$

$$\left(P_0 - \frac{\rho gh}{2}\right)hL + T\cdot L = P\cdot hL$$

$$\Rightarrow P_0h - \frac{\rho gh^2}{2} + T = P\cdot h$$

$$\therefore\ h = \sqrt{\frac{2T}{\rho g}}$$

EXAMPLE 8.9

A metallic sphere of radius 1.0×10^{-3} m and density 1.0×10^{4} kg/m^3 enters a tank of water, after a free fall through a distance of h in the earth's gravitational field. If its velocity remains unchanged after entering water, determine the value of h. Given: coefficient of viscosity of water = 1.0×10^{-3} N-s/m^2, $g = 10$ m/s^2 and density of water = 1.0×10^{3} kg/m^3.

Sol. The velocity attained by the sphere in falling freely from a height h is

$$v = \sqrt{2gh} \qquad ...(i)$$

This is the terminal velocity of the sphere in water. Hence by Stokes' law, we have

$$v = \frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}$$

where r is the radius of the sphere, ρ is the density of the material of the sphere $\sigma(=1.0\times10^3\ \text{kg/m}^3)$ is the density of water and η is coefficient of viscosity of water.

$$\therefore \quad v = \frac{2\times(1.0\times10^{-3})^2(1.0\times10^4-1.0\times10^3)\times10}{9\times1.0\times10^{-3}} = 20\ \text{m/s}$$

From Eq. (i), we have

$$h = \frac{v^2}{2g} = \frac{20\times20}{2\times10} = 20\ \text{m}$$

EXAMPLE 8.10

A sphere is dropped under gravity through a fluid of viscosity η. Taking the average acceleration as half of the initial acceleration, show that the time taken to attain the terminal velocity is independent of the fluid density.

Sol. Here, coefficient of viscosity of the fluid = η; density of the fluid = σ; density of the material of sphere = ρ. Let r be the radius of the sphere.

Now, mass of the sphere, $m = \frac{4}{3}\pi r^3\rho$

and mass of the fluid displaced, $m' = \frac{4}{3}\pi r^3\sigma$

Net downward force on the sphere,

F = weight of the sphere – weight of the fluid displaced

$$= mg - m'g = \frac{4}{3}\pi r^3\rho g - \frac{4}{3}\pi r^3\sigma g$$

or $$F = \frac{4}{3}\pi r^3(\rho-\sigma)g \qquad ...(i)$$

Suppose that the sphere falls with an initial acceleration a. Then, according to Newton's second law of motion,

$$F = ma = \left(\frac{4}{3}\pi r^3\rho\right)a \qquad ...(ii)$$

From Eqs. (i) and (ii), we have

or $$\frac{4}{3}\pi r^3(\rho-\sigma)g = \left(\frac{4}{3}\pi r^3\rho\right)a$$

or $$a = \left(\frac{\rho-\sigma}{\rho}\right)g$$

Now, terminal velocity of the ball,

$$v = \frac{2r^2(\rho-\sigma)g}{9\eta}$$

Suppose that the average acceleration of the sphere during the fall is a', till the drop acquires terminal velocity. Then, using the relation: $v = u + at$, we have

$$\frac{2r^2(\rho-\sigma)g}{9\eta} = 0 + a't = a't$$

But $$a' = \frac{a}{2} = \frac{(\rho-\sigma)g}{2\rho}$$

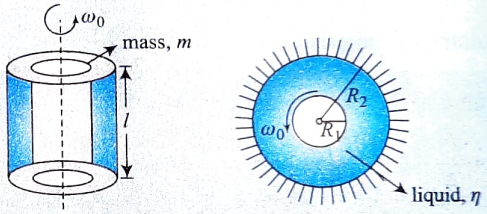
$$\frac{2r^2(\rho-\sigma)g}{9\eta} = \left(\frac{\rho-\sigma}{2\rho}\right)gt$$

or $$t = \frac{4r^2\rho}{9\eta}$$

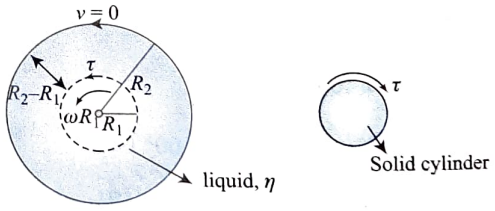
Hence, the time taken to acquire the terminal velocity is independent of the fluid-density.

EXAMPLE 8.11

Viscous liquid of viscosity η is filled between an outer fixed cylinder and an inner cylinder as shown in the figure. The central solid cylinder starts with initial angular velocity ω_0. Find out the time after which the angular velocity becomes half.



Sol. The velocity of the liquid layer in contact with outer fixed cylinder will be zero while the speed of the liquid layer attached with rotating cylinder will be ωR_1. The viscose force developed



$$|F| = \eta A\frac{dv}{dz}, \text{ where } \frac{dv}{dz} = \frac{\omega R_1 - 0}{R_2 - R_1}$$

$$F = \eta(2\pi R_1 l)\left(\frac{\omega R_1}{R_2 - R_1}\right)$$

The torque of this force about central axes,

$$\tau = FR_1 = \frac{2\pi\eta R_1^3\omega l}{R_2 - R_1}$$

$$I\alpha = \frac{2\pi\eta R_1^3\omega l}{R_2 - R_1} \Rightarrow \frac{MR_1^2}{2}\left(-\frac{d\omega}{dt}\right) = \frac{2\pi\eta R_1^3\omega l}{R_2 - R_1}$$

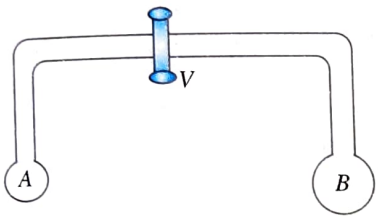
$$-\int_{\omega_0}^{\omega_0/2}\frac{d\omega}{\omega} = \frac{4\pi\eta R_1 l}{m(R_2 - R_1)}\int_0^t dt \Rightarrow t = \frac{m(R_2 - R_1)\ln 2}{4\pi\eta l R_1}$$

Exercises

Single Correct Answer Type

1. A water drop is divided into eight equal droplets. The pressure difference between inner and outer sides of the big drop
(1) will be the same as for smaller droplet
(2) will be half of that for smaller droplet
(3) will be one-fourth of that for smaller droplet
(4) will be twice of that for smaller droplet

2. The value V in the bent tube is initially kept closed. Two soap bubbles A (smaller) and B (larger) are formed at the two open ends of the tube. V is now opened and air can flow freely between the bubbles.



(1) There will be change in the size of the bubbles
(2) The bubbles will become of equal size
(3) A will become smaller and B will become larger
(4) The sizes of A and B will be interchanged

3. Two soap bubbles of different radii are in communication with each other. Then
(1) air follows from the larger bubble into smaller bubble till both bubbles acquire same size
(2) air follows from the smaller bubble into larger bubble and the larger bubble grows in size with decrease in size of the smaller bubble
(3) air does not flow but the sizes of the bubbles changes
(4) sizes of the bubbles remain unchanged

4. A paper disc of radius R from which a hole of radius r is cut out is floating in a liquid of the surface tension S. The force on the disc due to the surface tension is
(1) $S \times 2\pi R$ (2) $S \times 2\pi r$
(3) $S \times 2\pi (R - r)$ (4) $S \times 2\pi (R + r)$

5. If T is surface tension of soap solution, the amount of work done in blowing a soap bubble from a diameter D to a diameter $2D$ is
(1) $2\pi D^2T$ (2) $4\pi D^2T$
(3) $6\pi D^2T$ (4) $8\pi D^2T$

6. Work W is required to form a bubble of volume V from a given solution. What amount of work is required to be done to form a bubble of volume $2V$?
(1) W (2) $2W$
(3) $2^{1/3}W$ (4) $4^{1/3}W$

7. The surface energy of a liquid drop is E. It is sprayed into 1000 equal droplets. Then its surface energy becomes
(1) $1000E$ (2) $100E$
(3) $10E$ (4) E

8. In the previous question, the work done in spraying is
(1) $999E$ (2) $99E$
(3) $9E$ (4) E

9. A small metal ball of diameter 4 mm and density 10.5 g/cm^3 in dropped in glycerine of density 1.5 g/cm^3. The ball attains a terminal velocity of 8 cm s^{-1}. The coefficient of viscosity of glycerine is
(1) 4.9 poise (2) 9.8 poise
(3) 98 poise (4) 980 poise

10. A sphere of brass released in a long liquid column attains a terminal speed v_o. If the terminal speed is attained by a sphere of marble of the same radius and released in the same liquid is $n\ v_o$, then the value of n will be (Given: The specific gravities of brass, marble and liquid are 8.5, 2.5 and 0.8, respectively)
(1) $\frac{5}{17}$ (2) $\frac{17}{77}$
(3) $\frac{11}{31}$ (4) $\frac{17}{5}$

11. Between a plate of area 100 cm^2 and another plate of area 100 m^2 there is a 1 mm, thick layer of water, if the coefficient of viscosity of water is 0.01 poise, then the force required to move the smaller plate with a velocity 10 cm s^{-1} with reference to large plate is
(1) 100 dyn (2) 10^4 dyn
(3) 10^6 dyn (4) 10^9 dyn

12. A river 10 m deep is flowing at 5 m s^{-1}. The shearing stress between horizontal layers of the river is ($\eta = 10^{-3}$ SI units)
(1) 10^{-3} N/m^2 (2) 0.8×10^{-3} N/m^2
(3) 0.5×10^{-3} N/m^2 (4) 1 N/m^2

13. A 20 cm long capillary tube is dipped in water. The water rises up to 8 cm. If the entire arrangement is put in a freely falling elevator, the length of water column in the capillary tube will be
(1) 20 cm (2) 4 cm
(3) 10 cm (4) 8 cm

14. A glass rod of diameter $d = 2$ mm is inserted symmetrically into a glass capillary tube of radius $r = 2$ mm. Then the whole arrangement is vertically dipped into liquid having surface tension 0.072 N/m. The height to which liquid will rise on capillary is (Take $g = 10$ m s^{-2}, $\rho_{liq} = 1000$ kg/m^3. Assume contact angle to be zero of capillary tube to be long enough)
(1) 1.44 cm (2) 6 cm
(3) 4.86 cm (4) None of these

15. Two soap bubbles, one of radius 50 mm and the other of radius 80 mm, are brought in contact so that they have a common interface. The radius of the curvature of the common interface is
(1) 0.003 m (2) 0.133 m
(3) 1.2 m (4) 8.9 m

16. A straw 6 cm long floats on water. The water film on one side has surface tension of 50 dyn/cm. On the other slide, camphor reduces the surface tension to 40 dyn/cm. The resultant force acting on the straw is

(1) $(50 \times 6 - 40 \times 6)$ dyn (2) 10 dyn

(3) $\left(\frac{50}{6} - \frac{40}{6}\right)$ dyn (4) 90 dyn

17. Two glass plates are separated by water. If surface tension of water is 75 dyn/cm and the area of each plate wetted by water is 8 cm^2 and the distance between the plates is 0.12 mm, then the force applied to separate the two plates is

(1) 10^2 dyn (2) 10^4 dyn

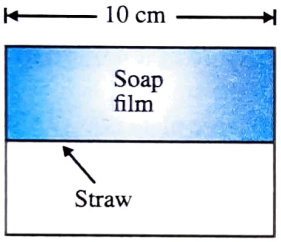
(3) 10^5 dyn (4) 10^6 dyn

18. A ring is cut from a platinum tube 8.5 cm internal and 8.7 cm external diameter. It is supported horizontally from the pan of a balance, so that it comes in contact with the water in a glass vessel. If an extra 3.103 gf is required to pull it away from water, the surface tension of water is

(1) 72 dyn/cm (2) 70.80 dyn/cm

(3) 63.35 dyn/cm (4) 60 dyn/cm

19. A soap film of surface tension 3×10^{-2} formed in a rectangular frame can support a straw as shown in figure. If $g = 10\ m\,s^{-2}$, the mass of the straw is



(1) 0.006 g (2) 0.06 g

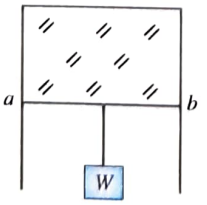
(3) 0.6 g (4) 6 g

20. The lower end of a capillary tube is at a depth of 12 cm and water rises 3 cm in it. The mouth pressure required to blow an air bubble at the lower end will be x cm of water column, where x is

(1) 12 (2) 15

(3) 3 (4) 9

21. A film of soap solution is trapped between a vertical frame and a light wire ab of length 0.1 m. If $g = 10\ m\,s^{-2}$, then the load W that should be suspended from the wire to keep it in equilibrium is



(1) 0.2 g (2) 0.3 g

(3) 0.4 g (4) 0.5 g

22. The angle of contact between glass and water is 0° and water (surface tension 70 dyn/cm) rises in a glass capillary up to 6 cm. Another liquid of surface tension 140 dyn/cm, angle of contact 60° and relative density 2 will rise in the same capillary up to

(1) 12 cm (2) 24 cm

(3) 3 cm (4) 6 cm

23. A hollow sphere has a small hole in it. On lowering the sphere in a tank of water, it is observed that water enters into the hollow sphere at a depth of 40 cm below the surface. Surface tension of water is 7×10^{-2} N/m. The diameter of the hole is

(1) $\frac{1}{28}$ mm (2) $\frac{1}{21}$ mm

(3) $\frac{1}{14}$ mm (4) $\frac{1}{7}$ mm

24. A capillary tube is attached horizontally to a constant pressure head arrangement. If the radius of the capillary tube is increased by 10%, then the rate of flow of the liquid shall change nearly by

(1) +10% (2) 46%

(3) –10% (4) –40%

25. A ball rises to the surface of a liquid with constant velocity. The density of the liquid is four time the density of the material of the ball. The frictional force of the liquid on the rising ball is greater than the weight of the ball by a factor of

(1) 2 (2) 3

(3) 4 (4) 6

26. A spherical ball falls through viscous medium with terminal velocity v. If this ball is replaced by another ball of the same mass but half the radius, then the terminal velocity will be (neglect the effect of buoyancy.)

(1) v (2) $2v$

(3) $4v$ (4) $8v$

27. Two parallel wires each of length 10 cm are 0.5 cm apart. A film of water is formed between them. If surface tension of water is 0.072 N/m, then the work done in increasing the distance between the wires by 1 mm is

(1) 1.44×10^{-5} J (2) 1.72×10^{-5} J

(3) 1.44×10^{-4} J (4) 1.72×10^{-4} J

28. The length of a needle floating on water is 2.5 cm. The minimum force in addition to its weight needed to lift the needle above the surface of water will be (surface tension of water is 0.072 N/m)

(1) 3.6×10^{-3} N (2) 10^{-2} N

(3) 9×10^{-4} N (4) 6×10^{-4} N

29. A solid sphere falls with a terminal velocity of 20 $m\,s^{-1}$ in air. If it is allowed to fall in vacuum,

(1) terminal velocity will be 20 $m\,s^{-1}$

(2) terminal velocity will be less than 20 $m\,s^{-1}$

(3) terminal velocity will be greater than 20 $m\,s^{-1}$

(4) no terminal velocity will be attained

30. A marble of mass x and diameter $2r$ is gently released in a tall cylinder containing honey. If the marble displaces mass y ($< x$) of the liquid, then the terminal velocity is proportional to

(1) $x + y$ (2) $x - y$

(3) $\frac{x+y}{r}$ (4) $\frac{x-y}{r}$

31. Water rises to a height h in a capillary tube of cross-sectional area A. The height to which water will rise in a capillary tube of cross-sectional area $4A$ will be

(1) h (2) $h/2$

(3) $h/4$ (4) $4h$

32. Neglecting the density of air, the terminal velocity obtained by a raindrop of radius 0.3 mm falling through the air of viscosity 1.8×10^{-5} N/m^2 will be

(1) 10.9 m/s (2) 8.3 m/s
(3) 9.2 m/s (4) 7.6 m/s

33. Water rises to a height of 2 cm in a capillary tube. If the tube is tilted 60° from the vertical, water will rise in the tube to a length of
(1) 4.0 cm (2) 2.0 cm
(3) 1.0 cm (4) water will not rise at all

34. A vessel, whose bottom has round holes with diameter 0.1 mm, is filled with water. The maximum height up to which water can be filled without leakage is
(1) 100 cm (2) 75 cm (3) 50 cm (4) 30 cm

35. Water rises to a height of 10 cm in a capillary tube and mercury falls to a depth of 3.42 cm in the same capillary tube. If the density of mercury is 13.6 g/c.c. and the angles of contact for mercury and water are 135° and 0°, respectively, the ratio of surface tension for water and mercury is
(1) 20 : 3 (2) 1 : 3
(3) 2 : 13 (4) 3 : 2

36. The velocity of small ball of mass M and density d_1 = when dropped a container filled with glycerine becomes constant after some time. If the density glycerine is d_2, the viscous force acting on ball is
(1) $\frac{Md_1 g}{d_2}$ (2) $Mg\left(1-\frac{d_2}{d_1}\right)$
(3) $\frac{M(d_1+d_2)}{g}$ (4) Md_1d_2

37. A cube with a mass = 20 g wettable water floats on the surface of water. Each face of the cube is $\alpha = 3$ cm long. Surface tension of water is 70 dyn/cm. The distance of the lower face of the cube from the surface of water is ($g = 980$ cm s^{-2})
(1) 2.3 cm (2) 4.6 cm
(3) 9.7 cm (4) 12.7 cm

38. Consider vertical parallel of semi-circular cross section dipped in a liquid. Assume that the wetting of the tube is complete. The force of surface tension on the flat part and on curved part of the tube are in the ratio
(1) 2 : π (2) 1 : π
(3) 3 : π (4) 4 : π

39. Two vertical parallel glass plates are partially submerged in water. The distance between the plates is d and the length is l. Assume that the water between the plates does not reach the upper edges of the plates and that the wetting is complete. The water will rise to height (ρ = density of water and σ = surface tension of water)
(1) $\frac{2\sigma}{\rho gd}$ (2) $\frac{\sigma}{2\rho gd}$
(3) $\frac{4\sigma}{\rho gd}$ (4) $\frac{5\sigma}{\rho gd}$

40. In the previous question, the force of attraction between the plates is
(1) $\frac{l\rho g}{2}h^2$ (2) $\frac{l\rho g}{2}h$
(3) $\frac{l\rho g}{2}$ (4) $\frac{l\rho g}{2h}$
(Here h represents the height to which the liquid rises.)

41. In Q.39, if water is replaced by another liquid of surface tension 2σ, then the force F of attraction between two plates would become
(1) $2F$ (2) $4F$
(3) $8F$ (4) $16F$

42. A number of droplets, each of radius r, combine to form a drop of radius R. If T is the surface tension, the rise in temperature will be
(1) $\frac{2T}{r}$ (2) $\frac{3T}{R}$
(3) $2T\left[\frac{1}{r}-\frac{1}{R}\right]$ (4) $3T\left[\frac{1}{r}-\frac{1}{R}\right]$

43. A drop of liquid of density ρ is floating half-immersed in a liquid of density d. If ρ is the surface tension the diameter of the drop of the liquid is
(1) $\sqrt{\frac{\sigma}{g(2\rho-d)}}$ (2) $\sqrt{\frac{2\sigma}{g(2\rho-d)}}$
(3) $\sqrt{\frac{6\sigma}{g(2\rho-d)}}$ (4) $\sqrt{\frac{12\sigma}{g(2\rho-d)}}$

44. A drop of water of volume V is pressed between the two glass placed so as to spread to an area A. If T is the surface tension, the normal force required to separate the glass plates is
(1) $\frac{TA^2}{V}$ (2) $\frac{2TA^2}{V}$
(3) $\frac{4TA^2}{V}$ (4) $\frac{TA^2}{2V}$

45. A thin square plate of side 5 cm is suspended vertically from a balance so that lower side just dips into water with side to surface. When the plate is clean ($\theta = 0°$), it appears to weigh 0.044 N. But when the plate is greasy ($\theta = 180°$), it appears to weigh 0.03 N. The surface tension of water is
(1) 3.5×10^{-2} N/m (2) 7.0×10^{-2} N/m
(3) 14.0×10^{-2} N/m (4) 1.08 N/m

46. A loop of 6.28 cm long thread is put gently on a soap film in a wire loop. The film is pricked with a needle inside the soap film enclosed by the thread. If the surface tension of soap solution is 0.030 N/m, then the tension in the thread is
(1) 1×10^{-4} N (2) 2×10^{-4} N
(3) 3×10^{-4} N (4) 4×10^{-4} N

47. A glass rod of radius r_1 is inserted symmetrically into a vertical capillary tube of radius r_2 such that their lower ends are at the same level. The arrangement is now dipped in water. The height to which water will rise into the tube will be (σ = surface tension of water, ρ = density of water)
(1) $\frac{2\sigma}{(r_2-r_1)\rho g}$ (2) $\frac{\sigma}{(r_2-r_1)\rho g}$
(3) $\frac{2\sigma}{(r_2+r_1)\rho g}$ (4) $\frac{2\sigma}{(r_2^2+r_1^2)\rho g}$

48. A large number of droplets, each of radius a, coalesce to form a bigger drop of radius b. Assume that the energy released in the process is converted into the kinetic energy

of the drop. The velocity of the drop is (σ = surface tension, ρ = density)

(1) $\left[\frac{\sigma}{\rho}\left(\frac{1}{a}-\frac{1}{b}\right)\right]^{1/2}$ (2) $\left[\frac{2\sigma}{\rho}\left(\frac{1}{a}-\frac{1}{b}\right)\right]^{1/2}$

(3) $\left[\frac{3\sigma}{\rho}\left(\frac{1}{a}-\frac{1}{b}\right)\right]^{1/2}$ (4) $\left[\frac{6\sigma}{\rho}\left(\frac{1}{a}-\frac{1}{b}\right)\right]^{1/2}$

49. The space between two large horizontal metal plates, 6 cm apart, is filled with a liquid of viscosity 0.8 N/m². A thin plate of surface area 0.01 m² is moved parallel to the length of the plate such that the plate is at a distance of 2 m from one of the plates and 4 cm from the other. If the plate moves with a constant speed of 1 m s^{-1}, then

(1) the layer of the fluid, which is having the maximum velocity, is lying mid-way between the plates

(2) the layer of the fluid, which is in contact with the moving plate, is having the maximum velocity

(3) the layer of the fluid, which is in contact with the moving plate and is on the side of farther plate, is moving with the maximum velocity

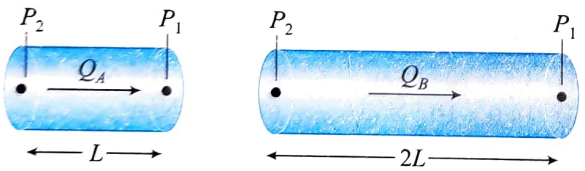
(4) the layer of the fluid, which is in contact with the moving plant and is on the side of nearer plate, is moving with the maximum velocity

50. In the previous question, the net force exerted by fluid on moving plate is

(1) 0.6 N backwards (2) 0.6 N forwards

(3) 0.2 N backwards (4) 0.2 N backwards

51. A viscous fluid is flowing through two horizontal pipes. The pressure difference $P_2 - P_1$ between the ends of each pipe is the same. The pipes have the same radius, although one is twice as long as the other. How does the volume flow rate Q_B in the longer pipe compare with the rate Q_A in the shorter pipe?



(1) Q_B is the same as Q_A.

(2) Q_B is twice as large as Q_A.

(3) Q_B is four times as large as Q_A.

(4) Q_B is one-half as large as Q_A.

52. A rain drop starts falling from a height of 2 km. It falls with a continuously decreasing acceleration and attains its terminal velocity at a height of 1 km. The ratio of the work done by the gravitational force in the first half and the second half of the drops journey is

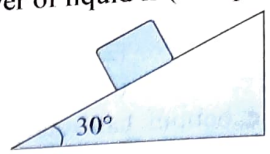
(1) 1 : 1 and the times of fall of the drop in the two halves is a : 1 (where $a > 1$)

(2) 1 : 1 and the times of fall of the drop in the two halves is a : 1 (where $a < 1$)

(3) a : 1 (where $a > 1$) and the times of fall of the drop in the two halves is 1 : 1

(4) a : 1 (where $a < 1$) and the times of fall of the drop in the two halves is 1 : 1

53. A cubical block (of side 2 m) of mass 20 kg slides on inclined plane lubricated with the oil of viscosity $\eta = 10^{-1}$ poise with constant velocity of 10 m/s (g = 10 m/s²). The thickness of layer of liquid is (10^{-1} poise = 10^{-2} N s m²)



(1) 2.5 mm (2) 4 mm

(3) 6 mm (4) 5.5 mm

54. A man is rowing a boat of mass m with a constant velocity 'v_0' in a river. The contact area of boat is 'A' and coefficient of viscosity is η. The depth of river is 'D'. Assume that velocity gradient is constant. The force required to row the boat is

(1) $\frac{\eta A v_0}{2D}$ (2) $\frac{2\eta A v_0}{D}$

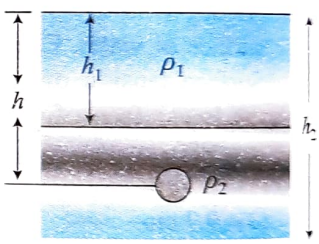
(3) $\frac{\eta A v_0}{D}$ (4) $\frac{3\eta A v_0}{2D}$

55. A soap bubble of radius R is surrounded by another soap bubble of radius $2R$, as shown. Take surface tension = S. Then, the pressure inside the smaller soap bubble, in excess of the atmospheric pressure, will be

(1) $4S/R$ (2) $3S/R$

(3) $6S/R$ (4) None of these

56. Calculate the pressure inside a small air bubble of radius r situated at a depth of h below the free surface of liquids of densities ρ_1 and ρ_2 and surface tensions T_1 and T_2. The thickness of first and second liquids are h_1 and h_2. Take atmosphere pressure = P_0.



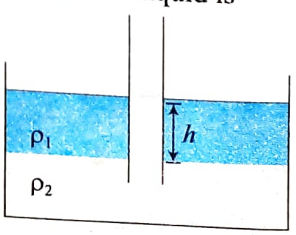
(1) $P_0 + \rho_1 g h_1 + \rho_2 g(h - h_1) - \frac{2T_2}{r}$

(2) $P_0 + \rho_1 g h_1 + \rho_2 g(h - h_1) + \frac{2T_2}{r}$

(3) $P_0 + \rho_1 g h_2 + \rho_2 g(h - h_1) + \frac{2T_2}{r}$

(4) None of these

57. A container is partially filled with a liquid of density ρ_2. A capillary tube of radius r is vertically inserted in this liquid. Now another liquid of density $\rho_1 (\rho_1 < \rho_2)$ is slowly poured in the container to a height h as shown. There is only denser liquid in the capillary tube. The rise of denser liquid in the capillary tube is also h. Assuming zero contact angle, the surface tension of heavier liquid is



(1) $r\rho_2 gh$ (2) $2\pi r\rho_2 gh$

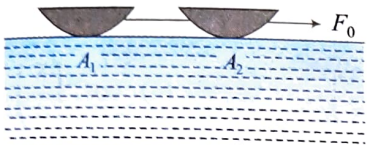
(3) $\frac{r}{2}(\rho_2-\rho_1)gh$ (4) $2\pi r(\rho_2-\rho_1)gh$

58. A glass plate of length 10 cm, breadth 1.54 cm and thickness 0.2 cm, weight 8.2 g in air. It is held vertically with the long side horizontal and the lower half under water. Surface tension of water is 7.3×10^{-2} Nm^{-1} and $g=9.8$ ms^{-2}. The apparent weight of the plate is

(1) 40.16×10^{-3} N (2) 80.16×10^{-3} N

(3) 30.12×10^{-3} N (4) 25.15×10^{-3} N

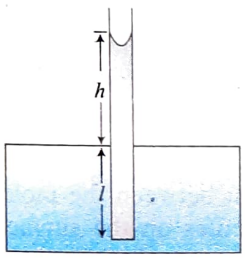
59. Two boats of base areas A_1 and A_2, connected by a string are being pulled by an external force F_0. The viscosity of water is η and depth of the water body is H. When the system attains a constant speed, the tension in the thread will be



(1) $F_0\left(\frac{A_1}{A_2}\right)$ (2) $F_0\frac{A_2}{(A_1+A_2)}$

(3) $F_0\frac{A_1}{(A_1+A_2)}$ (4) $F_0\left(\frac{A_2}{A_1}\right)$

60. Water rises to a height h in a capillary tube lowered vertically into water to a depth l as shown in the figure. The lower end of the tube is now closed, the tube is then taken out of the water and opened again. The length of the water column remaining in the tube will be



(1) $2h$ if $l\geq h$ and $l+h$ if $l\leq h$

(2) h if $l\geq h$ and $l+h$ if $l\leq h$

(3) $4h$ if $l\geq h$ and $l-h$ if $l\leq h$

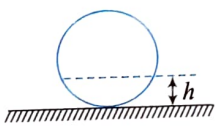
(4) $\frac{h}{2}$ if $l\geq h$ and $l+h$ if $l\leq h$

61. Two parallel glass plates are dipped partly in the liquid of density d keeping them vertical. If the distance between the plates is x, surface tension for liquid is T and angle of contact is θ then rise of liquid between the plates due to capillary will be

(1) $\frac{T\cos\theta}{xd}$ (2) $\frac{2T\cos\theta}{xdg}$

(3) $\frac{2T}{xdg\cos\theta}$ (4) $\frac{T\cos\theta}{xdg}$

62. A liquid is filled in a spherical container of radius R till a height h. At this positions the liquid surface at the edges is also horizontal. The contact angle is



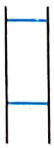
(1) 0 (2) $\cos^{-1}\left(\frac{R-h}{R}\right)$

(3) $\cos^{-1}\left(\frac{h-R}{R}\right)$ (4) $\sin^{-1}\left(\frac{R-h}{R}\right)$

Multiple Correct Answers Type

1. Excess pressure can be $(2T/R)$ for
   (1) spherical drop
   (2) spherical meniscus
   (3) cylindrical bubble in air
   (4) spherical bubble in water
2. If a liquid rises to the same height in two capillaries of the same material at the same temperature, then
   (1) the weight of liquid in both capillaries must be equal
   (2) the radius of meniscus must be equal
   (3) the capillaries must be cylindrical and vertical
   (4) the hydrostatic pressure at the base of capillaries must be same
3. Viscous force is somewhat like friction as it opposes, the motion and is non-conservative but not exactly so, because
   (1) it is velocity dependent while friction is not
   (2) it is velocity independent while friction is
   (3) it is temperature dependent while friction is not
   (4) it is independent of area is like surface tension while friction is dependent
4. When a capillary tube is dipped in a liquid, the liquid rises to a height h in the tube. The free liquid surface inside the tube is hemispherical in shape. The tube is now pushed down so that the height of the tube outside the liquid is less than h. Then
   (1) the liquid will come out of the tube like in a small fountain
   (2) the liquid will ooze out of the tube slowly
   (3) the liquid will fill the tube but not come out of its upper end
   (4) the free liquid surface inside the tube will not be hemispherical
5. A vertical glass capillary tube, open at both ends, contains some water. Which of the following shapes may not be taken by the water in the tube?

(1) (2)

(3)  (4)

6. If n drops of a liquid, each with surface energy E, join to form a single drop, then
   (1) some energy will be released in the process
   (2) some energy will be absorbed in the process
   (3) the energy released or absorbed will be $E(n-n^{2/3})$
   (4) the energy released or absorbed will be $nE(2^{2/3}-1)$

7. Eight identical droplets, each falling under gravity in the earth's atmosphere with terminal velocity v, combine to form a single drop. The terminal velocity of single drop and drag force on it are calculated. Find the correct statement from the following.
   (1) Final terminal velocity is $4v$
   (2) Final terminal velocity is $8v$
   (3) Drag force is increased to four times
   (4) Drag force is increased to eight times

8. A sphere is dropped into a viscous liquid of viscosity η from some height. If the density of material and liquid are ρ and σ respectively ($\rho > \sigma$) then which of the following is/are incorrect.
   (1) The acceleration of the sphere just after entering the liquid is $g\left(\frac{\rho - \sigma}{\rho}\right)$.
   (2) Time taken to attain terminal speed $t \propto \rho^0$
   (3) At terminal speed, the viscous force is maximum
   (4) At terminal speed, the net force acting on the sphere is zero

9. Two separate bubbles (radii 0.002 m and 0.004 m) formed of the same soap solution (surface tension 0.07 Nm^{-1}) comes together to form a double bubble. The internal film common to both the bubbles has radius R. Then,
   (1) $R = 0.004$ m
   (2) $R = 0.003$ m
   (3) internal film is concave towards smaller bubble
   (4) internal film is concave towards bigger bubble

10. The velocity of liquid (v) in steady flow at a location through cylindrical pipe is given by $v = v_0\left(1 - \frac{r^2}{R^2}\right)$, where r is the radial distance of that location from the axis of the pipe and R is the inner radius of pipe, if $R = 10$ cm, volume rate of flow through the pipe is $\pi/2 \times 10^{-2}\ m^3 s^{-1}$ and the coefficient of viscosity of the liquid is 0.75 N s m^{-2}
    (1) Magnitude of viscous force per unit area at $r = 4$ cm is 6 N/m^2
    (2) Maximum velocity of flow of liquid is 1 m/s
    (3) Viscous force per unit area increases with radial distance
    (4) Viscous force per unit area decreases with radial distance

11. A water drop and a mercury drop is taken inside identical horizontal conical glass pipes. Then
    (1) the water drop tends to move towards the narrow end
    (2) the water drop tends to move towards the wide end
    (3) the mercury drop tends to move towards the wide end
    (4) the mercury drop tends to move towards the narrow end

Linked Comprehension Type

For Problems 1–2

A lead sphere of 1.0 mm diameter and relative density 11.20 attains a terminal velocity of 0.7 $cm\,s^{-1}$ in a liquid of relative density 1.26.

1. Determine the coefficient of dynamic viscosity of the liquid.
   (1) 0.45 N/m^2 (2) 0.85 N/m^2
   (3) 0.56 N/m^2 (4) 0.77 N/m^2

2. What is the value of the Reynolds number?
   (1) 0.01 (2) 0.03
   (3) 0.15 (4) 0.26

For Problems 3–4

A long capillary tube of radius 0.2 mm is placed vertically inside a beaker of water.

3. If the surface tension of water is 7.2×10^{-2} N/m and the angle of contact between glass and water is zero, then determine the height of the water column in the tube.
   (1) 3 cm (2) 9 cm
   (3) 7 cm (4) 5 cm

4. If the tube is now pushed into water so that only 5.0 cm of its length is above the surface, then determine the angle of contact between the liquid and glass surface.
   (1) $\cos^{-1}\left(\frac{4}{5}\right)$ (2) $\cos^{-1}\left(\frac{5}{7}\right)$
   (3) $\cos^{-1}\left(\frac{3}{5}\right)$ (4) $\cos^{-1}\left(\frac{5}{4}\right)$

For Problems 5–7

An oil of relative density 0.9 and viscosity 0.12 kg/ms flows through a 2.5 cm diameter pipe with a pressure drop of 38.4 kN/m^2 in a length of 30 m. Determine

5. Determine the discharge
   (1) $2.16\times10^{-4}\ m^3/s$ (2) $2.9\times10^{-3}\ m^3/s$
   (3) $1\times10^{-4}\ m^3/s$ (4) $2\times10^{-4}\ m^3/s$

6. Determine the shear stress at the pipe wall
   (1) $8\times10^{-6}\ N/m^2$ (2) $3.9\times10^{-6}\ N/m^2$
   (3) $2.3\times10^{-6}\ N/m^2$ (4) $10.6\times10^{-6}\ N/m^2$

7. Determine the power required to maintain the flow
   (1) 2.2 W (2) 3.84 W
   (3) 5.6 W (4) 9.3 W

For Problems 8–10

Molecular forces exist between the molecules of a liquid in a container. The molecules on the surface have unequal force leading to a tension on the surface. If this is not compensated by a force, the equilibrium of the liquid will be a difficult task. This leads to an excess pressure on the surface. The nature of the meniscus can inform us of the direction of the excess pressure. The angle of contact of the liquid decided by the forces between the molecules, air and container can make the angle of contact.

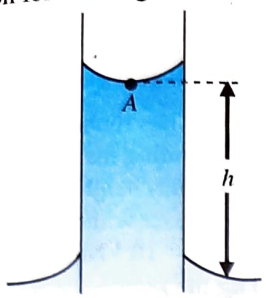
8. The direction of the excess pressure in the meniscus of a liquid of angle of contact $2\pi/3$ is
   (1) upward (2) downward
   (3) horizontal (4) cannot be determined

9. If the excess pressure in a soap bubble is p, the excess pressure in an air bubble is
   (1) $\frac{p}{2}$ (2) p
   (3) $2p$ (4) $4p$

10. In a meniscus of radius r, with excess pressure p in atmospheric pressure p_0, the force experienced is
    (1) $(p - p_0)\pi r^2$ (2) $(p - p_0)2\pi r$
    (3) $p\pi r^2$ (4) $p_0 2\pi r$

For Problems 11–13

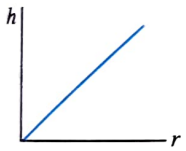
Figure shows a capillary tube of radius r dipped into water. The atmospheric pressure is P_0 and the capillary rise of water is h. s is the surface tension for water–glass.

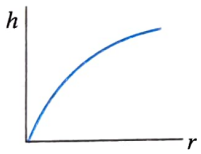


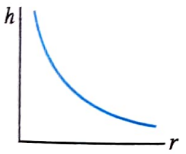
11. The pressure inside water at the point A (lowest point of the meniscus) is
(1) P_0 (2) $P_0 + \frac{2s}{r}$
(3) $P_0 - \frac{2s}{r}$ (4) $P_0 - \frac{4s}{r}$

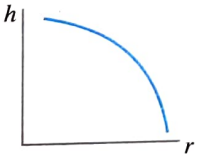
12. Initially, $h = 10$ cm. If the capillary tube is now inclined at 45°, the length of water rising in the tube will be
(1) 10 cm (2) $10\sqrt{2}$ cm
(3) $\frac{10}{\sqrt{2}}$ cm (4) none of these

13. Which of the following graphs may represent the relation between the capillary rise h and the radius r of the capillary?

(1) 

(2) 

(3) 

(4) 

For Problems 14–15

A man starts (from rest $u = 0$) rowing his stationary cuboidal boat of base area $A = 10$ m^2. The driving force on the boat due to rowing is 100 N in the direction of motion. The depth of the lake is 10 m and the combined mass of man and the boat to be 150 kg and velocity gradient uniform. [Take coefficient of viscosity of water = 15 poise]

14. Find the maximum velocity that the boat can achieve.
(1) 50 m/s (2) 75 m/s
(3) $\frac{100}{3}$ m/s (4) $\frac{200}{3}$ m/s

15. The time (in seconds) in which he will attain half of this maximum velocity is
(1) 150 ln 2 s (2) 80 ln 2 s
(3) 100 ln 2 s (4) 200 ln 2 s

Matrix Match Type

1. Match the following:

Column I	Column II
i. Splitting of bigger drop into small droplets	a. Temperature increases
ii. Formation of bigger drop from small droplets	b. Temperature decreases
iii. Spraying of liquid	c. Surface energy increases
	d. Surface energy decrease

2. A glass plate of length 10 cm breadth 4 cm and thickness 0.4 cm weighs 20 g in air. It is held vertically with long side horizontal and half the plate immersed in water. What will be its apparent weight? Surface tension of water = 70 J dyn/cm.

Column I	Column II
i. Buoyant force due to liquid	a. 20 gf
ii. Force due to surface tension	b. 8 gf
ii. Apparent weight of the plate	c. 1.5 gf
	d. 13.5 gf

3. Column II depends on physical quantity/physical law given in Column I. Match the following columns and select the correct option from the codes given below.

Column I	Column II
i. stroke's law	a. radius
ii. terminal velocity	b. density of material of the body
iii. excess pressure inside mercury drop	c. coefficient of viscosity
iv. viscous force on a plate moving horizontally on a liquid surface	d. surface tension
	e. velocity gradient

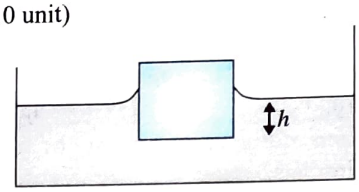
Codes

	i	ii	iii	iv
(1)	a, c	a, b, c	a, d	c, e
(2)	a, b, c	c, e	a, c	a, d
(3)	a, d	a, c	c, e	a, b, c, d
(4)	c	a, c	a	a, d

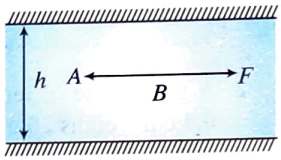
Numerical Value Type

1. A cube of side a and mass m just floats on the surface of water as shown in the figure. The surface tension and density of water are T and ρ_w, respectively. If angle of contact between cube and water surface is zero, find the

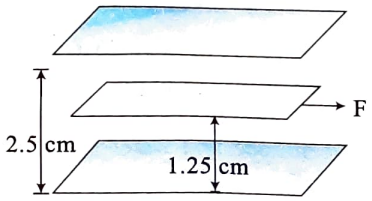
distance h (in metres) between the lower face of cube and surface of the water.
(Take $m = 1$ kg, $g = 10$ m s^{-2}, $aT = \frac{10}{4}$ unit and $\rho_w a^2 g = 10$ unit)



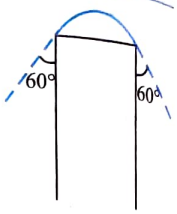
2. A thin plate AB of large area A is placed symmetrically in a small gap of height h filled with water of viscosity η_0 and the plate has a constant velocity v by applying a force F as shown in the figure. If the gap is filled with some other liquid of viscosity $0.75\ \eta_0$, at what minimum distance (in cm) from top wall should the plate be placed in the gap, so that the plate can again be pulled at the same constant velocity V, by applying the same force F? (Take $h = 20$ cm)



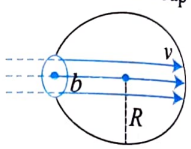
3. The diameter of a gas bubble formed at the bottom of a pond is $d = 4$ cm. When the bubble rises to the surface, its height is (tension of water = $T = 0.07$ N m^{-1}.)
4. n drops of water, each of radius 2 mm, fall through air at a terminal velocity of 8 cm s^{-1}. If they coalesce to form a single drop, then the terminal velocity of the combined drop is 32 cm s^{-1}. Find n.
5. A substance breaks down under a stress of 10^5 Pa. If the density of the wire is 2×10^3 kg/m^3, find the minimum length of the wire which will break under its own weight ($g = 10$ m s^{-2}).
6. The capillaries of radius r and $2r$ and lengths l and $2l$ respectively are joined in series. The streamline flow is maintained in capillaries. The pressure difference across first and second capillary is P_1 and P_2 respectively. Find P_1/P_2.
7. The coefficient of viscosity of blood is 2.12×10^{-3} Pa s and its density is 2.12×10^{-3} Pa s. Find the blood flow rate (in $\pi \times 10^{-6}$ m^3 s^{-1}), corresponding to the largest average velocity in an artery of radius 2×10^{-3} m for the flow to be laminar.
8. A plate of area 2 m^2 is made to move horizontally with a speed of 2 ms^{-1} by applying a horizontal tangential force over the free surface of a liquid. The depth of the liquid is 1 m and the liquid in contact with the bed is stationary. Coefficient of viscosity of liquid = 0.01 poise. Find the tangential force needed to move the plate (in N).
9. A space 2.5 cm wide between two large plane surfaces is filled with oil. Force required to drag a very thin plate of area 0.5 m^2 just midway the surfaces at a speed of 0.5 ms^{-1} is 1N. Find the coefficient of viscosity of oil (in $\times 10^{-3}$ kg-sm^{-2}).



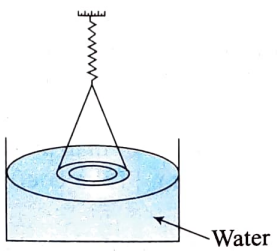
10. A soap bubble is being blown on a tube of radius 1 cm. The surface tension of the soap solution is 0.05 Nm^{-1} and the bubble makes 60° with the tube as shown in the figure. Find the excess of pressure over the atmosphere pressure in the tube (in Pa)



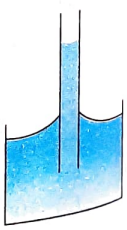
11. A bowl whose bottom has round holes of diameter 1 mm is filled with water. Assuming that surface tension acts only at holes, find the maximum height (in cm) up to which water can be filled in the vessel without leakage. (Given, surface tension of water = 75×10^{-3} Nm^{-1}, $g = 10$ ms^{-2} and density of water = 1000 kg m^{-3}).
12. A soap film is made by dropping a circular frame of radius b in soap solution. A bubble is formed by blowing air with speed v in the form of a cylinder. Air stops after striking surface of soap bubble. Density of air is ρ. The radius of the bubble is $R > b$. The radius R of the bubble when the soap bubble separates from the ring is $\frac{\beta T}{\rho v^2}$. Find the value of β. (where, surface tension of liquid is T).



13. A metallic sphere of radius 1.0×10^{-3} m and density 10×10^4 kg/m^3 enters a tank of water, after a free fall through a distance of h in the earth's gravitational field. If its velocity remains unchanged after entering water, determine the value of h (in m). Given: coefficient of viscosity of water = 1.0×10^{-3} N-s/m^2, $g = 10$ m/s^2 and density of water = 1.0×10^3 kg/m^3.
14. A small steel ball falls through a syrup at a constant speed of 10 cm s^{-1}. If the steel ball is pulled upwards with a force equal to twice its effective weight, find the terminal velocity of the ball (in cm/s) in this case.
15. A circular ring has inner and outer radii equal to 20 mm and 50 mm respectively. Mass of the ring is $m = 0.8$ g. It gently pulled out vertically from a water surface by a sensitive spring. When the spring is stretched 4 cm from its equilibrium position the ring is on verge of being pulled out from the water surface. If spring constant is $k = 0.9$ Nm^{-1} find the surface tension of water.



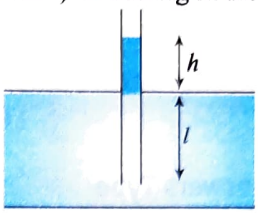
16. A long thin walled capillary tube of mass $M = 10$ g and radius $r = 7$ mm is partially immersed in a liquid of surface tension $T = 0.75$ Nm^{-1}. The angle of contact for the liquid and the tube wall is 60°. How much force is needed to hold the tube vertically? Neglect buoyancy force on the tube.



17. A tapering glass capillary tube A of length 0.1 m has diameters 10^{-3} m and 5×10^{-4} m at the ends. When it is just immersed in a liquid at 0°C with larger radius in contact with liquid surface, the liquid rises 8×10^{-2} m in the tube. In another experiment, in a cylindrical glass capillary tube B, when immersed in the same liquid at 0°C, the liquid rises to 6×10^{-2} m height. The rise of liquid in tube B is only 5.5×10^{-2} m when the liquid is at 50°C. Find the rate at which the surface tension changes (in $\times 10^{-4}$ N/m °C) with temperature considering the change to be linear. The density of liquid is $(1/14) \times 10^4$ kg/m^3 and the angle of contact is zero. Effect of temperature on the density of liquid and glass is negligible.

18. There is an air bubble of radius 1.0 mm in a liquid of surface tension 0.075 N/m and density 1000 kg/m^3. The bubble is at a depth of 10 cm below the free surface. By what amount is the pressure (in Pa) inside the bubble greater than the atmospheric pressure? (Take $g = 9.8$ m/s^2)

19. The capillary tube is dipped in water vertically. It is sufficiently long so that water rises to maximum height $h = 10$ cm in the tube. The length of the portion immersed in water is $l > h$. The lower end of the tube is closed, the tube is taken out and opened again. Then, find the length of the water column (in cm) remaining in the tube.



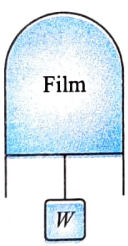
Archives

JEE MAIN

Single Correct Answer Type

1. Work done in increasing the size of a soap bubble from a radius of 3 cm to 5 cm is nearly. (Surface tension of soap solution = 0.03 Nm^{-1})
(1) 4π mJ (2) 0.2π mJ
(3) 2π mJ (4) 0.4π mJ **(AIEEE 2011)**

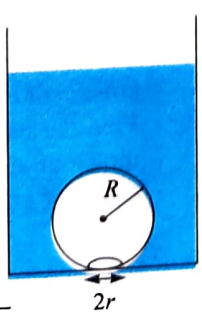
2. A thin liquid film formed between a U-shaped wire and a light slider supports a weight of 1.5×10^{-2} N (see figure). The length of the slider is 30 cm and its weight negligible. The surface tension of the liquid film is



(1) 0.0125 Nm^{-1} (2) 0.1 Nm^{-1}
(3) 0.05 Nm^{-1} (4) 0.025 Nm^{-1}
(AIEEE 2012)

3. Assume that a drop of liquid evaporates by decrease in its surface energy, so that its temperature remains unchanged. What should be the minimum radius of the drop for this to be possible? The surface tension is T, density of liquid is ρ and L is its latent heat of vaporization.
(1) $\sqrt{\frac{T}{\rho L}}$ (2) $\frac{T}{\rho L}$
(3) $\frac{2T}{\rho L}$ (4) $\frac{\rho L}{T}$ **(JEE Main 2013)**

4. On heating water, bubbles being formed at the bottom of the vessel detach and rise. Take the bubbles to be spheres of radius R and making a circular contact of radius r with the bottom of the vessel. If $r << R$ and the surface tension of water is T, the value of r just before bubbles detach is (density of water is ρ_w)



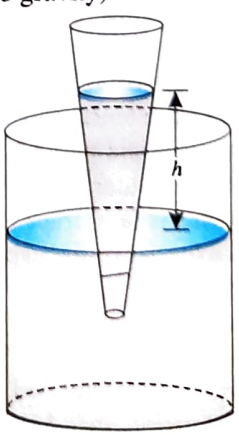
(1) $R^2\sqrt{\frac{\rho_w g}{T}}$ (2) $R^2\sqrt{\frac{3\rho_w g}{T}}$
(3) $R^2\sqrt{\frac{\rho_w g}{6T}}$ (4) $R^2\sqrt{\frac{\rho_w g}{3T}}$ **(JEE Main 2014)**

5. The following observations were taken for determining surface tension T of water by capillary method:
Diameter of capillary, $D = 1.25 \times 10^{-2}$ m
Rise of water, $h = 1.45 \times 10^{-2}$ m
Using $g = 9.80$ m/s^2 and the simplified relation $T = \frac{rhg}{2} \times 10^3$ N/m, the possible error in surface tension is closest to
(1) 2.4% (2) 10%
(3) 0.15% (4) 1.5%
(JEE Main 2017)

JEE ADVANCED

Single Correct Answer Type

1. A glass capillary tube is of the shape of truncated cone with an apex angle α so that its two ends have cross sections of different radii. When dipped in water vertically, water rises in it to a height h, where the radius of its cross section is b. If the surface tension of water is S, its density is ρ, and its contact angle with glass is θ, the value of h will be (g is the acceleration due to gravity)



(1) $\frac{2S}{b\rho g}\cos(\theta - \alpha)$ (2) $\frac{2S}{b\rho g}\cos(\theta + \alpha)$
(3) $\frac{2S}{b\rho g}\cos(\theta - \alpha/2)$ (4) $\frac{2S}{b\rho g}\cos(\theta + \alpha/2)$
(JEE Advanced 2014)

2. Consider an expanding sphere of instantaneous radius R whose total mass remains constant. The expansion is such that the instantaneous density ρ remains uniform throughout the volume. The rate of fractional change in density $\left(\frac{1}{\rho}\frac{d\rho}{dt}\right)$ is constant. The velocity v of any point on the surface of the expanding sphere is proportional to

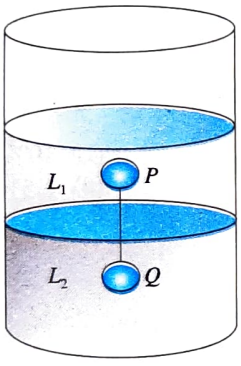
(1) R^3 (2) $\frac{1}{R}$

(3) R (4) $R^{2/3}$

(JEE Advanced 2017)

Multiple Correct Answers Type

1. Two spheres P and Q of equal radii have densities ρ_1 and ρ_2, respectively. The spheres are connected by a massless string and placed in liquids L_1 and L_2 of densities σ_1 and σ_2 and viscosities η_1 and η_2, respectively. They float in equilibrium with the sphere P in L_1 and sphere Q in L_2 and the string being taut (see figure). If sphere P alone in L_2 has terminal velocity $\vec{V}_P$ and Q alone in L_1 has terminal velocity $\vec{V}_Q$, then



(1) $\frac{|\vec{V}_P|}{|\vec{V}_Q|}=\frac{\eta_1}{\eta_2}$ (2) $\frac{|\vec{V}_P|}{|\vec{V}_Q|}=\frac{\eta_2}{\eta_1}$

(3) $\vec{V}_P\cdot\vec{V}_Q>0$ (4) $\vec{V}_P\cdot\vec{V}_Q<0$

(JEE Advanced 2015)

2. A uniform capillary tube of inner radius r is dipped vertically into a beaker filled with water. The water rises to a height h in the capillary tube above the water surface in the beaker. The surface tension of water is σ. The angle of contact between water and the wall of the capillary tube is θ. Ignore the mass of water in the meniscus. Which of the following statements is (are) true?

(1) For a given material of the capillary tube, h decreases with increase in r.

(2) For a given material of the capillary tube, h is independent of σ.

(3) If this experiment is performed in a lift going up with a constant acceleration, then h decreases.

(4) h is proportional to contact angle θ.

(JEE Advanced 2018)

3. Consider a thin square plate floating on a viscous liquid in a large tank. The height h of the liquid in the tank is much less than the width of the tank. The floating plate is pulled horizontally with a constant velocity u_0. Which of the following statements is (are) true?

(1) The resistive force of liquid on the plate is inversely proportional to h

(2) The resistive force of liquid on the plate is independent of the area of the plate

(3) The tangential (shear) stress on the floor of the tank increases with u_0

(4) The tangential (shear) stress on the plate varies linearly with the viscosity η of the liquid

(JEE Advanced 2018)

Linked Comprehension Type

For Problems 1–3

When liquid medicine of density ρ is to be put in the eye, it is done with the help of a dropper. As the bulb on the top of the dropper is pressed, a drop forms at the opening of the dropper. We wish to estimate the size of the drop. We first assume that the drop formed at the opening is spherical because that requires a minimum increase in its surface energy. To determine the size, we calculate the net vertical force due to the surface tension T when the radius of the drop is R. When the force becomes smaller than the weight of the drop, the drop gets detached from the dropper.

(IIT-JEE 2010)

1. If the radius of the opening of the dropper is r, the vertical force due to the surface tension on the drop of radius R (assuming $r << R$) is

(1) $2\pi rT$ (2) $2\pi RT$

(3) $2\pi r^2T/R$ (4) $2\pi R^2T/r$

2. If $r=5\times10^{-4}$m, $\rho=10^3$ kg m^{-3}, $g=10$ m s^{-12}, $T=0.11$ Nm^{-1}. the radius of the drop when it detaches from the dropper is approximately

(1) 1.4×10^{-3} m (2) 3.3×10^{-3} m

(3) 2.0×10^{-3} m (4) 4.1×10^{-3} m

3. After the drop detaches, its surface energy is

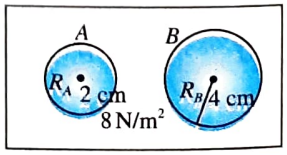
(1) 1.4×10^{-6} J (2) 2.7×10^{-6} J

(3) 5.4×10^{-6} J (4) 8.1×10^{-6} J

Numerical Value Type

1. Two soap bubbles A and B are kept in a closed chamber where the air is maintained at pressure 8 N/m^2. The radii of bubbles A and B are 2 cm and 4 cm, respectively. Surface tension of the soap water used to make bubbles is 0.04 N/m. Find the ratio nB/nA, where nA and nB are the number of moles of air in bubbles A and B, respectively. (Neglect the effect of gravity.)

(IIT-JEE 2009)



2. Consider two solid spheres P and Q each of density 8 gm cm^{-3} and diameters 1 cm and 0.5 cm, respectively. Sphere P is dropped into a liquid of density 0.8 gm cm^{-3} and viscosity $\eta = 3$ poiseulles. Sphere Q is dropped into a liquid of density 1.6 gm cm^{-3} and viscosity $\eta =$ 2 poiseulles. The ratio of the terminal velocities of P and Q is

(JEE Advanced 2016)

3. A drop of liquid of radius $R = 10^{-2}$ m having surface tension $S = \frac{0.1}{4\pi}$ Nm^{-1} divides itself into K identical drops. In this process the total change in the surface energy $\Delta U = 10^{-3}$ J. If $K = 10^{\alpha}$ then the value of α is _____.

(JEE Advanced 2017)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)	2. (3)	3. (2)	4. (4)	5. (3)
6. (4)	7. (3)	8. (3)	9. (2)	10. (2)
11. (1)	12. (3)	13. (1)	14. (1)	15. (2)
16. (1)	17. (3)	18. (1)	19. (3)	20. (2)
21. (4)	22. (3)	23. (3)	24. (2)	25. (2)
26. (2)	27. (1)	28. (1)	29. (4)	30. (4)
31. (2)	32. (1)	33. (1)	34. (4)	35. (3)
36. (2)	37. (1)	38. (1)	39. (1)	40. (1)
41. (2)	42. (4)	43. (4)	44. (2)	45. (2)
46. (3)	47. (1)	48. (4)	49. (2)	50. (1)
51. (4)	52. (1)	53. (2)	54. (3)	55. (3)
56. (2)	57. (3)	58. (2)	59. (3)	60. (1)
61. (2)	62. (2)			

Multiple Correct Answers Type

1. (1),(2),(3),(4)	2. (1),(2)	3. (1),(3)
4. (3),(4)	5. (1),(2),(3)	6. (1),(3)
7. (1),(4)	8. (1),(2),(3)	9. (1),(3)
10. (1),(2),(3)	11. (1),(3)	

Linked Comprehension Type

1. (4)	2. (1)	3. (3)	4. (2)	5. (3)
6. (1)	7. (2)	8. (1)	9. (1)	10. (3)
11. (3)	12. (2)	13. (3)	14. (4)	15. (3)

Matrix Match Type

1. i. → b., c.; ii. → a., d.; iii. → b., c.
2. i. → b.; ii. → c.; iii. → d.
3. (1)

Numerical Value Type

1. (2)	2. (5)	3. (5)	4. (8)	5. (5)
6. (8.00)	7. (4.00)	8. (0.004)	9. (25)	10. (10)
11. (3.00)	12. (4.00)	13. (20)	14. (10)	15. (0.08)
16. (0.133)	17. (1.4)	18. (1130)	19. (20.00)	

ARCHIVES

JEE Main

Single Correct Answer Type

1. (4)	2. (4)	3. (3)	4. (None)	5. (4)

JEE Advanced

Single Correct Answer Type

1. (4) 2. (3)

Multiple Correct Answers Type

1. (1),(3) 2. (1),(3) 3. (1),(3),(4)

Linked Comprehension Type

1. (3) 2. (1) 3. (2)

Numerical Value Type

1. (6) 2. (3) 3. (6)